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ANALYZING CARBON TRADE PERMITS IN OPTIMIZATION FRAMEWORK

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ABSTRACT

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Analyzing Carbon trade permits in Optimization framework

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The threats caused by global warming motivate different stake holders to deal with and control them. This Master's thesis focuses on analyzing carbon trade permits in optimization framework. The studied model determines optimal emission and uncertainty levels which minimize the total cost. Research questions are formulated and answered by using different optimization tools. The model is developed and calibrated by using available consistent data in the area of carbon emission technology and control. Data and some basic modeling assumptions were extracted from reports and existing literatures . The data collected from the countries in the Kyoto treaty are used to estimate the cost functions. Theory and methods of constrained optimization are briefly presented. A two-level optimization problem (individual and between the parties) is analyzed by using several optimization methods. The combined cost optimization between the parties leads into multivariate model and calls for advanced techniques. Lagrangian, Sequential Quadratic Programming and Differential Evolution (DE) algorithm are referred to. The role of inherent measurement uncertainty in the monitoring of emissions is discussed. We briefly investigate an approach where emission uncertainty would be described in stochastic framework. MATLAB software has been used to provide visualizations including the relationship between decision variables and objective function values. Interpretations in the context of carbon trading were briefly presented. Suggestions for future work are given in stochastic modeling, emission trading and coupled analysis of energy prices and carbon permits.

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1 Introduction

In recent years our planet, earth has faced drastic climate changes. There has been global warming that makes our planet hotter than one century ago. Several environmental risks such as drought, flooding, spread of tropical diseases, destruction of cost line due to high ocean tides, melting of ice covering peaks of mountains, raging forest fire, hurricanes and storms have brought worrisome to the mass of people. Data shows that, over last 100 years, the temperature of air near by earth's surface has risen by almost 1 degree Celsius scale and for 20 warmest years, 19 have occurred since 1980 and three hottest years ever observed, have occurred in the last eight years [19]. The warming of planet earth has been caused by high accumulation of green house gases (GHG) which absorb more heat from the sun. Most of greenhouse gases are related to human activities such as heavy industries and intensive agriculture. Different treaties have been signed by industrial developed and fast developing countries to curb the situation. Among those treaties, Kyoto Protocol has become successful to address and give solution of reducing global warming threats. This Protocol embraces flexible mechanisms of alleviating global warming such as Emission Trading, Clean Development Mechanism (CDM) and Joint Implementation [20].

In this section, three main points will be put in the central of our discussion. The first one will be the background of the thesis, where by a reader will find exactly what motivated the author to the topic and the tools author has been using to analyze research data in the context of carbon trading. The second point will be the objective of the thesis where by main goals and research questions are outlined and how they will be solved explicitly. The last point is the structure of the thesis in brief, where all sections are outlined shortly. After this short introduction the background of the thesis will be discussed in the following subsection.

1.1 Background of the Thesis

Drastic climate change due to high concentrations of greenhouse gases (GHG) in the atmosphere is one of the most severe environmental risks. Most of green house gases are caused by human activities such as industries and agriculture. Most of these gases are categorized into six types [5], which are carbondioxide (CO_2), methane (CH_4), nitrous oxide (N_2O), hydrofluorocarbons (HFCs), Perfluorocarbons (PFCs) and sulfur hexafluoride (SF_6).

The global efforts to alleviate the threats of climate changes have been made by several international conventions. The United Nations Framework Convention on Climate Change [6] that was conducted at Rio de Janeiro, Brazil in 1992 aimed at reinforcing her Parties to stabilize greenhouse gas concentrations in the atmosphere at a level that would prevent dangerous anthropogenic interference with the climate system. Later under the same convention, Kyoto Protocol was established at Kyoto, Japan in December, 11^{th}, 1997. The protocol came into effect in February, 16^{th}, 2005 after Russian ratification. It specifies to its members an emission level which has not to be exceeded within the commitment period of 2008-2012. Under Kyoto Protocol, industrialized countries [7] agreed principally to cut down emission on average of about 5.2 percent below 1990 levels.

Article 5 of Kyoto protocol [8], requires her Parties to conduct the national system of estimating anthropogenic emission and prepare the inventory for reporting emissions by sources and their removal by sinks. The guidelines for such national system have been specified in International Panel on Climate Change [9] . However the protocol seeks from her members to report the emission to her Convention Secretariat and comply to their levels of endowment but it does not regard uncertainty which is associated with reported emission.

All emission estimates according to [10] contain uncertainty due to errors in measurement instruments, natural variability of emission generating process and bias expert judgements. However [3] pointed that uncertainties are generally caused by emission factors and activity data reflecting GHG related activities. Uncertainty in emission factors arise from lack of sufficient knowledge about processes generating emissions, lack of relevant measurements and thus inappropriate generalizations. Depending on the source generating emission, [11] and [12] reported that uncertainty of CO_2 from energy sources is small, around 5 percent. Other pollutants are reported to have much more uncertainty, usually more than 20 percent as pointed in [13]. These include N_2O from agricultural soil, PFCs and SF_6 from aluminium production, CH_4 from landfills and N_2O from road traffic. The implementation of Kyoto Protocol need high quality emission inventories to ensure that parties comply to their commitments and right reducing measures of emissions are taken. Uncertainty reporting is a crucial in the context of emission trading. Kyoto protocol in response, from its article 17 introduced emission trading to facilitate achievement of national agreed reduction targets. Annex B countries to the protocol allowed to sell their excess emission reductions as it is potential for cost-efficiency.

Carbon market performance under inventory reporting has been explained deeply by [1] and [2]. Both assumed that uncertainty has to be associated with emissions to oversee the compliance with Kyoto targets. In order to meet these targets one has to invest in emission reduction or monitoring uncertainty or by buying permits.

Reducing emission levels create costs investment in technology process improvements and renovations of process installations. Reduction of uncertainty is another source of costs. Improving measurement accuracy and extending the scope of emission monitoring is possible only by investment into improved technology. Hence smaller uncertainty means high cost level.

Several optimization methods have been used to solve the optimal value such as emission levels of different Parties or Regions under consideration. Classical Lagrangian, Sequential Quadratic Programming (SQP) and Differential evolution methods have been applied widely to analyze to solve the optimal values and to analyze relationship existing between different decision variables in the model. In the next subsection which is the objective of the thesis, these methods will be discussed briefly on how they are going to solve our optimization problems.

1.2 Objective of the Thesis

Primary objective of this master's thesis is to analyze two schemes in Emission trading. The first scheme is when there is no transaction, i.e. no trade of carbon permits but each Party carry its own initiative to cut down emissions and relative uncertainty. The second scheme of our analysis will be upon the situation where by the trade of permits is done, by carbon permits being exchanged between potential buyers and sellers. In both schemes we are going to find and compare total costs for cutting down both emissions and relative uncertainty.

The secondary objectives of our work are to analyze:

- the *nature* of our optimal solution.
- effect of introducing relative uncertainty to the cost function.
- emission uncertainty in stochastic framework.

1.3 Research Questions

Our research questions have been extracted from objectives and will be answered immediately to the next coming sections. The research questions are as follows:

Research Question 1

Is there any significant impact between the situations when carbon trade permits are traded or not traded?
To answer this question cost functions are introduced. These functions are based on the data collected from the countries of Kyoto treaty (confer table 1). We are going to use the method of Lagrange multiplier, Sequential Quadratic Programming (SQP) and Differential Evolution (DE) to calculate the optimal points of both emissions and relative uncertainty and compare the total cost for reducing emissions and relative uncertainty in case there is no transactions and when carbon permits are bought or traded.

Research Question 2

Does our solution attains local or global minimum?

The nature of the solution will be determined by using first order conditions (KKT) and sufficient second order conditions.

Research Question 3

Does inclusion of relative uncertainty to the cost function has an impact in analyzing the cost of reducing uncertainty?

We are going to replace absolute uncertainty with relative uncertainty and find the relationship between emissions and relative uncertainty.

Research Question 4

Does stochastic analysis of Emission uncertainty important in modeling emissions and relative uncertainty?

We are going to study an alternative formulation in modeling of total emissions when the uncertainty intervals are replaced by probability distributions.

After describing how are we going to answer our research questions, the next subsection will describe briefly how the thesis is structured.

1.4 Structure of the Thesis

The thesis work comprises of seven sections. It begins with introduction section in brief describing the background, objective and research questions related to our master's thesis. In this section some important points such as Lagrangian, Sequential Quadratic Programming (SQP), Differential Evolution optimization methods, local and global minimum points and costs of reducing both emission and relative uncertainty are briefly reviewed.

Analysis of necessary and sufficient conditions for optimality of general constrained minimization problem have been presented in section 2. In this section, Lagrangian of constrained problem, active constraints, constraint qualification conditions and Kuhn Tucker conditions for optimality have been deeply examined. The end of this section is marked by brief explanation of saddle point conditions which are more restrictive than Kuhn Tucker conditions.

Methodology and modeling have been presented in section 3. The section describes how the model has been chosen, parameter estimates of the model, introduction of relative uncertainty to the model and analysis of local minima. Assumptions used in estimating the cost parameters of the model have been clearly described. Data and source from which the data have extracted are presented briefly in section 4. Section 5 entails extensively several methods have been used to come up with the solution of our optimization problem. Classical Lagrangian for optimization, Sequential Quadratic Programming (SQP) and Differential Evolution(DE) methods have been adequately described and used to solve parameters such as optimal emission level and relative emission uncertainty to the situations where permits are not transacted and when permits are allowed to be bought or sold within the participants to the carbon trade. In addition to the optimal number of permits each

Party should be allocated are solved and illustrated using MATLAB plots. The relationship between decision variables and objective function value has been explicitly presented. The section is ending by evaluating the model results by giving brief interpretations in the context of Carbon Trading.
Section 6 is concerned about stochastic emission uncertainty, where by uncertainty related to emission levels are explicitly analyzed in stochastic framework. The final section is conclusion and future work followed by references.

2 Constrained Optimization

In this section we are going to analyze necessary and sufficient conditions for optimality of general constrained minimization problem(CP). We are going to examine deeply the Lagrangian of constrained problem, 'active constraints', constraint qualification conditions and Kuhn Tucker conditions for optimality.

2.1 Lagrangian for constrained problem

Let us consider the following general constrained minimization problem (CP) as it is pointed in [17]:

$$\text{(CP)} \qquad \min f(x) = f(x_1, ..., x_n) \tag{2.1}$$

$$\text{s.t } c_i(x) = 0, \qquad i = 1, ..., m_e \tag{2.2}$$

$$c_i(x) \geq 0, \qquad i = m_e + 1, ..., m \tag{2.3}$$

The constraints $c(x) = (c_i(x), ..., c_m(x))^T$ in (2.2) and (2.3) is a column vector. We are going to define important terminologies in the context of constrained optimization as follows:

Definition 1. *(Feasible set)*
*The **feasible set** $\Omega \subset \mathbb{R}^n$ is the set of all points which satisfy all given constraints.*

According to [16], p.308, active and inactive set of constraints in constrained optimization can be explained briefly as follows:

Definition 2. *(Active set)*
The active set $\mathcal{A}(x)$ at any feasible point x consist of the set of equality constraint

indices $\mathcal{E}$ together with indices of inequality constraints i such that $c_i(x) = 0$, that is, $\mathcal{A}(x) = \mathcal{E} \cup \{i \in \mathcal{I} | c_i(x) = 0\}$.

We can say that the inequality constraint $i \in \mathcal{I}$ is said to be *active* at a feasible point x if $c_i(x) = 0$ and *inactive* if the strict inequality $c_i(x) > 0$ is satisfied. Furthermore all equality constraints are active at every feasible point x.

The Lagrangian function of the constrained problem (CP) can be written as:

$$L(x, \lambda) = f(x) - \lambda^T c(x) = f(x) - \sum_{i=1}^{m} \lambda_i c_i(x) \tag{2.4}$$

where $\lambda = (\lambda_1, ..., \lambda_m)^T$ is a vector of Lagrange multipliers. The first order partial derivative with respect to x gives:

$$\nabla_x L(x^*, \lambda^*) = \nabla f(x^*) - \sum_{i=1}^{m} \lambda_i{}^* \nabla c_i(x^*) \tag{2.5}$$

2.2 Optimality conditions

In this subsection we are going to discuss necessary and sufficient conditions for optimality. Before looking for optimality conditions of constrained optimization problem we have to check if linear independence constraint qualification (LICQ) holds. This has been explained in [16], p. 320 that:

Definition 3. *Given x is the feasible point and $\mathcal{A}(x)$ is the set of active constraints defined in Definition 2 then the linear independence constraint qualification (LICQ) it is said to hold if the gradients of active constraints are linearly independent.*

A feasible point x mentioned in Definition 3 is known as *regular point*.

2.2.1 Necessary conditions for optimality

The necessary conditions for x^* to be a local minimizer of constrained problem (CP) are also called the *first order conditions*. According to [16], p. 321 the necessary conditions for optimality are stated in the following theorem:

Theorem 2.1. *(First-Order Necessary Conditions)*
Suppose x^* is the local minimizer of constrained problem (CP) in (2.1), (2.2) and

(2.3) and both objective function f and the constraints c_i are continuously differentiable and that linear independence constraint qualification (LICQ) holds then there exist Lagrange multipliers $\lambda^* = ({\lambda_1}^*, ..., {\lambda_m}^*)$ such that the following conditions are satisfied at (x^*, λ^*):

- *Feasibility:*

$$\begin{aligned} c_i(x^*) &= 0 \qquad i = 1, ..., m_e \\ c_i(x^*) &\geq 0 \qquad i = m_e + 1, ..., m \end{aligned} \tag{2.6}$$

- *Stationarity*

$$\nabla_x L(x^*, \lambda^*) = 0 \quad \text{or} \quad \nabla f(x^*) = \sum_{i=1}^{m} \lambda_i^* \nabla c_i(x^*) \tag{2.7}$$

- *Complementarity*

$${\lambda_i}^* c_i(x^*) = 0 \quad i = m_e + 1, ..., m \tag{2.8}$$

- *Dual feasibility*

$$\lambda_i^* \geq 0 \quad i = m_e + 1, ..., m \tag{2.9}$$

These conditions from theorem 2.1 are known as Kuhn-Tucker (KT) or sometimes Karush Kuhn Tucker (KKT) conditions. The point x that satisfies the KT conditions is called KT-point or KKT- point. It is pointed in [18] that the conditions in (2.9) are for dual feasibility that means that the Lagrange multipliers that correspond to active constraint can be zero.

On top of that, stationarity and complementarity conditions shown in (2.7) and (2.8), mean that the gradient $\nabla f(x^*)$ is the *linear* combination of gradients of the active constraints at x^*. According to [16] the conditions in (2.8) are called complementary in the sense that either constraint i is active or $\lambda_i^* = 0$ or possibly both. It is obvious that the lagrange multipliers correspond to inactive constraints are zero. This leads us to define the special case for complementarity as follows:

Definition 4. *(Strict Complementarity).*
Given that $\mathcal{A}(x^*)$ is the set of active constraints at optimal point x^* and $\mathcal{I}$ is an index of inactive constraints and λ^* satisfy the KT conditions,we can say that strict complementary conditions holds if exactly one of λ_i^* and $c_i(x^*)$ is zero for each index $i \in \mathcal{I}$. In other words we can say that $\lambda_i^* > 0$ for each $i \in \mathcal{I} \cap \mathcal{A}(x^*)$.

2.2.2 Sufficient conditions for Optimality

These are conditions which will guarantee our solution to the local minimum solution for constrained minimization problem. They include in both first order (KKT) and second order conditions. These conditions are summarized in the following theorem as it is pointed in [17]:

Theorem 2.2. *(Sufficient Conditions).*
x^* is local minimum point on constrained problem (CP) if there exist Lagrange multipliers $\lambda^* = ({\lambda_1}^*, ..., {\lambda_m}^*)$ such that the following conditions hold:

1. *KKT-conditions of Theorem 2.1.*
2. *Second order conditions:*

For every non-zero vector $y \in \mathbb{R}^n$ such that;

$y^T \nabla c_i(x^*) = 0$ *for all equality constraints* $i = 1, ..., m_e$.

$y^T \nabla c_i(x^*) = 0$ *for all inequality constraints with* ${\lambda_i}^* > 0$,

$y^T {\nabla_x}^2 L(x^*, \lambda^*) y > 0.$

The condition $y^T {\nabla_x}^2 L(x^*, \lambda^*) y > 0$ pointed in Theorem 2.2 implies that the symmetric matrix ${\nabla_x}^2 L(x^*, \lambda^*)$ is positive definite.

2.2.3 Saddle point conditions

These are more restrictive conditions than those which have been mentioned in Theorem 2.2 and they determine whether the constrained problem (CP)has attained global minimum point or not. These conditions are presented in [17] and being summarized in the following theorem;

Theorem 2.3. *(Saddle point conditions).*
If (x^*, λ^*) is a saddle point of the Lagrangian of (CP), i.e. if $\lambda_i{}^* \geq 0$ for, $i = m_e + 1, ..., m$ and $L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*)$ for all $x \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}^m$ with $\lambda_i \geq 0$ for $i = m_e + 1, ..., m$, then x^* is the global minimum point.

3 Methodology and Modeling

Our optimization model will focus to minimize costs of reducing reported emission and its associated uncertainties. In our case we shall incorporate in both absolute and relative uncertainties.
The necessary set of variables can be defined as follows. Let

i = Parties or countries participating under Kyoto Protocol,for $i = 1, 2, 3, ..., N$.

x_i = reported emission in every party i.

ϵ_i = volume of absolute uncertainty emission.

$C_i(x_i)$ =total costs for Party i of keeping reported emission on level, x_i.

$\mathcal{F}_i(\epsilon_i)$ =total costs for Party i of keeping absolute uncertainty on the level ϵ_i

K_i =Kyoto target of emission to each party i and

y_i =number of emission permits accrued by Party i (might be positive for net purchaser, or negative for net supplier of permit).

Each Party faces a two step optimization problem. The first step optimization problem is for each Party to carry its own individual task to decide whether to abate emissions or to invest in monitoring the volume of absolute uncertainty emissions. The second step optimization is for the Party (or country) to decide whether or not to exchange the number of emission permits with other Parties [1].

For individual optimization, the least cost of reducing reported emission, $C_i(x_i)$ as well as monitoring the volume of absolute uncertainty, $\mathcal{F}_i(\epsilon_i)$, is given by:

$$f_i(y_i) = \min_{x_i,\epsilon_i}[C_i(x_i) + \mathcal{F}_i(\epsilon_i)] \tag{3.1}$$

$$\text{s.t } x_i + \epsilon_i \leq K_i + y_i \tag{3.2}$$

Both cost functions $C_i(x_i)$ and $\mathcal{F}_i(\epsilon_i)$ are assumed to be positive, decreasing and convex in x_i and ϵ_i respectively. Furthermore these functions are assumed to be continuously differentiable [3]. The convexity of $f_i(y_i)$ is assured since it is a minimum sum of two convex functions $C_i(x_i)$ and $\mathcal{F}_i(\epsilon_i)$.
This is according to the following lemma in [17] as;

Lemma 3.1. *(Sum of convex functions).*
If $f(x)$ and $g(x)$ are both convex functions, hence $f(x) + g(x)$ is also convex.

The marginal costs $C_i'(x_i)$ and $\mathcal{F}_i'(\epsilon_i)$ are both negative in x_i and ϵ_i respectively so as to be positive in reducing x_i and ϵ_i. The theorem proved in [4] will be helpful to examine how objective value of optimization problem changes as the result of the change of its parameters and is stated here under as follows:

Theorem 3.2. *(Envelope theorem) :*
Suppose $M(a) = max f(x, a)$ gives the maximized value of objective function f as a function of parameter a then $M(a) = f(x(a), a)$ and $M(a)$ changes as parameter a changes, namely $\frac{dM(a)}{da} = \frac{\partial f(x^*, a)}{\partial a}|_{x^*=x(a)}$ at optimal point x^*

By substituting equation (3.2) into (3.1) and eliminating x_i we have;

$$f_i(y_i) = \min_{\epsilon_i}[(C_i(K_i + y_i - \epsilon_i)) + \mathcal{F}_i(\epsilon_i)] \tag{3.3}$$

Then by applying theorem 3.2 in equation (3.3) we get;

$$f_i'(y_i) = \frac{\partial}{\partial y_i}\min_{\epsilon_i}[(C_i(K_i + y_i - \epsilon_i)) + \mathcal{F}_i(\epsilon_i)] = C_i'(x_i*) \tag{3.4}$$

If substituting (3.2) into (3.1) and eliminating ϵ_i we have;

$$f_i(y_i) = \min_{x_i}[C_i(x_i) + \mathcal{F}_i(K_i + y_i - x_i)] \tag{3.5}$$

The use of envelope theorem to (3.5) gives;

$$f_i'(y_i) = \frac{\partial}{\partial y_i}\min_{\epsilon_i}[C_i(x_i) + (\mathcal{F}_i(K_i + y_i - x_i))] = \mathcal{F}_i'(\epsilon_i*) \tag{3.6}$$

where by x_i^* and ϵ_i^* are optimal solutions of subproblem (3.1) and (3.2), i.e. the optimal reported emission and optimal volume of absolute uncertainty respectively. As equation (3.4) is equal to (3.6) then we obtain the optimality condition $C_i'(x_i^*) = \mathcal{F}_i'(\epsilon_i^*)$, that imply that the marginal costs for cutting down reported emissions to x_i^* is equal to marginal cost of reducing the volume of absolute uncertainty to ϵ_i^* at optimal conditions. By minimizing (3.1) subject to linear constraint (3.2) and setting Lagrangian λ_i we obtain the Lagrangian function;

$$\ell(x_i, \epsilon_i, \lambda_i) = C_i(x_i) + \mathcal{F}_i(\epsilon_i) - \lambda_i(x_i + \epsilon_i - K_i - y_i) \tag{3.7}$$

Then the first order partial derivatives of (3.7) gives;

$$\frac{\partial \ell}{\partial x_i} = C_i'(x_i) - \lambda_i = 0 \tag{3.8}$$

$$\frac{\partial \ell}{\partial \epsilon_i} = \mathcal{F}_i'(\epsilon_i) - \lambda_i = 0 \tag{3.9}$$

$$\frac{\partial \ell}{\partial \lambda_i} = x_i + \epsilon_i - K_i - y_i = 0 \tag{3.10}$$

Solving (3.8) and (3.9) we obtain:

$$\lambda_i = C_i'(x_i^*) = \mathcal{F}_i'(\epsilon_i^*) \tag{3.11}$$

from which λ_i is Lagrangian multiplier and is interpreted as the shadow price, i.e. the willingness of Party i to pay for emitting one more unit of reported emission, x_i or volume of absolute uncertainty, ϵ_i by considerably relaxing constraint (3.2) by one unit. At optimal conditions, shadow price is equal to the marginal costs of reducing reported emission as well as reducing volume of absolute uncertainty. Since the market will not be in equilibrium, the shadow prices λ_i will differ considerably between buyers and sellers of permits reflecting potential for trade. This will automatically lead to second optimization problem to find permit distribution among the participants so as to equalize the shadow price among them. The aggregate cost of reaching the Kyoto targets is defined as the sum of individual costs as follows;

Suppose the aggregate cost function is given by:

$$F(y_1, ..., y_N) = \sum_{i=1}^{N} f_i(y_i) \tag{3.12}$$

It follows that our optimization problem will be:

$$\min_{y_i} F(y_1, ..., y_N) \tag{3.13}$$

$$\text{s.t} \quad \sum_{i=1}^{N} y_i = 0 \tag{3.14}$$

By setting Lagrangian multiplier μ to (3.13) and (3.14) and solving the first order condition we have;

$$L(y_1, ..., y_N, \mu) = \sum_{i=1}^{N} f_i(y_i) - \mu \sum_{i=1}^{N} y_i \tag{3.15}$$

$$\frac{\partial L}{\partial y_i} = f_i'(y_i) - \mu = 0 \tag{3.16}$$

$$\frac{\partial L}{\partial \mu} = -\sum_{i=1}^{N} y_i = 0 \tag{3.17}$$

Solving equation (3.16) we obtain the first order condition:

$$f_i'(y_i) = \mu, \forall i \tag{3.18}$$

Condition (3.18) imply that the marginal cost of permits, y_i, shall in equilibrium equal to a specific level μ to all participants. It is obvious that by combining (3.4) and (3.6) we deduce that:

$$f_i'(y_i) = C_i'(x_i^*) = \mathcal{F}_i'(\epsilon_i^*) \tag{3.19}$$

This shows that the only necessary condition to bring the permit market into equilibrium is for permit price equal to both marginal costs for reducing reported emission and volume of absolute uncertainty.

3.1 Introduction of Relative uncertainty to the Model

We aim at investigating the effects of introducing the relativity uncertainty to the cost function (3.1). We claim that this approach is suitable for analyzing the costs of reducing uncertainty that involved to the inventory of non-CO_2 GHG such as methane, CH_4, nitrous oxide, (N_2O) and their aggregate in combination with CO_2.

As it was mentioned earlier, uncertainty of emission factors depend on emission source and the knowledge about processes generating emission, then it was reported to [11] and [12] that CO_2 emission factors from energy related sources is 5 percent. It is reported in [3]that, other GHG depending on the emission source, have more uncertainties, for example N_2O from agricultural sources is up to 100 percent, while N_2O from combustion is up to 200 percent.

Let $\mathcal{H}_i(R_i)$ be the costs for reducing relative uncertainty, R_i. It is assumed that relative uncertainty is given by $R_i = \epsilon_i/x_i$. The total abatement costs is the sum of $\mathcal{H}_i(R_i)$ and emission reduction costs, $C_i(x_i)$. Under Kyoto protocol, as pointed in [3] we are needed to express uncertainties in absolute terms. That is emission level x_i plus absolute uncertainty $\epsilon = x_i.R_i$ shall not exceed the Kyoto target, K_i increased or decreased by a certain specific level of Permit, y_i (see condition (3.2)).

Let $Z_i\left(x_i, R_i\right)$ represents the total cost for abating both emissions, x_i and relative uncertainty, R_i such that:

$$Z_i\left(x_i, R_i\right) = C_i(x_i) + \mathcal{H}_i(R_i) \tag{3.20}$$

Then after introducing the relative uncertainty into trade system then equation (3.1)and (3.2) become;

$$\mathcal{G}_i\left(x_i, R_i\right) = \min_{x_i, R_i} Z_i\left(x_i, R_i\right) \tag{3.21}$$

$$\text{s.t } x_i + x_i.R_i \leq K_i + y_i \tag{3.22}$$

The approach showing in (3.21) and (3.22) strongly reflect the dependence between both emissions and their associated uncertainty. As it has been noticed, the constraint (3.22) is non-linear in contrast to (3.2). It is assumed that both functions $C_i(x_i)$ and $\mathcal{H}_i(R_i)$ display the usual economic properties: that is they are convex, decreasing and continuous differentiable. This implies that both x_i and R_i should be positive to reflect the reality and it is assumed that $K_i + y_i$ should be strict positive and Parties do not sell more permit than their Kyoto compliances. The Lagrange function of (3.21) and (3.22) is:

$$L\left(x_i, R_i, \lambda_i\right) = C_i(x_i) + \mathcal{H}_i(R_i) - \lambda_i\left(x_i + x_i.R_i - K_i - y_i\right), \tag{3.23}$$

whose first order conditions give;

$$\frac{\partial L}{\partial x_i} = C_i'(x_i) - \lambda_i - R_i.\lambda_i = 0 \tag{3.24}$$

$$\frac{\partial L}{\partial R_i} = \mathcal{H}_i'(R_i) - \lambda_i.x_i = 0 \tag{3.25}$$

$$\frac{\partial L}{\partial \lambda_i} = x_i + x_i.R_i - K_i - y_i = 0 \tag{3.26}$$

Solving (3.24) and (3.25) for optimality we get

$$\lambda_i = \frac{C_i'(x_i^*)}{1 + R_i^*} = \frac{\mathcal{H}_i'(R_i^*)}{x_i^*}, \tag{3.27}$$

where by x_i^* and R_i^* are optimal levels of emissions and relative uncertainties and Lagrange multiplier, λ_i is interpreted as the permit shadow price.

With first order conditions of (3.23) it is assumed that the cost function $\mathcal{H}_i'(R_i)$ is independent of emission level x_i, that is relative uncertainty does not change in case of change in emissions. However from (3.27) we notice that the marginal cost ratio $\frac{C_i'(x_i)}{\mathcal{H}_i'(R_i)}$ depends on both optimal level of emission, x_i^* and optimal relative uncertainty R_i^* while in case of independent emission and absolute uncertainty stated in (3.11) reveals that the ratio of marginal costs is 1.

The cost function $Z_i(x_i, R_i)$ in (3.21) is the minimum sum of two convex functions subject to the non-linear constraint (3.22) with respect to the variables x_i and R_i from the fact that $\epsilon_i = x_i.R_i$.

3.2 Analysis of local minima

To have more insight of the minimum of Lagrange function, then the second derivative of $Z_i(x_i, R_i)$ has to be analyzed so as to check the existence of several local minima. Taking into consideration that countries need not to over-comply to Kyoto targets,we take a case of equality constraint (3.22). We express our goal function (3.21) to depend only on x_i. Now constraint (3.22) becomes:

$$x_i + x_i.R_i - K_i - y_i = 0 \tag{3.28}$$

By making R_i the subject from (3.28) we obtain:

$$R_i = \frac{K_i + y_i - x_i}{x_i} \tag{3.29}$$

Substituting (3.29) into (3.21) we get:

$$\mathcal{G}_i\left(x_i\right) = C_i\left(x_i\right) + \mathcal{H}_i\left(\frac{K_i + y_i - x_i}{x_i}\right) \tag{3.30}$$

The first order derivative of (3.30) gives:

$$\frac{d\mathcal{G}_i}{dx_i} = C_i'\left(x_i\right) + \left[\frac{-x_i - \left(K_i + y_i - x_i\right)}{{x_i}^2}\right].\mathcal{H}_i'\left(\frac{K_i + y_i - x_i}{x_i}\right) \tag{3.31}$$

$$= C_i'\left(x_i\right) - \frac{\left(K_i + y_i\right)}{{x_i}^2}.\mathcal{H}_i'\left(\frac{K_i + y_i - x_i}{x_i}\right) \tag{3.32}$$

Setting the first derivative to zero and from (3.29), it follows that:

$$\frac{K_i + y_i}{{x_i}^2} = \frac{C_i'\left(x_i\right)}{\mathcal{H}_i'\left(R_i\right)} \tag{3.33}$$

The second order derivative with respect to x_i becomes:

$$\frac{d^2\mathcal{G}_i}{d{x_i}^2} = C_i''\left(x_i\right) + 2\frac{x_i\left(K_i + y_i\right)}{{x_i}^4}\mathcal{H}_i'\left(\frac{K_i + y_i - x_i}{x_i}\right) + \left(\frac{K_i + y_i}{{x_i}^2}\right)^2\mathcal{H}_i''\left(\frac{K_i + y_i - x_i}{x_i}\right)$$

$$= C_i''\left(x_i\right) + 2\frac{\left(K_i + y_i\right)}{{x_i}^3}\mathcal{H}_i'\left(R_i\right) + \left(\frac{K_i + y_i}{{x_i}^2}\right)^2\mathcal{H}_i''\left(R_i\right) \tag{3.34}$$

From condition (3.33) which was valued at $\frac{d\mathcal{G}_i}{dx_i} = 0$ we make both $\mathcal{H}_i'(R_i)$ and x_i the subject:

$$\mathcal{H}_i'\left(R_i\right) = \frac{{x_i}^2 C_i'\left(x_i\right)}{K_i + y_i} \tag{3.35}$$

$$x_i = \sqrt{\frac{\mathcal{H}_i'\left(R_i\right)}{C_i'\left(x_i\right)}\left(K_i + y_i\right)} \tag{3.36}$$

Substituting (3.35) into (3.34) we have:

$$\frac{d^2\mathcal{G}_i}{dx_i{}^2}|_{\frac{d\mathcal{G}_i}{dx_i}=0} = C_i''(x_i) + \left(\frac{C_i'(x_i)}{\mathcal{H}_i'(R_i)}\right)^2 \mathcal{H}_i''(R_i) + 2\frac{(K_i+y_i)}{x_i{}^3}\frac{x_i{}^2 C_i'(x_i)}{K_i+y_i}$$

$$= C_i''(x_i) + \left(\frac{C_i'(x_i)}{\mathcal{H}_i'(R_i)}\right)^2 \mathcal{H}_i''(R_i) + 2\frac{C_i'(x_i)}{x_i} \quad (3.37)$$

By substituting (3.36) into (3.37) we have:

$$\frac{d^2\mathcal{G}_i}{dx_i{}^2}|_{\frac{d\mathcal{G}_i}{dx_i}=0} = C_i''(x_i) + \left(\frac{C_i'(x_i)}{\mathcal{H}_i'(R_i)}\right)^2 \mathcal{H}_i''(R_i) + 2\sqrt{\frac{C_i'(x_i)}{\mathcal{H}_i'(R_i)}}\frac{C_i'(x_i)}{\sqrt{K_i+y_i}} \quad (3.38)$$

The first two terms of (3.38) are positive since the cost function $C_i(x_i)$ and $\mathcal{H}_i(R_i)$ are convex. The third term can be negative since both marginal costs $C_i'(x_i)$ and $\mathcal{H}_i'(R_i)$ are negative. Now depending on magnitude of $C_i'(x_i)$ and $\mathcal{H}_i'(R_i)$, $\frac{d^2\mathcal{G}_i}{dx_i{}^2}|_{\frac{d\mathcal{G}_i}{dx_i}=0}$ can be negative. This implies that the problem (3.21), (3.22) can be non-convex and could have several local minima.

Without the loss of generality we can derive the goal function that depends on relative uncertainty, R_i and its corresponding second order derivative as follows: Making x_i the subject from (3.28) we have:

$$x_i = \frac{K_i + y_i}{1 + R_i} \quad (3.39)$$

Substituting (3.39) into (3.21) we get:

$$\mathcal{P}_i(R_i) = C_i\left(\frac{K_i+y_i}{1+R_i}\right) + \mathcal{H}_i(R_i) \quad (3.40)$$

The first order derivative of (3.40) gives:

$$\frac{d\mathcal{P}_i}{dR_i} = \frac{-(K_i+y_i)}{(1+R_i)^2}C_i'\left(\frac{K_i+y_i}{1+R_i}\right) + \mathcal{H}_i'(R_i) \quad (3.41)$$

Setting the first order derivative to zero and making use of (3.39) we find that:

$$\frac{\mathcal{H}_i'(R_i)}{C_i'(x_i)} = \frac{K_i+y_i}{(1+R_i)^2} \quad (3.42)$$

From (3.41) the second order derivative becomes;

$$\frac{d^2\mathcal{P}_i}{dR_i{}^2} = \frac{2(1+R_i)(K_i+y_i)}{(1+R_i)^4}C_i'(x_i) - \frac{(K_i+y_i)}{(1+R_i)^2}.\frac{-(K_i+y_i)}{(1+R_i)^2}C_i''(x_i) + \mathcal{H}_i''(R_i)$$

$$= \mathcal{H}_i''(R_i) + \left(\frac{K_i+y_i}{(1+R_i)^2}\right)^2 C_i''(x_i) + 2\frac{(K_i+y_i)}{(1+R_i)^3}C_i'(x_i) \tag{3.43}$$

Solving $\frac{d^2\mathcal{P}_i}{dR_i{}^2}$ at $\frac{d\mathcal{P}_i}{dR_i} = 0$ we manipulate (3.42), that will give:

$$(1+R_i) = \sqrt{\frac{C_i'(x_i)}{\mathcal{H}_i'(R_i)}}.\sqrt{K_i+y_i}$$

$$\Longrightarrow (1+R_i)^3 = \frac{C_i'(x_i)}{\mathcal{H}_i'(R_i)}(K_i+y_i)\sqrt{\frac{C_i'(x_i)}{\mathcal{H}_i'(R_i)}}.\sqrt{K_i+y_i} \tag{3.44}$$

Substituting (3.44) into (3.43) and make the use of (3.42) we finally get the expression:

$$\frac{d^2\mathcal{P}_i}{dR_i{}^2}\Big|_{\frac{d\mathcal{P}_i}{dR_i}=0} = \mathcal{H}_i''(R_i) + \left(\frac{\mathcal{H}_i'(R_i)}{C_i'(x_i)}\right)^2 C_i''(x_i) + 2\sqrt{\frac{\mathcal{H}_i'(R_i)}{C_i'(x_i)}}.\frac{\mathcal{H}_i'(R_i)}{\sqrt{K_i+y_i}} \tag{3.45}$$

Since both $\mathcal{H}_i(R_i)$ and $C_i(x_i)$ are assumed to be convex then the first two terms to the right of (3.45) are positive. That is $\mathcal{H}_i''(R_i)$ and $C_i''(x_i)$ are strictly greater than zero. The third term is negative when both $\mathcal{H}_i'(R_i)$ and $C_i'(x_i)$ are negative. Depending on magnitude of marginal costs, $\mathcal{H}_i'(R_i)$ and $C_i'(x_i)$ we can signal $\frac{d^2\mathcal{P}_i}{dR_i{}^2}$ at $\frac{d\mathcal{P}_i}{dR_i} = 0$ to be either positive or negative. This implies that problem (3.21) can be non-convex and could have several local minima.

The analysis of second order is very important in the context of carbon permit market as we can not achieve global minimum cost solution. The market might be locked to local minima.

3.3 Choice of cost function

In order to reach local minimum conditions we considered a convex function for emission reduction to be:

$$C_i(x_i) = \begin{cases} b_i (x_i - a_i)^2 & \text{for } x_i \in [0, a_i] \\ \\ 0 & \text{for } x_i > a_i \end{cases} \tag{3.46}$$

where a_i is initial emission or 'Business-As-Usual' (BAU). If $x_i = a_i$, no cost or emission regulation is taken into account to reduce emission. This reflects baseline emission and is also known as business as usual (BAU) as it is pointed in [3]. In the same manner we formulate cost function for reducing the relative uncertainty as;

$$\mathcal{H}_i(R_i) = \begin{cases} d_i (R_i - R_{0,i})^2 & \text{for } R_i \in [0, R_{0,i}] \\ \\ 0 & \text{for } R_i > R_{0,i} \end{cases} \tag{3.47}$$

where by $R_{0,i}$ is initial volume of relative uncertainty. If $R_i = R_{0,i}$, it means that no cost is incurred to reduce relative uncertainty and $R_{0,i}$ will reflect the baseline for relative uncertainty.

These cost functions have been proposed by several authors [1], [2], and [3]. Quadratic curve reflects the typical well known feature of increasing marginal costs. The parameter values in the model are derived from available data in the countries of Kyoto treaty (refer table 1).

Our claim is that although (3.46) and (3.47) are convex (downward) functions, they can not achieve the least cost solution. In other words there exist local maxima as an indication of these functions to exhibit non-convexity to some points. To support our claim let us consider both functions $\mathcal{H}_i(R_i)$ and $C_i(x_i)$ by writing (3.30) in terms of (3.46) and (3.47) to get:

$$\mathcal{T}_i(x_i) = b_i (x_i - a_i)^2 + d_i \left(\frac{K_i + y_i - x_i}{x_i} - R_{0,i} \right)^2 \tag{3.48}$$

Setting to zero the first order derivative of (3.48) gives:

$$\frac{d\mathcal{T}_i}{dx_i} = 2b_i (x_i - a_i) + 2d_i \left(\frac{K_i + y_i - x_i}{x_i} - R_{0,i} \right) \left(\frac{-x_i - (K_i + y_i - x_i)}{{x_i}^2} \right) = 0 \tag{3.49}$$

Manipulating (3.49) by dividing it by 2 and multiplying by ${x_i}^3$ we get:

$$b_i\left(x_i-a_i\right){x_i}^3-d_i\left(K_i+y_i-x_i-x_iR_{0,i}\right)\left(K_i+y_i\right)=0$$

$$\Longrightarrow b_i\left(x_i-a_i\right){x_i}^3-d_i\left(K_i+y_i\right)^2+d_i\left(1+R_{0,i}\right)\left(K_i+y_i\right)x_i=0 \tag{3.50}$$

$$\Longrightarrow \left(x_i-a_i\right){x_i}^3+\frac{d_i}{b_i}\left(1+R_{0,i}\right)\left(K_i+y_i\right)x_i-\frac{d_i}{b_i}\left(K_i+y_i\right)^2=0$$

By normalizing x_i upon a_i we find that $\frac{x_i}{a_i}=u_i$ from which:

$$x_i=a_iu_i \tag{3.51}$$

Substituting (3.51) into (3.50) we get:

$$0={a_i}^4{u_i}^4-{a_i}^4{u_i}^3+\frac{d_i}{b_i}\left(1+R_{0,i}\right)\left(K_i+y_i\right)a_iu_i-\frac{d_i}{b_i}\left(K_i+y_i\right)^2$$

$$\Rightarrow 0={u_i}^4-{u_i}^3+\frac{d_i}{{a_i}^3b_i}\left(1+R_{0,i}\right)\left(K_i+y_i\right)u_i-\frac{d_i}{b_i{a_i}^4}\left(K_i+y_i\right)^2$$

$$\Rightarrow 0={u_i}^4-{u_i}^3+\frac{d_i}{{a_i}^3b_i}\left(1+R_{0,i}\right)\left(K_i+y_i\right)u_i-\frac{d_i}{{a_i}^3b_i}\left(1+R_{0,i}\right)\left(K_i+y_i\right)\frac{K_i+y_i}{a_i(1+R_{0,i})} \tag{3.52}$$

Now we let two dimensionless parameters α_i and γ_i represent:

$$\alpha_i=\frac{d_i}{{a_i}^3b_i}\left(1+R_{0,i}\right)\left(K_i+y_i\right) \tag{3.53}$$

$$\gamma_i=\frac{K_i+y_i}{a_i(1+R_{0,i})} \tag{3.54}$$

Substituting Equations (3.53) and (3.54) into (3.52) we have:

$$\begin{aligned}&0={u_i}^4-{u_i}^3+\alpha_iu_i-\alpha_i\gamma_i\\&\Rightarrow 0={u_i}^3(u_i-1)+\alpha_i(u_i-\gamma_i),\ \text{for } u_i>0,\ \alpha_i>0 \text{ and } \gamma_i>0\end{aligned} \tag{3.55}$$

To have minima solutions, let us write Equation (3.55) as:

$$\alpha_i(u_i-\gamma_i)-{u_i}^3(1-u_i)=0 \tag{3.56}$$

We have to consider the expression $\alpha_i(u_i - \gamma_i)$ as the tangent line to ${u_i}^3(1-u_i)$ in order to determine parameters α_i and γ_i which might help us to draw the conclusion about the minima. The function $f''(u_i) = {u_i}^3(1-u_i)$ has two point of inflexion from the fact that $f''(u_i) = 6u_i - 12{u_i}^2 = 0$ gives us $u_i = 0$ or $u_i = 0.5$. When $u_i < 1/2$ implies that ${u_i}^3(1-u_i)$ is convex and if $u_i > 1/2$ the expression is concave. Our focus will be when $u_i = 0.5$. The slope at this point, $f'(u_i)|_{u_i=0.5} = 3{u_i}^2 - 4{u_i}^3|_{u_i=0.5} = 0.25$. That is $f(0.5) = 0.5^3(1-0.5) = 1/16$ and clearly the line passes through $(1/2, 1/16)$. Furthermore we can find that:

$$\alpha_i(u_i - \gamma_i) = 1/4u_i - 1/16 \tag{3.57}$$

$$\alpha_i = \gamma_i = 0.25 \tag{3.58}$$

Equation (3.56) exhibits only one positive solution $u_i = 1/2$ for $\alpha_i = 0.25$ and $\gamma_i = 0.25$. For values of α_i and γ_i less than 0.25 , Equation (3.56) exhibits more than one solution. This is showed in figure 1 where by $f(u_i) = \alpha_i(u_i - \gamma_i) - {u_i}^3(1-u_i)$, $g(u_i) = \alpha_i(u_i - \gamma_i)$ and $h(u_i) = u_i^3(1-u_i)$.

Figure 2 represents functions $f(u_i)$, $g(u_i)$ and $h(u_i)$ with $\alpha_i = \gamma_i = 0.2 < 0.25$. In fact $f(u_i)$ exhibits more than one positive solution at this range of $\alpha_i = \gamma_i = 0.2 < 0.25$ which are $u_i = 0.7236, 0.4472$ and 0.2764 and it is concave when $u_i < 0.5$ and convex when $u_i > 0.5$, while $h(u_i)$ is convex when $u_i < 0.5$ and is concave when $u_i > 0.5$.

On the other hand, figure 3 shows functions $f(u_i)$, $g(u_i)$ and $h(u_i)$ with $\alpha_i = \gamma_i = 0.3 > 0.25$. Function $f(u_i)$ exhibits a positive real solution at $u_i = 0.5477$ and it is concave when $u_i < 0.5$ and convex when $u_i > 0.5$, while function $h(u_i)$ is convex when $u_i < 0.5$ and it is concave when $u_i > 0.5$.

It is pointed in [3] that parameter γ_i in the context of carbon market is interpreted as the ratio between Kyoto emission plus traded permit to the BAU emission level

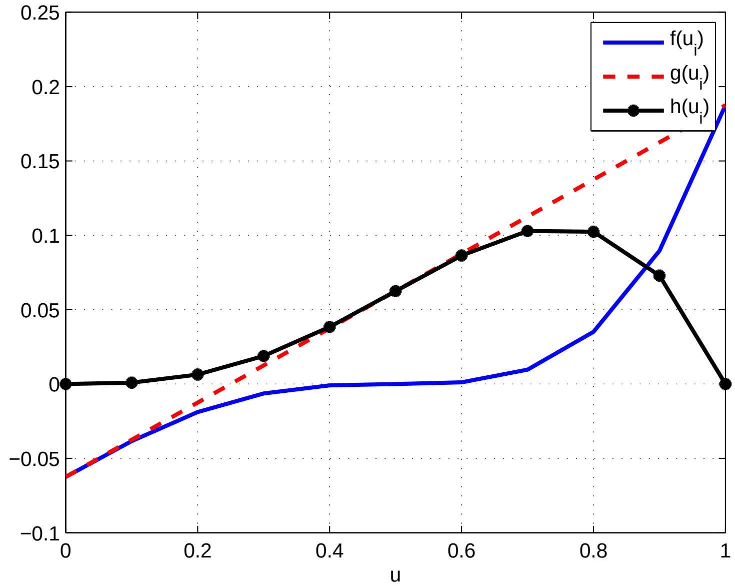

Figure 1: Convex and concave behavior revealed by first order derivative of goal function, $\mathcal{G}_i(x_i)$ in Equation (3.30) expressed in terms variable $\dfrac{x_i}{a_i} = u_i$ and parameters, $\alpha_i = \gamma_i = 0.25$)

plus absolute uncertainty.Thus;

$$\gamma_i = \frac{K_i + y_i}{a_i(1+R_i)}$$

$$= \frac{K_i\left(1+\frac{y_i}{K_i}\right)}{a_i(1+R_i)} \tag{3.59}$$

$$= \frac{K_i}{a_i}(1+\tau_i)\,(1+R_i)^{-1}$$

By binomial expansion we find that;

$$\gamma_i = \frac{K_i}{a_i}(1+\tau_i)\left(1 - R_i + {R_i}^2 + \ldots\right)$$

$$\approx \frac{K_i}{a_i}\left(1+\tau_i - R_i - R_i\tau_i + \ldots\right) \tag{3.60}$$

$$\approx \frac{K_i}{a_i}\left(1+\tau_i - R_i\right)$$

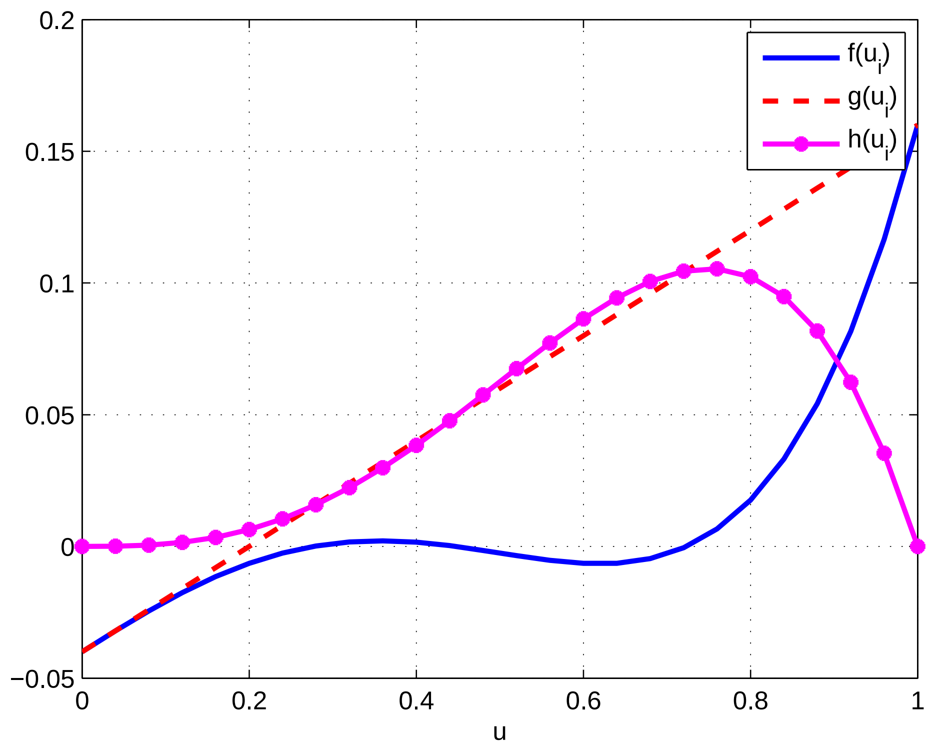

Figure 2: Convex and concave behavior revealed by first order derivative of goal function, $\mathcal{G}_i(x_i)$ in Equation (3.30) expressed in terms variable $\frac{x_i}{a_i} = u_i$ and parameters, $\alpha_i = \gamma_i = 0.2 < 0.25$)

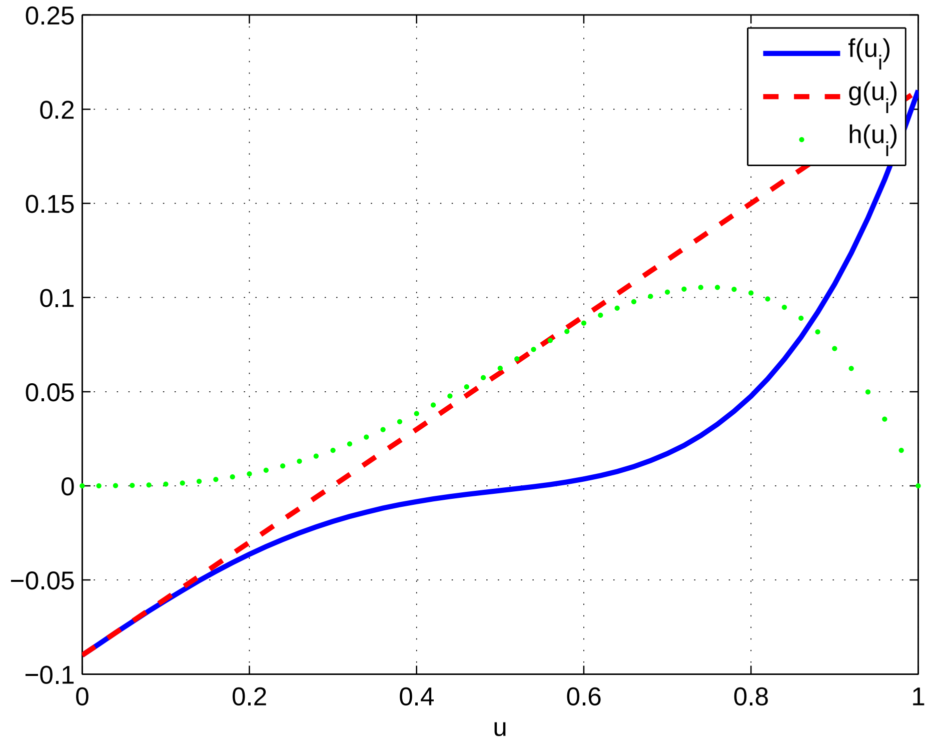

Figure 3: Convex and concave behavior revealed by first order derivative of goal function, $\mathcal{G}_i(x_i)$ in Equation (3.30) expressed in terms variable $\dfrac{x_i}{a_i} = u_i$ and parameters, $\alpha_i = \gamma_i = 0.3 > 0.25$)

where by $\frac{K_i}{a_i}$ is a ratio of agreed Kyoto Protocol to Business as Usual emission and $\tau_i = \frac{y_i}{K_i}$ is the ratio of traded emission permits to Kyoto Protocol targets. If for sure $\gamma_i < 0.25$ is approximately as saying more than 75 percent of BAU emission level has been reduced.

3.4 Estimates of cost parameters

Due to the fact that the information about costs for reducing relative uncertainty is limited then parameter d_i from Equation (3.47) can be obtained by assuming that the costs of relative uncertainty reduction at any level R_i^1 relative to initial uncertainty $R_{0,i}$ are dependent on costs of emission reduction according to the following formula:

$$\frac{\partial C_i(x_i)}{\partial x_i}|_{x_i={x_i}^1} = \frac{\partial \mathcal{H}_i(R_i)}{\partial R_i}|_{R_i={R_i}^1} \cdot \frac{1}{a_i} \tag{3.61}$$

with

$$\frac{{x_i}^1}{a_i} = \frac{{R_i}^1}{R_{0,i}} \tag{3.62}$$

This formulation originated from [1] that marginal cost of absolute uncertainty reduction $\mathcal{F}_i'(\epsilon_i)$ at any level relative to the initial uncertainty $\epsilon_{0,i}$ is the same as marginal cost of emission reduction $C_i'(x_i)$ at the same percentage of BAU (Business As Usual) level. That is :

$$\frac{\partial \mathcal{F}_i}{\partial \epsilon_i}|_{\epsilon_i={\epsilon_i}^1} = \frac{\partial C_i}{\partial x_i}|_{x_i={x_i}^1} \tag{3.63}$$

with $\frac{{\epsilon_i}^1}{\epsilon_{0,i}} = \frac{{x_i}^1}{a_i}$, where the cost function for absolute uncertainty $\mathcal{F}_i(\epsilon_i)$ is downside function.

Now from Equations (3.46) and (3.47) we find that;

$$C_i'(x_i)\,|_{x_i={x_i}^1} = \frac{1}{a_i}\mathcal{H}_i'(R_i)\,|_{R_i={R_i}^1}$$

$$\Rightarrow 2b_i\left({x_i}^1 - a_i\right) = 2\frac{d_i}{a_i}\left({R_i}^1 - R_{0,i}\right) \tag{3.64}$$

$$\Rightarrow b_i\left({x_i}^1 - a_i\right) = \frac{d_i}{a_i}\left({R_i}^1 - R_{0,i}\right)$$

It follows from (3.62) that $R_i{}^1 = \frac{x_i{}^1 R_{0,i}}{a_i}$, and substituting it to Equation (3.64) we find :

$$b_i \left(x_i{}^1 - a_i\right) = \frac{d_i}{a_i}\left(\frac{x_i{}^1 R_{0,i}}{a_i} - R_{0,i}\right)$$

$$b_i a_i{}^2 \left(x_i{}^1 - a_i\right) = d_i R_{0,i} \left(x_i{}^1 - a_i\right) \tag{3.65}$$

$$\Rightarrow d_i = \frac{a_i{}^2 b_i}{R_{0,i}}$$

Since parameters to the right of Equation (3.65) are available then parameter d_i of Equation (3.47) can be evaluated.

4 Data

We used data from [3], where the cost function parameters were estimated with model fitting methods. Available measured data from Kyoto regions were used to derive cost functions for reducing emission level and uncertainty. Data rooted from cost reducing functions which were derived by the help of MERGE Model. MERGE is an abbreviation stands for 'A model for Evaluating the Regional and Global Effects of GHG Reduction Policies' and was developed by Manne and Richel as it is quoted from [14]. Five Kyoto regions were considered for estimating the cost functions. These regions are US, OECDE (OECD Europe), Japan, CANZ (Canada, Australia and New Zealand combined) and EEFSU(Eastern Europe and Former Soviet Union combined). The parameters for emissions and uncertainty reduction which were projected with reference to year 2010 are shown in table 1. Energy related carbon emission reductions were only considered and monetary unit was US dollar of 1997.

5 Solution to the Model

In this section, three optimization methods which are Lagrangian, Sequential Quadratic Programming (SQP) and Differential Evolution (DE) have been used to solve the optimal values of emission levels , relative uncertainty and number of allocated permits to each Party or Region. Decision variables are plotted against the objective function value to establish the relationship between them that is very important for various interpretations in carbon trading context.

	Kyoto target	Initial emissions (BAU)	Cost function parameter	Initial uncertainty
Variable	K_i	a_i	b_i	$R_{0,i}$
Unit	MtC/yr	MtC/yr	$MU\$/(MtC/yr)^2$	
US	1251	1820.3	0.2755	0.13
OECDE	860	1038.0	0.9065	0.20
Japan	258	350.0	2.4665	0.15
CANZ	215	312.7	1.1080	0.20
EEFSU	1314	898.6	0.7845	0.30
Total	3898	4419		

Table 1: Projections with reference to 2010 of Kyoto target together with cost parameters of emissions and uncertainty reductions. Source:[3]

5.1 Model Results

In this subsection and few that follow we are going to discuss different approaches which have been used to come with very important results in the context of carbon trading. First, Lagrangian was used thoroughly in optimizing both emission and relative uncertainty. Second, sequential quadratic programming algorithm has been used to find optimal emission, relative uncertainty and optimal distribution vectors among the Parties participating in Kyoto treaty. It is gradient based method that guarantees local optimal solution. Third, Differential Evolution (DE) to find optimal emissions, relative uncertainty and optimal distribution of permits. On top of that it has been used to come up with important visualizations such as relationship between decision variables and objective function values that has important implications in carbon trading. Finally, stochastic uncertainty approach has been introduced for modelling of inherent uncertainty in the monitoring of emissions.

We were interested also to compare our results with those from [3] who used the search method called sequential bilateral trading, where the permit trade were selected randomly, i.e. $[y_1, ..., y_5]$ so as to solve the optimal permit vector by changing $[y_1, ..., y_5]$ in stepwise manner (+1, -1 stages, i.e a pairwise trade of permits where one Party gains buy selling permits to another).

Relative uncertainty R_i can be expressed in terms of reported emission, x_i and absolute uncertainty, ϵ_i from the fact that $R_i = \dfrac{\epsilon_i}{x_i}$. For our case we solved Equations (3.21) and (3.22) by using Lagrangian function (3.23) and its first order partial derivatives (3.24), (3.25) and (3.26) which were equated to zero to get the optimal solutions for both reported emission, x_i and uncertainty R_i. The cost functions for reducing both emissions, $C_i(x_i)$ and relative uncertainty, $\mathcal{H}_i(R_i)$ have been defined in Equations (3.46) and (3.47).

Two important cases were considered. The first case we considered each Party to optimize its own level of emission and relative uncertainty when there was no transaction carried on (that is, the situation where there is no permit trade, $y_i = 0$). This implies that the constraint (3.22) will be $x_i + x_i.R_i = K_i$ just for each Party to comply to Kyoto target, K_i.

The second case was the situation where by permits, y_i were traded or bought. In both cases the optimal Equations (3.24), (3.25) and (3.26) were considered as a system of nonlinear equations, $F(\mathbf{x}) = 0$. We used an *inline function* and optimset command which display iterations for solving $F(\mathbf{x}) = 0$ from MATLAB software. The approximate solutions were found in such a way that, $F(\mathbf{x})$ is nearly to zero. The results for both cases are shown to the tables (2) and (3) respectively. Some variables are defined in tables 2 and 3 with abbreviations as follows:

- **Ems**=Emissions
- **Runc**=Relative uncertainty
- **Sp**=Shadow price
- **MC**=Marginal cost of emission reduction
- **MR**=Marginal cost of uncertainty reduction
- **TC**=Total cost for reducing both emissions and relative uncertainty
- **Ptr**=Permits traded

5.2 Sequential Quadratic Programming (SQP)

In this subsection we are going to use sequential quadratic programming (SQP) method to solve the optimal emissions, x_i, relative uncertainty,R_i, and optimal distribution of emission permits, y_i among the Parties under Kyoto Treaty. Our

	Ems	**Runc**	**Sp**	**MC**	**MR**	**TC**	α_i	γ_i
Variable	x_i^*	R_i^*	λ_i	$C_i'(x_i)$	$\mathcal{H}_i'(R_i)$	$C_i(x_i^*) + \mathcal{H}_i(R_i^*)$		
Units	MtC/yr	-	$\$/tC$	$\$/tC$	$	$MUS\$$	-	-
US	1134.9	0.1000	-342.6	-376.86	-388820	135740	5.9738	0.6082
OECDE	738.4	0.1647	-466.4	-543.22	-344390	87450	4.9711	0.6904
JAPAN	230.4	0.1199	-526.9	-590.08	-121400	37110	5.6514	0.6410
CANZ	185.6	0.1583	-243.1	-281.58	-45120	18840	4.1254	0.5730
EEFSU	991.2	0.3257	109.6	145.30	108640	8120	6.3365	1.1248
Total						287260		

Table 2: Optimal values of both emissions, x_i and uncertainty, R_i in the situation before trade i.e, $y_i = 0$

	Ems	**Runc**	**Sp**	**MC**	**Ptr**	**TC**	α_i	γ_i
Variable	x_i^*	R_i^*	λ_i	$C_i'(x_i)$	y_i	$C_i(x_i^*)+ \mathcal{H}_i(R_i^*)$		
Units	MtC/yr	-	$\$/tC$	$\$/tC$	-	$MUS\$$	-	-
US	1312.4	0.1000	-252.9	-278.2	201	77388	6.9336	0.7059
OECDE	873.96	0.1774	-252.6	-297.4	169	26887	5.9480	0.8261
JAPAN	291.63	0.1316	-254.4	-287.9	72	9085	7.2286	0.8199
CANZ	181.36	0.1579	-251.35	-291.0	-5	20073	4.0294	0.5596
EEFSU	696.86	0.2585	-251.51	-316.5	-437	35565	4.2292	0.7507
Total					0	168998		

Table 3: The optimal values of emissions and relative uncertainty when the permits were traded

quadratic subproblem is:

$$\begin{aligned}\mathcal{G}_i\left(x_i, R_i\right) &= \min_{x_i,R_i}[Z_i\left(x_i, R_i\right)] \\ & \text{s.t } x_i + x_i.R_i \leq K_i + y_i \\ & \sum_{i=1}^{5}\left(y_i\right) = 0, \quad y_i \in (-\infty, +\infty) \\ & 0 \leq x_i \leq a_i, \quad 0 \leq R_i \leq R_{0,i}\end{aligned} \tag{5.1}$$

The function $Z_i\left(x_i, R_i\right)$ is defined by:

$$Z_i\left(x_i, R_i\right) = \sum_{i=1}^{5}\left[b_i\left(x_i - a_i\right)^2 + d_i\left(R_i - R_{0,i}\right)^2\right] \tag{5.2}$$

Sequential Quadratic Programming (SQP) is a gradient based method whose function to be minimized and the constraints must both be continuous. It guarantees local solution and represents the state of art in non linear programming methods as it gives efficiency, accuracy, and percentage of successful solution over a large number of test problems [27].

Our problem (5.1) is Medium- Scale Optimization of the general form:

$$\min_{d\in\mathcal{R}^n} \quad q(d) = \frac{1}{2}d^T H d + c^T d$$

$$A_i d = b_i \qquad i = 1, ..., m_e$$

$$A_i d \leq b_i \qquad i = m_e + 1, ..., m$$

where by a MATLAB function *fmincon* can be used to solve it.The MATLAB function, *fmincon* attempts to find the minimum of scalar function of several variables starting at an initial estimate . *fmincon* uses Sequential Quadratic Programming (SQP) method. In this method, the function solves the Quadratic Programming (QP) at each iteration. According to [28], [29] and [30] an estimate of the Hessian (H) of the Lagrangian is updated at each iteration by using BFGS (Broyden, Fletcher, Goldfarb and Shanno) quasi Newton formula. The general SQP algorithm, according to [17] is as follows:

1. Choose a starting point x_0 and matrix B_0 approximating the Hessian of the Lagangian. Set $k = 0$.

2. Terminate if x_k satisfies the optimality conditions or if $k > k\max$.

3. Solve the problem QP_k:

$$\min_p \Phi_k(p) = p^T g_k + \frac{1}{2} p^T B_k p$$
$$\text{s.t} \quad A_k p = -c_k$$

   Denote the solution by p_k.

4. Set $x_{k+1} = x_k + p_k$.

5. Calculate an estimate for the Lagrange multipliers u_{k+1} and $B_{k+1} = B(x_{k+1}, u_{k+1})$. Set $k + 1$ and go to 2.

5.3 Implementation of solution using SQP

During the implementation of the solution, bounds from which the optimal values have to be found were defined as shown in optimization problem (5.1). Initial value was set. Equality constraint was implemented in the working file Separate function files were created. These are objective function value and non-linear constraints files. The optimizer *fmincon* was commanded to call both objective and non-linear constraint functions. The maximum number of iteration and function evaluation were set to 10^6 for more accuracy and precision. After 19 iteration the following results were displayed:

	Ems	**Runc**	**Ptr**	**TC**
Variable	x_i^*	R_i^*	y_i	$C_i(x_i^*)+$ $\mathcal{H}_i(R_i^*)$
Units	MtC/yr	-	-	$MUS\$$
US	1313.3	0.1064	202.07	74731
OECDE	874.04	0.1774	169.10	26861
JAPAN	292.2	0.1317	73	8949
CANZ	180.78	0.1579	-5.6789	20244
EEFSU	696.11	0.2584	-438.03	35824
Total			0	166609

Table 4: The optimal values of emissions, emission permits and relative uncertainty after transaction using SQP method

Rounding up the permits, the optimal permit vector will be $y_i = [202\ , 169\ , 73\ , -6\ , -438]$ for regions US, OECDE, JAPAN, CANZ and EEFSU respectively. Negative sign means sellers of permits while positive sign of permits means the buyers of permits. The aggregate sum of cost for all regions was lower than when you compare to the situation where the trade of permits is taking place, i.e. $166609 MUS\$$ compared to $287260 MUS\$$. More interpretations can b e drawn, for example, emission levels when permits were not transacted were slightly lower compared to the situation when the permits were traded. US, OECDE and Japan emission levels increase because of buying more permits to pollute while in case of CANZ and EEFSU emissions decrease because they are sellers of carbon permits. The results are too close to those were found by using sequential bilateral trade pointed in [3].

Some other classical non-derivative methods such as Nelder-mead algorithm can be used after using advanced techniques of changing from constrained to unconstrained optimization.

5.4 Differential Evolution (DE) algorithm

Differential Evolution (DE) algorithm was first introduced by Price and Storn in 1995 and can be classified as an *evolutionary optimization algorithm* [22]. There are several versions of DE. For our case we are going to use a particular version of differential evolution which is known as DE/rand/l/bin scheme to solve our optimization problem:

$$\mathcal{G}_i(x_i, R_i) = \min_{x_i, R_i} Z_i(x_i, R_i) \tag{5.3}$$

$$\text{s.t } x_i + x_i.R_i \leq K_i + y_i$$

The function $Z_i(x_i, R_i)$ is defined by:

$$Z_i(x_i, R_i) = b_i(x_i - a_i)^2 + d_i(R_i - R_{0,i})^2 \tag{5.4}$$

DE/rand/l/bin, according to [23] is the technical name where by:

- rand, means the base vector is *rand*omly chosen.
- l, means l vector difference is added to base vector.

- bin, means the number of parameters donated by the mutant vector, closely follow *bin*omial distribution.

In general we use this scheme to optimize a function, f of the form:

$$f(x) : \mathcal{R}^D \longrightarrow \mathcal{R} \tag{5.5}$$

The optimization goal is to minimize the value of this objective function $f(X)$;

$$min(f(x)) \tag{5.6}$$

by optimizing the values of parameters:

$$X = (x_1, ..., x_D), X \in \mathcal{R}^D, \tag{5.7}$$

where X is a vector comprised of D objective function parameters. Parameters of objective function are also subject to lower and higher boundary constraints, $x^{(L)}$ and $x^{(U)}$ respectively:

$$x_j^{(L)} \leq x_j \leq x_j^{(U)}, j = 1, ..., D \tag{5.8}$$

The important aspect we should put into consideration is that, Differential Evolution (DE) like any other *Evolutionary algorithm* mimics the natural evolution mechanisms such as reproduction, gene crossover and mutation, survival of fittest and so on [17]. The general Evolutionary algorithm can be shown in the following pseudocode fashion [24]:

```
---------------------------------------------------------
BEGIN
  INITIALIZE population with random candidate solutions;
  EVALUATE each candidate;
  REPEAT UNTIL (TERMINATION CONDITION is satisfied) DO
    1. SELECT parents;
    2. RECOMBINE pairs of parents;
    3. MUTATE the resulting offspring;
    4. EVALUATE new candidates;
    5. SELECT individuals for the next generation;
  OD
END
----------------------------------------------------------
```

This general scheme of evolutionary algorithm as flow chart is presented in figure 4:

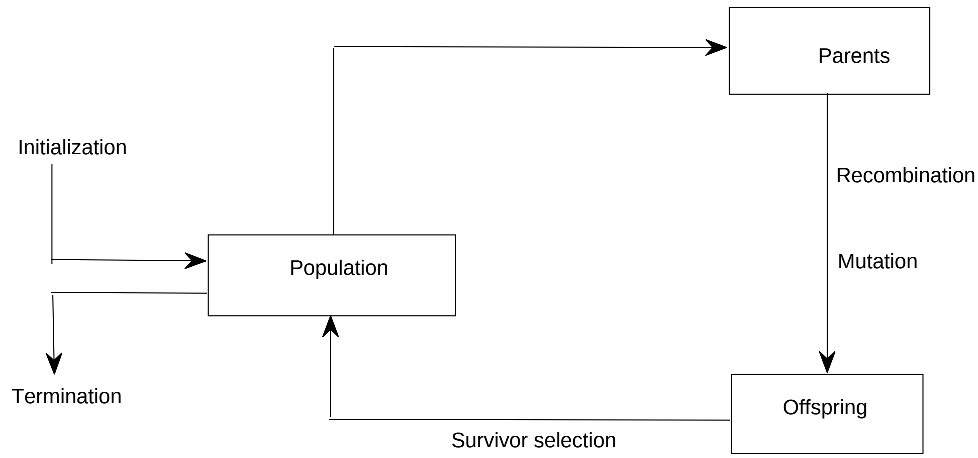

Figure 4: The general scheme of Evolutionary algorithm as flow-chart

For our case we are going to apply DE/rand/l/bin strategy (**Algorithm** 5.1) which is the representative of DEGS (Differential Evolution with Global Selection) in convex unimodal problems and has the following control parameters: Crossover rate, $CR \in [0,1]$, mutation factor $F \in]0,1+]$ and the population size NP. D is the dimension of the problem (the length of population vectors $\vec{x_{i,g}}$) [25].

Unimodal problem according to [24] is defined as:

Definition 5. *(Unimodal problem)*
A problem is said to be ***unimodal*** *if there is only one point which is fitter than all of its neighbors.*

5.5 Implementation of solution using DE

Our solution to the constrained optimization problem 5.3 has been implemented by using function 'sol=cde(func,limits)'. This function implements Differential Evolution (DE) strategy, **DE/rand/l/bin**. The name of the function to be minimized

Algorithm 5.1 DE/rand/l/bin (DEGS)

Randomly initialize population, $g = 1$
while termination criterion not met **do**
  for $i = 1;i \leq NP;i = i+1$ **do**
    Randomly pick $r_0, r_1, r_2 \in \{1, 2, ..., NP\}$, $r_0 \neq r_1 \neq r_2 \neq i$
    Randomly pick $j_{rand} \in \{1, 2, ..., D\}$
    for $j = 1;j \leq D;j = j+1$ **do**
      $v_{j,i,g} = x_{j,r_0,g} + F.\left(x_{j,r_1,g} - x_{j,r_2,g}\right)$ (Mutation)
      $u_{j,i,g} = \begin{cases} v_{j,i,g} & \text{if } rand[0,1] \leq CR \vee j = j_{rand} \quad \text{(Crossover)} \\ x_{j,i,g} & \text{otherwise} \end{cases}$
    end for
  end for
  for $i = 1; i \leq NP;i = i+1$ **do**
    $\vec{x_{i,g}} + 1 = \begin{cases} \vec{u_{i,g}} & \text{if } f(\vec{u_{i,g}}) \leq f(\vec{x_{i,g}}) \quad \text{(Selection)} \\ \vec{x_{i,g}} & \text{otherwise} \end{cases}$
  end for
  $g = g + 1$ (update generation counter)
end while

and initialization of limits of decision variables are given as input. Output is the best found solution.

The population size, NP was set to 100 while keeping updating generation counter from $G = 1$ to the maximum number of $G_{max} = 300$ so as to be assured of getting the best solution. Both mutation factor F and Crossover rate, CR were set to 0.9

The limits in which the best solution will be sought were provided by considering the following facts:

- The optimal emission levels, x_i^*, should be between 0 and a_i which is Business As Usual (BAU), i.e. $0 \leq x_i^* \leq a_i$. BAU is emission without restrictions and is shown in table 1.
- Optimal Relative uncertainty, R_i^*, should not exceed initial uncertainty, $R_{0,i}$ presented in table 1, i.e. $0 \leq R_i^* \leq R_{0,i}$.
- For economic reasons, Permits traded, y_i, whether negative (for potential sellers) or positive (for potential buyers) should not exceed the Kyoto targets, K_i set to each Party or Region, i.e $-K_i \leq y_i \leq K_i$.
- Permits in all regions should sum up to 0, i.e. $\sum y_i = 0$. This means that no more permits should be left in the market for transaction.

The results presented in table 5 were found by using Differential Evolution algorithm: We were very interested to find the number of permits each Party can buy to emit

	Ems	**Runc**	**Sp**	**MC**	**Ptr**	**TC**	α_i	γ_i
Variable	x_i^*	R_i^*	λ_i	$C_i'(x_i)$	y_i	$C_i(x_i^*)+$ $\mathcal{H}_i(R_i^*)$		
Units	MtC/yr	-	$\$/tC$	$\$/tC$	-	$MUS\$$	-	-
US	1304.3	0.0998	-253.3	-278.6	200	79768	6.9288	0.7054
OECDE	881.73	0.1672	-251.4	-293.4	170	27300	5.9538	0.8269
JAPAN	293.61	0.1126	-261.8	-291.3	70	10660	7.1848	0.8149
CANZ	182.26	0.1385	-249.7	-284.3	-4	20902	4.0486	0.5623
EEFSU	699.19	0.2146	-250.6	-304.4	-436	46593	4.2340	0.7516
Total					0	195883		

Table 5: The optimal values of emissions, relative uncertainty and permits found by using Differential Evolution algorithm 5.1

more or to sell so as to gain profit from the buyers. For illustration we called (evoked) the ***history*** from initial generation to maximum generation where by the optimal number of permits can be found. For example after 17 generations, US had been allocated 200 permits as it is shown in figure 5:

The maximum number of generations from which the optimal value of permits will be sought, is found to the point where our figure starts to **stabilize** and forming **horizontal line**.

We extended our knowledge of using DE in solving also the optimal solution of problem (5.3) when there is no transaction, i.e, $y_i = 0$. The parameter x_1, R_1 and objective function value, $\mathcal{G}_1(x_1, R_1)$, for US were plotted against number of generations in figure 6 and figure 7 as follows:

The optimal values, x_1^* , R_1^* and $\mathcal{G}_1(x_1, R_1)$ which is objective function value were found to be $1134.9 MtC/yr$, 0.1023 and $134800 MUS\$$ after 95, 129 and 112 number of **generations** respectively (confer figure 6 and 7).

We were also interested to find how the parameter for emission, x_1 and parameter for relative uncertainty, R_1 are related to objective function value (cost), $\mathcal{G}_1(x_1, R_1)$. The parameter, x_1 was found to be *negatively correlated* to objective function value, $\mathcal{G}_1(x_1, R_1)$ as it can be clearly seen from the left panel of figure 6 and the separate figure 7. The relationship between emission parameter, x_1 and objective function value (total cost) was plotted in figure 8 as follows:

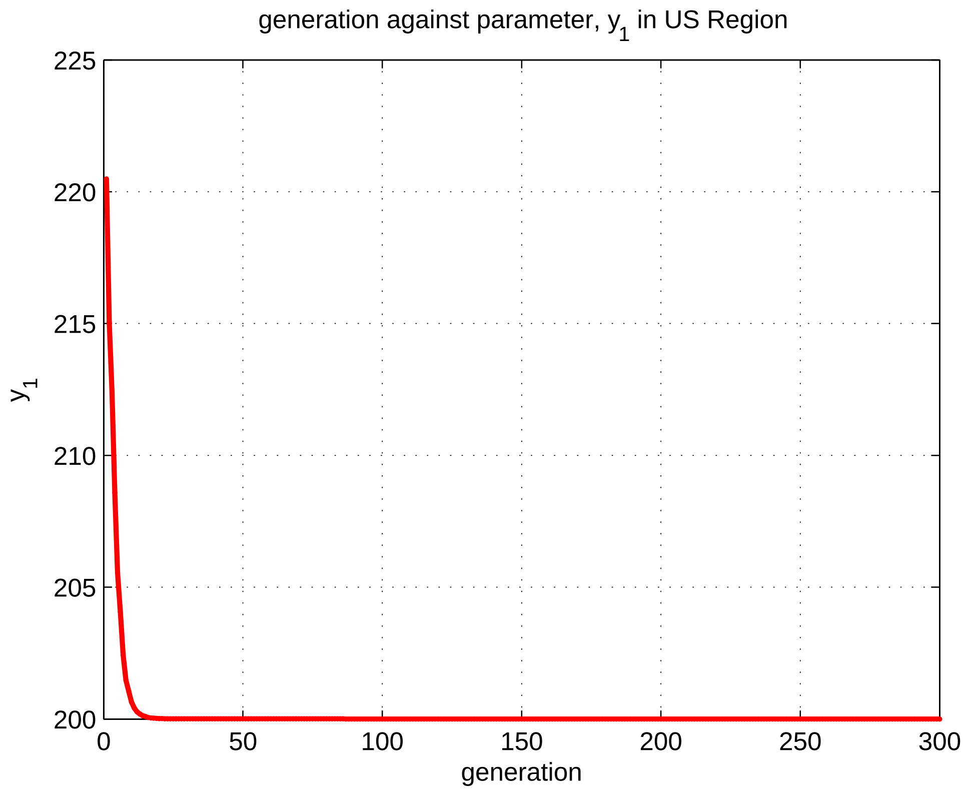

Figure 5: The number of generations against permits in US Region

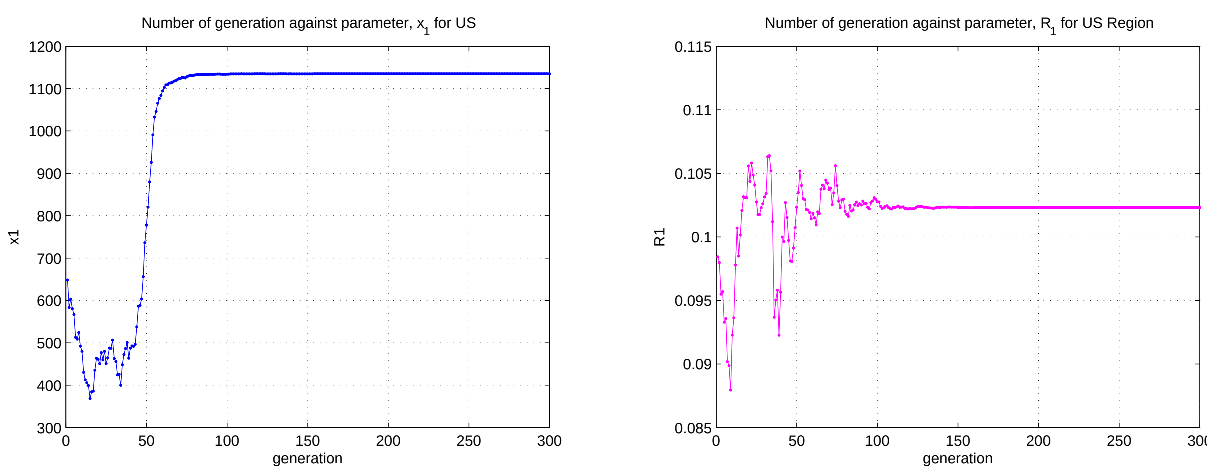

Figure 6: Parameter value, x_1 and R_1 plotted against number of generation in US Region

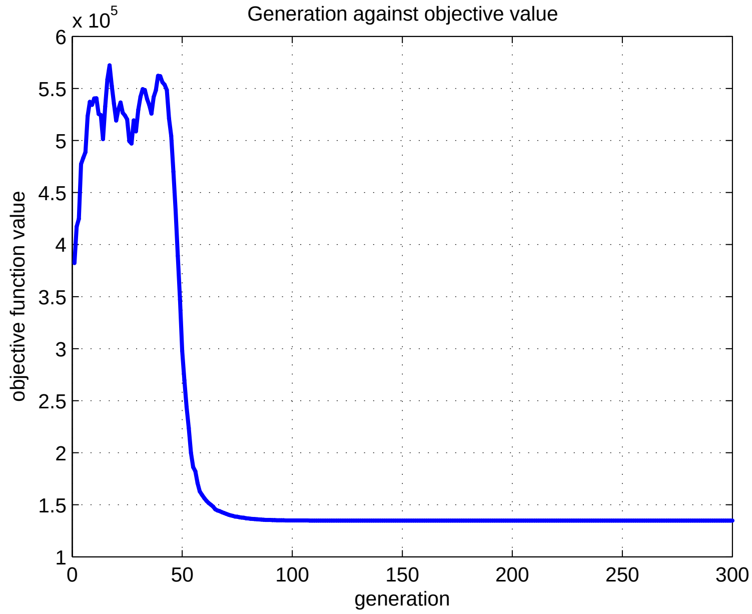

Figure 7: Objective function value, $\mathcal{G}_1(x_1, R_1)$ against number of generation

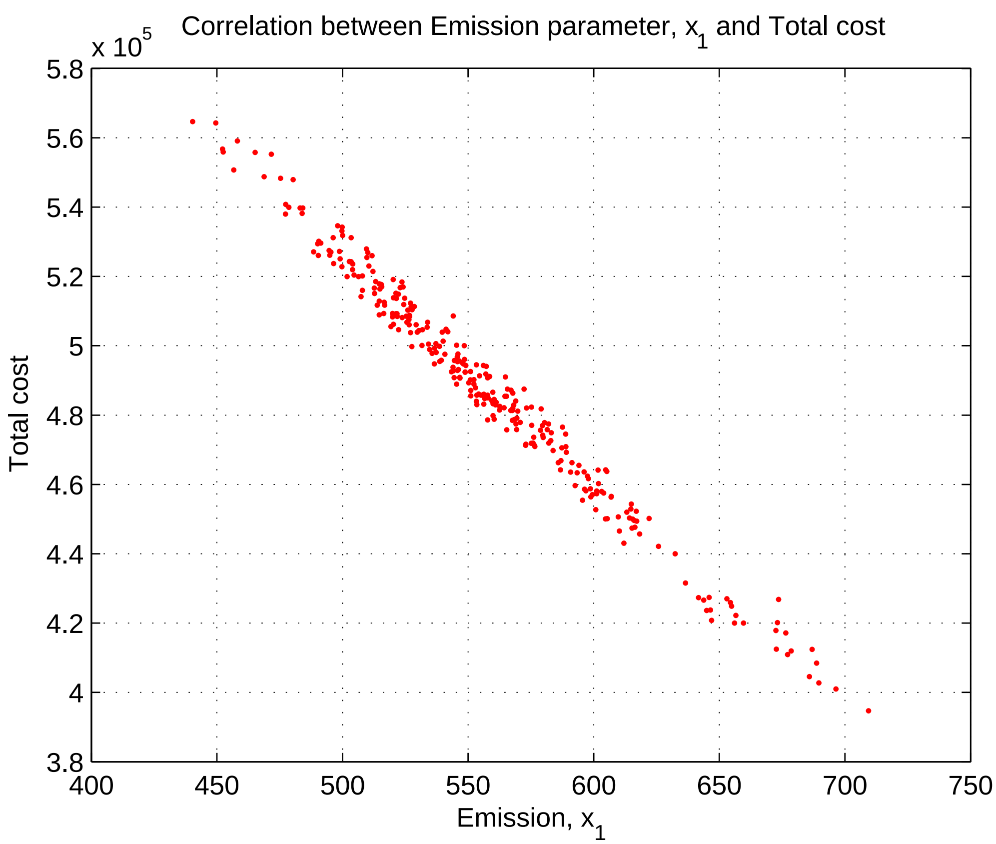

Figure 8: Objective function value, against parameter, x_1

We calculated also *correlation coefficient*, ρ, for $-1 \leq \rho \leq 1$, from history of generations, to check the extent by which the two parameters showing in figure 8 are correlated. Correlation matrix, $R(i,j)$ was found by using the formula:

$$R(i,j) = \frac{C(i,j)}{\sqrt{C(i,i)C(j,j)}} \tag{5.9}$$

where by C is *covariance matrix.* Suppose, G_1 is objective function value for US, then correlation matrix, $R(x_1, G_1)$ was found to be:

$$R(x_1, G_1) = \begin{pmatrix} & x_1 & G_1 \\ x_1 & 1.0000 & -0.9996 \\ G_1 & -0.9996 & 1.0000 \end{pmatrix} \tag{5.10}$$

Correlation coefficients are off-diagonal elements of matrix 5.10. That is correlation coefficient of parameters x_1 and G_1 is $\rho = -0.9996$. This is to show that the two parameters are *strongly negative* correlated. An **implication** to carbon trading context is that as emission decreases, the cost of abating (reducing) it increases and vice-versa.

We plotted also the relationship between relative uncertainty, R_1 and objective function value, G_1 in figure 9 as follows:

Figure 9 shows that parameter, R_1 has no correlation with objective function value, G_1. This indicates that relative uncertainty, R_1 for US Region has little or no effect (contribution) to objective function value. Correlation matrix for parameters R_1 and G_1 was calculated and found to be:

$$R(x_1, G_1) = \begin{pmatrix} & R_1 & G_1 \\ R_1 & 1.0000 & -0.0214 \\ G_1 & -0.0214 & 1.0000 \end{pmatrix} \tag{5.11}$$

where by its corresponding correlation coefficient is, $\rho = -0.0214$.

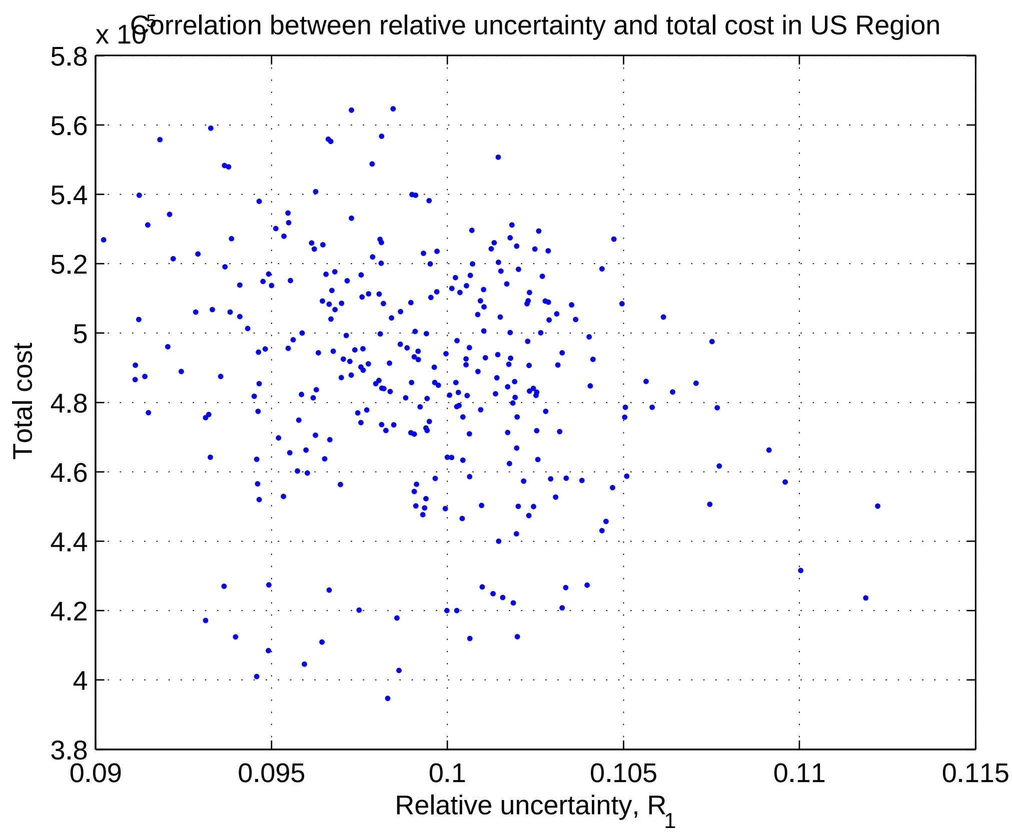

Figure 9: Objective function value, against parameter, R_1

Although results found by using Differential Evolution algorithm in table 5 are converging to those found by classical Lagrangian method in table 3, we have to take a note that DE is heuristic method and it does not guarantee global solution [17]. Sometimes Evolutionary algorithm gives both feasible and infeasible solutions. However it is impractical to find the objective function value, for an infeasible solution, but two solutions according to [26] are compared at a time by using the following scenarios:

- Any feasible solution is preferred to any infeasible solution.
- Among two feasible solutions, the one having a better objective function value is preferred.
- Among two infeasible solutions, the one having a smaller constraint violation is preferred.

5.6 Employment of Kuhn Tucker conditions to the Model

In this subsection we are going to analyze Kuhn Tucker(KT) conditions (sometimes called Karush-Kuhn-Tucker(KKT)) according to the results presented in table 2 and table 3 respectively. This is generalization of Lagrangian method to determine the necessary conditions for optimality. According to [16], p.330, these conditions tell us much on how the first order derivatives of objective function is related to the active constraints at a solution x_i. First we are going to analyze the situation when there is no trade of permits. This implies that constraint (3.22) becomes $x_i + x_i R_i = K_i$.

Let us consider the optimization problem presented in Equations (3.21) and (3.22). By substituting the cost functions (3.46) and (3.47) and adding non negativity constraints, $x_i \geq 0$ and $R_i \geq 0$. Suppose that we present the function $\mathcal{W}_i(x_i, R_i)$ such that:

$$\mathcal{W}_i(x_i, R_i) = b_i \left(x_i - a_i\right)^2 + d_i \left(R_i - R_{0,i}\right)^2 \tag{5.12}$$

It follows that the optimization problem becomes;

$$\begin{aligned} \mathcal{U}_i(x_i, R_i) &= \min_{x_i,R_i} \mathcal{W}_i(x_i, R_i) \\ &\text{s.t } x_i + x_i R_i = K_i \\ &x_i \geq 0 \\ &R_i \geq 0 \end{aligned} \tag{5.13}$$

The Lagrangian function of Problem (5.13) can be written as follows;

$$L(x_i, R_i, \lambda_i, \mu_i, \Lambda_i) = b_i \left(x_i - a_i\right)^2 + d_i \left(R_i - R_{0,i}\right)^2 - \lambda_i(x_i + x_i R_i - K_i) - \mu_i x_i - \Lambda_i R_i \tag{5.14}$$

The following Kuhn-Tucker conditions apply:

- Primal feasibility

$$\begin{aligned} &x_i + x_i R_i = K_i \\ &x_i \geq 0 \\ &R_i \geq 0 \end{aligned} \tag{5.15}$$

- Stationarity

$$\frac{\partial L}{\partial x_i} = 2b_i(x_i - a_i) - \lambda_i - \lambda_i R_i - \mu_i = 0 \tag{5.16}$$

$$\frac{\partial L}{\partial R_i} = 2d_i(R_i - R_{0,i}) - \lambda_i x_i - \Lambda_i = 0 \tag{5.17}$$

- Dual feasibility

$$\mu_i \geq 0 \tag{5.18}$$

$$\Lambda_i \geq 0 \tag{5.19}$$

- Complementary slackness

$$\mu_i x_i = 0 \tag{5.20}$$

$$\Lambda_i R_i = 0, \text{for } i = 1, 2, ..., 5 \tag{5.21}$$

It is clearly that $K_i \neq 0$ and it follows from (5.15) which can be written as $x_i(1 + R_i) = K_i$, that affirms $x_i \neq 0$. From Equation (5.20) implies that $\mu_i = 0$. It is practically that reported emission can not be equal to Kyoto target, $x_i \neq K_i$ and immediately it follows that from (5.15) that $R_i \neq 0$. Hence from (5.21) it implies that $\Lambda_i = 0$.

By substituting $\mu_i = 0$ and $\Lambda_i = 0$ into (5.16) and (5.17) and make use of equality constraint (5.15) we shall get the system of the following three equations:

$$\begin{aligned} 2b_i(x_i - a_i) - \lambda_i - \lambda_i R_i &= 0 \\ 2d_i(R_i - R_{0,i}) - \lambda_i x_i &= 0 \\ x_i + x_i R_i - K_i &= 0 \end{aligned} \tag{5.22}$$

Solving the system (5.22) we shall end up with optimal values for emission , relative uncertainty and shadow prices presented in table 2.

We have noted that $x_i \neq 0$ and $R_i \neq 0$. This implies that the only **active constraints** in (5.15) is $x_i + x_i R_i = K_i$.

5.6.1 KT sufficient second order conditions

In order to examine for sufficient KT second order conditions we need to find the Hessian matrix by considering Equations (5.16) and (5.17) as follows:

$$\nabla^2(x_i, R_i, \lambda_i) = \begin{bmatrix} 2b_i & -\lambda_i \\ -\lambda_i & 2d_i \end{bmatrix} \tag{5.23}$$

The values for b_i and λ_i are obvious presented in table 1 and 2 respectively. In order to get the values for d_i we have to apply relation (3.65) where the values for initial uncertainty $R_{0,i}$ are also provided in table 1. Then the values for d_i for $i = 1, 2, ..., 5$ separately are:

$d_1 = 7022100, d_2 = 4883500, d_3 = 2014300, d_4 = 541700$ and $d_5 = 2111600$

We have now to apply determinant and eigenvalue tests so as to deduce the nature of our minima points. From Equation (5.23) and for $i = 1$, let us call $a_{11} = 2b_1$. The determinants of a_{11} and a_{22} are given by:

$$|a_{11}| = 2b_1 = 2 * 0.2755 = 0.551 > 0, \tag{5.24}$$

$$|a_{22}| = \begin{vmatrix} 2b_1 & -\lambda_1 \\ -\lambda_1 & 2d_1 \end{vmatrix} = 4b_1d_1 - {\lambda_1}^2 = 7620979.44 > 0 \tag{5.25}$$

For $i = 2$ it follows that:

$$|a_{11}| = 2b_2 = 2 * 0.9065 = 1.813 > 0, \tag{5.26}$$

$$|a_{22}| = \begin{vmatrix} 2b_2 & -\lambda_2 \\ -\lambda_2 & 2d_2 \end{vmatrix} = 4b_2d_2 - {\lambda_2}^2 = 17490042.04 > 0 \tag{5.27}$$

For $i = 3$ we have:

$$|a_{11}| = 2b_3 = 2 * 2.4665 = 4.933 > 0, \tag{5.28}$$

$$|a_{22}| = \begin{vmatrix} 2b_3 & -\lambda_3 \\ -\lambda_3 & 2d_3 \end{vmatrix} = 4b_3d_3 - {\lambda_3}^2 = 19595460.19 > 0 \tag{5.29}$$

For $i = 4$ we find that:

$$|a_{11}| = 2b_4 = 2 * 1.1080 = 2.216 > 0, \tag{5.30}$$

$$|a_{22}| = \begin{vmatrix} 2b_4 & -\lambda_4 \\ & \\ -\lambda_4 & 2d_4 \end{vmatrix} = 4b_4d_4 - {\lambda_4}^2 = 2341716.79 > 0 \tag{5.31}$$

For i=5, we have:

$$|a_{11}| = 2b_5 = 2 * 0.7845 = 1.569 > 0, \tag{5.32}$$

$$|a_{22}| = \begin{vmatrix} 2b_5 & -\lambda_5 \\ & \\ -\lambda_5 & 2d_5 \end{vmatrix} = 4b_5d_5 - {\lambda_5}^2 = 6614188.64 > 0 \tag{5.33}$$

In order to use eigenvalue test, let the values of eigenvalues be η_i and manipulating matrix in Equation (5.23) it will give us:

$$\begin{vmatrix} 2b_i - \eta_i & -\lambda_i \\ & \\ -\lambda_i & 2d_i - \eta_i \end{vmatrix} = 0 \tag{5.34}$$

The determinant of Equation (5.34) gives us:

$${\eta_i}^2 - 2(b_i + d_i)\eta_i + 4b_id_i - {\lambda_i}^2 = 0 \tag{5.35}$$

Now we have to solve eigenvalues, η_i from the set of quadratic equations in (5.35) separately for $i = 1, ..., 5$ as follows:
For $i = 1$ we have quadratic equation:

$$\begin{aligned} {\eta_1}^2 - 2(b_1 + d_1)\eta_1 + 4b_1d_1 - {\lambda_1}^2 = 0 \\ \Longrightarrow {\eta_1}^2 - 14044200.551\eta_1 + 7620979.44 = 0 \end{aligned} \tag{5.36}$$

Equation (5.36) gives $\eta_1 = (1.4044).10^7 > 0$ or $\eta_1 = 0.5426 > 0$

For $i = 2$ it follows;

$$\begin{aligned} {\eta_2}^2 - 2(b_2 + d_2)\eta_2 + 4b_2d_2 - {\lambda_2}^2 = 0 \\ \Longrightarrow {\eta_2}^2 - 9767001.813\eta_2 + 17490042.04 = 0 \end{aligned} \tag{5.37}$$

Equation (5.37) gives $\eta_2 = (9.7670).10^6 > 0$ or $\eta_2 = 1.7907 > 0$

For $i = 3$, we have quadratic equation:

$$\eta_3{}^2 - 2(b_3 + d_3)\eta_3 + 4b_3d_3 - \lambda_3{}^2 = 0$$
$$\implies \eta_3{}^2 - 4028604.933\eta_3 + 19595460.19 = 0 \tag{5.38}$$

By solving (5.38) we get $\eta_3 = (4.0286).10^6 > 0$ or $\eta_3 = 4.8641 > 0$
For $i = 4$, we have:

$$\eta_4{}^2 - 2(b_4 + d_4)\eta_4 + 4b_4d_4 - \lambda_4{}^2 = 0$$
$$\implies \eta_4{}^2 - 1083402.216\eta_4 + 2341716.79 = 0 \tag{5.39}$$

Solving (5.39) we get $\eta_4 = (1.0834).10^6 > 0$ or $\eta_4 = 2.1615 > 0$

Lastly, for $i = 5$ we find that:

$$\eta_5{}^2 - 2(b_5 + d_5)\eta_5 + 4b_5d_5 - \lambda_5{}^2 = 0$$
$$\implies \eta_5{}^2 - 4223201.569\eta_5 + 6614188.64 = 0 \tag{5.40}$$

Solving (5.40) we get $\eta_5 = (4.2232).10^6 > 0$ or $\eta_5 = 1.5662 > 0$
Since all roots of quadratic equations (5.36, 5.37, 5.38, 5.39, 5.40) are positive implies that all eigenvalues are also positive. Hessian matrix (5.23) is positive definite and eigenvalues are all positive, concluding that $(x^*, R^*) = (x_i{}^*, R_i{}^*)$ are local minimum points.

Furthermore to be assured that our solution attains local minimum let us test if the following relation exists according to our optimal values we have:

$$L(x^*, R^*, \lambda, \mu, \Lambda) \leq L(x^*, R^*, \lambda^*, \mu^*, \Lambda^*) \leq L(x, R, \lambda^*, \mu^*, \Lambda^*) \tag{5.41}$$

In-fact from (5.14) when substituting the optimal values from table 2 and cost parameters from table 1 we find that:

$$L(x^*, R^*, \lambda^*, \mu^*, \Lambda^*) = b_i\left(x_i{}^* - a_i\right)^2 + d_i\left(R_i{}^* - R_{0,i}\right)^2 - \lambda_i{}^*(x_i{}^* + x_i{}^* R_i{}^* - K_i) \tag{5.42}$$

$$- \mu_i{}^* x_i{}^* - \Lambda_i{}^* R_i{}^*$$

$\Longrightarrow L(x_1^*, R_1^*, \lambda_1{}^*, \mu_1{}^*, \Lambda_1{}^*) = 136637,\ L(x_2^*, R_2^*, \lambda_2{}^*, \mu_2{}^*, \Lambda_2{}^*) = 87450,$

$L(x_3^*, R_3^*, \lambda_3{}^*, \mu_3{}^*, \Lambda_3{}^*) = 37120,\ L(x_4^*, R_4^*, \lambda_4{}^*, \mu_4{}^*, \Lambda_4{}^*) = 18840$ and

$L(x_5^*, R_5^*, \lambda_5{}^*, \mu_5{}^*, \Lambda_5{}^*) = 8120$

Again we can find separately the values of expression $L(x_i^*, R_i^*, \lambda_i, \mu_i, \Lambda_i)$ for $i = 1, ..., 5$ as follows:

$$L(x_1^*, R_1^*, \lambda_1, \mu_1, \Lambda_1) = 135740 - 1134.9\mu_1 - 0.1000\Lambda_1 \tag{5.43}$$

$$L(x_2^*, R_2^*, \lambda_2, \mu_2, \Lambda_2) = 87450 - 738.4\mu_2 - 0.1647\Lambda_2 \tag{5.44}$$

$$L(x_3^*, R_3^*, \lambda_3, \mu_3, \Lambda_3) = 37110 - 230.4\mu_3 - 0.1199\Lambda_3 \tag{5.45}$$

$$L(x_4^*, R_4^*, \lambda_4, \mu_4, \Lambda_4) = 18840 - 185.6\mu_4 - 0.1583\Lambda_4 \tag{5.46}$$

$$L(x_5^*, R_5^*, \lambda_5, \mu_5, \Lambda_5) = 8120 - 991.2\mu_5 - 0.3257\Lambda_5 \tag{5.47}$$

Depending on the choice of $\mu_i \geq 0$ and $\Lambda_i \geq 0$ from conditions (5.18) and (5.19) it follows that:

$$L(x^*, R^*, \lambda, \mu, \Lambda) \leq L(x^*, R^*, \lambda^*, \mu^*, \Lambda^*) \tag{5.48}$$

We also have to find the value of expression $L(x, R, \lambda^*, \mu^*, \Lambda^*)$ by substituting to it the values of shadow prices λ_i from table 2 as follows:

$$L(x, R, \lambda^*, \mu^*, \Lambda^*) = b_i\left(x_i - a_i\right)^2 + d_i\left(R_i - R_{0,i}\right)^2 + \lambda_i{}^*\left(x_i + x_i R_i - K_i\right) \tag{5.49}$$

But $x_i + x_i R_i - K_i \longrightarrow 0$ and from the fact that the value of d_i is large enough we will end up with:

$$L(x, R, \lambda^*, \mu^*, \Lambda^*) = b_i\left(x_i - a_i\right)^2 + d_i\left(R_i - R_{0,i}\right)^2 \tag{5.50}$$

This signals that:

$$L(x^*, R^*, \lambda^*, \mu^*, \Lambda^*) \leq L(x, R, \lambda^*, \mu^*, \Lambda^*) \tag{5.51}$$

Hence from relations (5.48) and (5.51) and by transitive property we get the general expression:

$$L(x^*, R^*, \lambda, \mu, \Lambda) \leq L(x^*, R^*, \lambda^*, \mu^*, \Lambda^*) \leq L(x, R, \lambda^*, \mu^*, \Lambda^*) \tag{5.52}$$

We claim from the general expression in (5.52) that $(x^*, R^*) = ({x_i}^*, {R_i}^*)$ are local minimum points of the problem. That is our solution reach the local minimum.

5.7 Evaluation of the Model Results

Table 2 presents the optimal values of Equations (3.21) and (3.22) for both emissions, x_i and relative uncertainties, R_i each Party should choose so as to comply with Kyoto Protocol. In this situation no transaction is done in the sense that carbon permits are not bought or traded. The shadow prices, λ_i are not in equilibrium. They differ from one region to another to reflect the potential for trade. This means that each region has its own willing to pay for an extra unit of emission it emits, i.e. by relaxing constraint (3.22) by one unit. In future Party members will see the benefit of trading permits so as to equilibrate and bringing down the shadow price.

The shadow prices for US, OECDE, Japan and CANZ are negative in the sense that those regions are positively willing to pay amounts of shadow prices shown in table 2 for an extra unit of emission, that is for every $1tC$. Further more it is pointed in [15] p.686 that the necessary condition for minimization problem is for λ values to be non-positive. The shadow price of EEFSU is positive in the sense that the region does not need to carry any emission reduction strategy to comply with Kyoto Protocol. For the same argument according to [1] the marginal costs $C_i'(x_i)$ and $\mathcal{H}_i'(R_i)$ for reducing emissions and relative uncertainty respectively are negative for regions US, OECDE, Japan and CANZ so as to be in positive sense in reducing emissions and relative uncertainty. These regions are willing to pay for an additional unit of reduced emissions or relative uncertainty. The aggregated total cost for reducing both emissions and relative uncertainty before any transaction is done was found to be 287260 MU$.

The values for α_i and γ_i were calculated following the relations (3.53), (3.54) and (3.65). The values for α_i range from 4.1254 for CANZ to 6.3365 for EEFSU, while the values for γ_i range from 0.5730 for CANZ to 1.1248 for EEFSU. The interesting feature here is in region EEFSU that has $\gamma_i > 1$. This implies that it does not need to have any initiative to abate emissions so as to comply with Kyoto targets. This follows direct from Equation (3.60) from which $a_i < K_i$ (confer table 1), that means the BAU assigned to EEFSU is lower than Kyoto targets. Another interesting feature is that, the smaller the value of γ_i the higher the need to abate the reduction in order to comply with Kyoto targets. For instances CANZ has task to reduce $1 - 0.5730 = 0.427 \approx 43\%$ from the BAU emission levels, while OECDE has to reduce $1 - 0.6904 \approx 31\%$ from its BAU emission level. Both α_i and γ_i are greater than threshold 0.25 that was stated in Equation (3.58) to show that from economic point of view, Parties will not be willing to abate to the threshold. It is pointed in [3] that, since values of $\alpha_i > 3$ and $\gamma_i > 0.45$ then in our case non-convex solution does not exist. Then it is possible our solution to attain local minimum.

It follows from table 3 that permits are now bought or traded within the Party members.Both CANZ and EEFSU accrue negative amount of permits, y_i which implies that they are net seller of permits in the carbon market, while US, OECDE and Japan accrue positive amount of permits to indicate that they are potential buyers of permits in the carbon market. The shadow prices for all regions is roughly equilibrated to 252$/tC. The total cost for reducing both emissions and relative uncertainty is 168998MUS$ lower than the total cost in the situation when there was no trade of permits (confer table 2). This has made a slight increase of optimal emission level of both emissions, x_i and relative uncertainty R_i for potential buyers of permits which are US, OECDE and Japan compared to the situation when there was no trade of permits.

The value of γ_i for EEFSU after the trade of permit is now $0.7507 < 1$. This means that EEFSU is now obliged to take part in emission reduction compared to the situation when there was no trade. We can signal also the slight increase of γ_i to those regions which are potential buyers of permits. This indicates that they have a slight relief of emission reduction and they will be motivated to buy permits in the market than to invest in other programmes of cutting down emissions. Likewise even in the context of trading permits the values of α_i and γ_i are greater than 3 and 0.45 respectively. This implies that non-convex solution does not exist and our optimal solutions will be locked to local minimum.

However from emission cost reducing function in (3.46) is a decreasing function from zero emission to Business As Usual (BAU). The cost function becomes zero from BAU onwards as a reflection of not having any restrictions or regulations to counter emitters. It indicates that costs for cutting down emissions increases as emission decreases and vice versa up to BAU. Afterwards costs stabilize to zero. This is shown briefly in figure 10 as an example from US with regard to data shown in table 1.

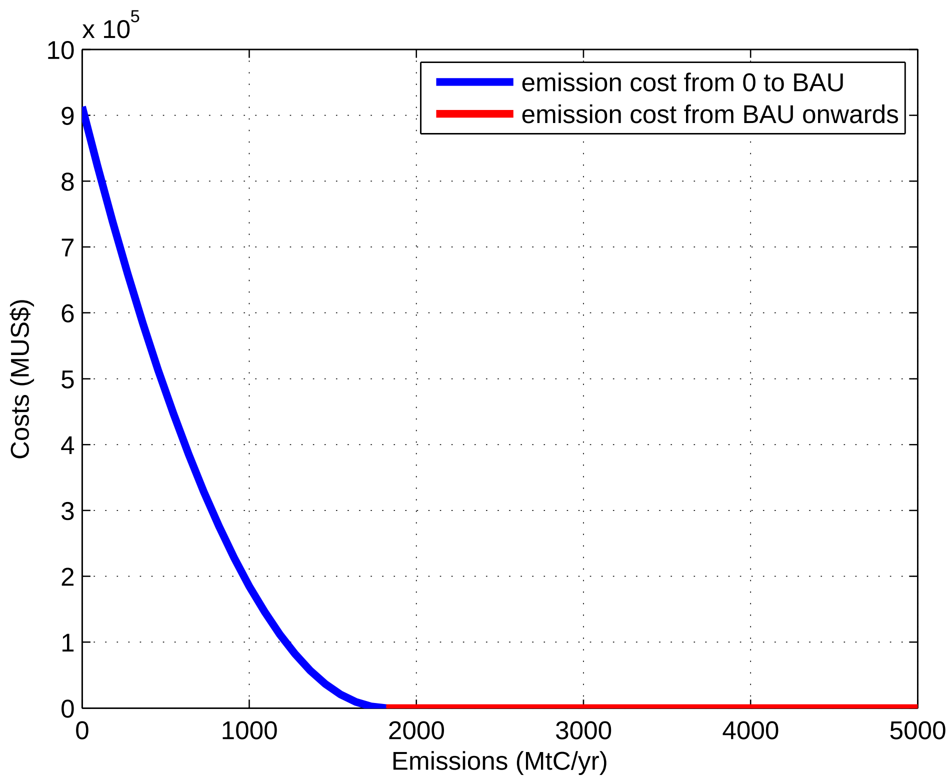

Figure 10: Costs for reducing emissions against reported emissions for US with reference to Equation (3.46)

It is also obvious for the case of uncertainty reduction cost function (3.47) that is a decreasing function. This indicates that the cost for monitoring relative uncertainty increases as volume of uncertainty decreases and vice versa to initial uncertainty and becomes zero onwards (Confer figure 11).

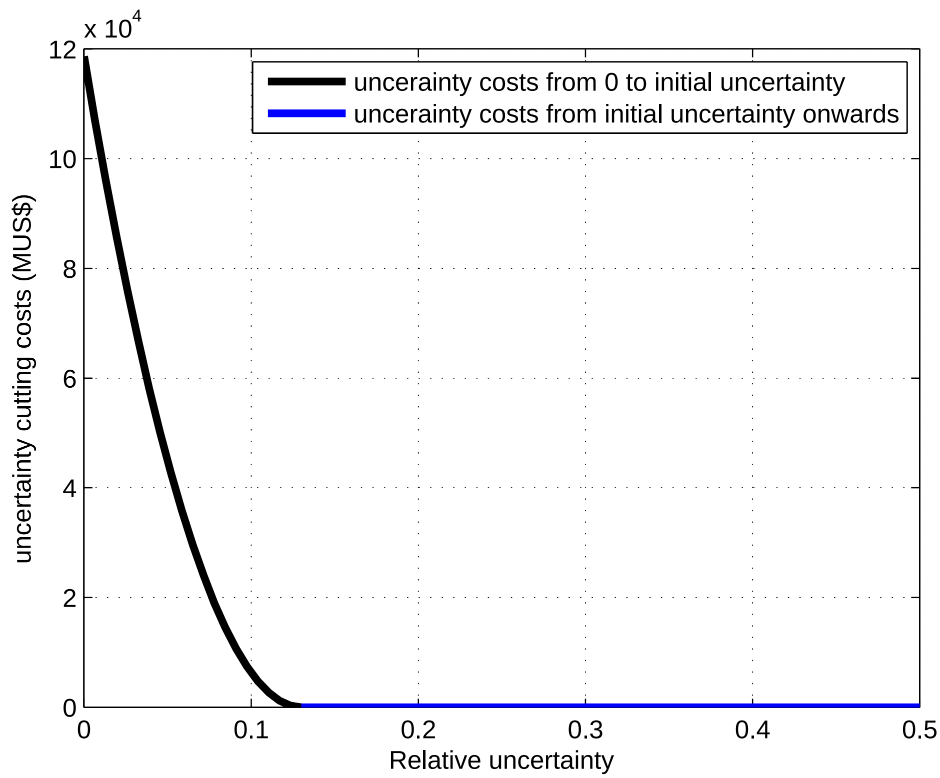

Figure 11: Costs for monitoring uncertainty against reported relative uncertainty for US with reference to Equation (3.47) and data from table 1

We can get clear picture when holding all regions' cost functions together. In figure 12, given that Marginal cost, MC $= \dfrac{\partial C}{\partial X}$, where C and X are both cost for reducing emission and reported emission respectively it is fundamental that, if all regions will be forced to cut down emissions as much as possible, JAPAN would have incurred high Marginal abatement costs for emission than other regions. It is also pointed in [21] those regions with high marginal abatement cost are considered to be potential buyers of permits while those which have low marginal abatement cost are potential sellers of permits.

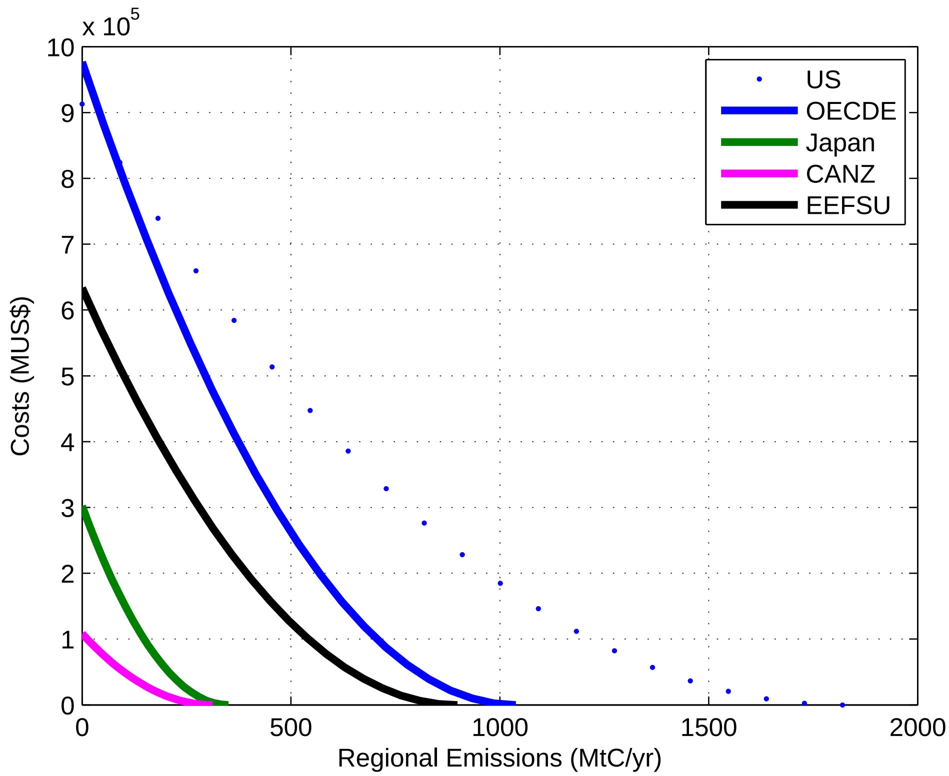

Figure 12: Costs against regions' reported emissions hold together with reference to data from table 1

In figure 13 all cost functions for uncertainty are decreasing functions to zero. Again since Marginal cost, $MC = \dfrac{\partial \mathcal{H}}{\partial R}$, where $\mathcal{H}$ and R are costs for reducing uncertainty, and reported relative uncertainty respectively then it follows that US has high marginal abatement cost for uncertainty than other regions.

The concepts of Abatement cost function and Marginal Abatement cost (MAC) has been defined by [21] as follows:

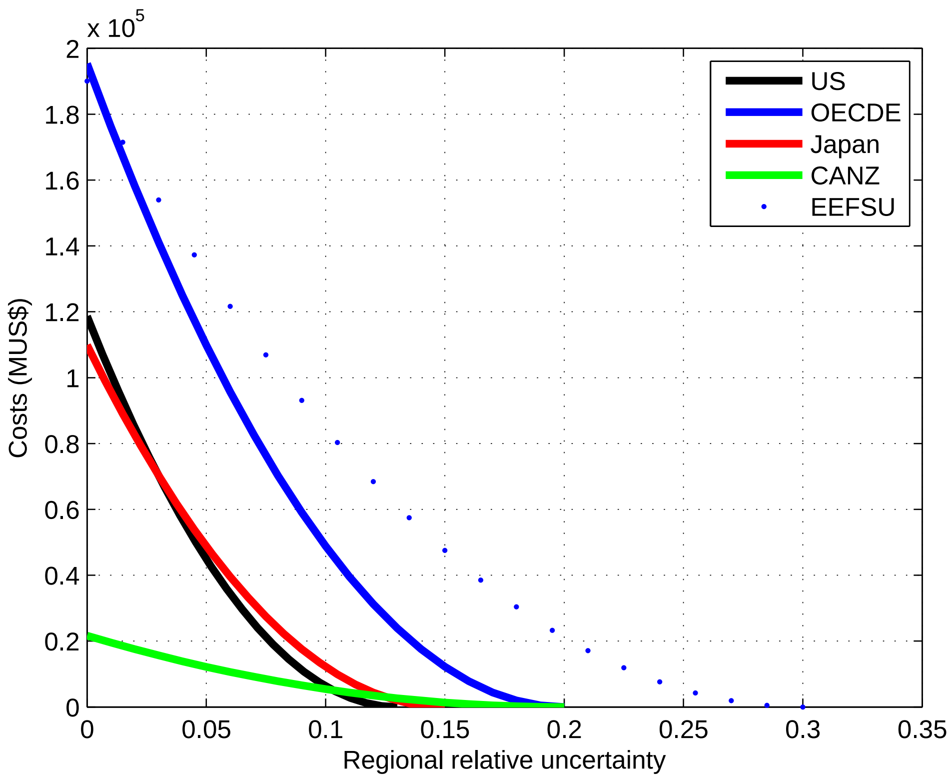

Figure 13: Costs against regions' reported relative uncertainty hold together with reference to data from table 1

Definition 6. *(Abatement cost function and MAC)*
Abatement cost function describes cost that related to emission reduction. Then Marginal Abatement Cost (MAC) is defined as the minimum possible cost related to emission reduction.

It is further pointed in [21] that both economic theory and empirical results indicates that MAC is a grown function of emission reduction level. Let us take an example of US. Emission for US was targeted to 1135MtC/yr with a MAC of 377.6$/tC. If the right to emit is decreased from that level Marginal abatement Cost (MAC) is higher than the targeted one. That means it is cheaper to buy additional permits to emit more than to introduce new pollution reduction investments (Confer figure 14).

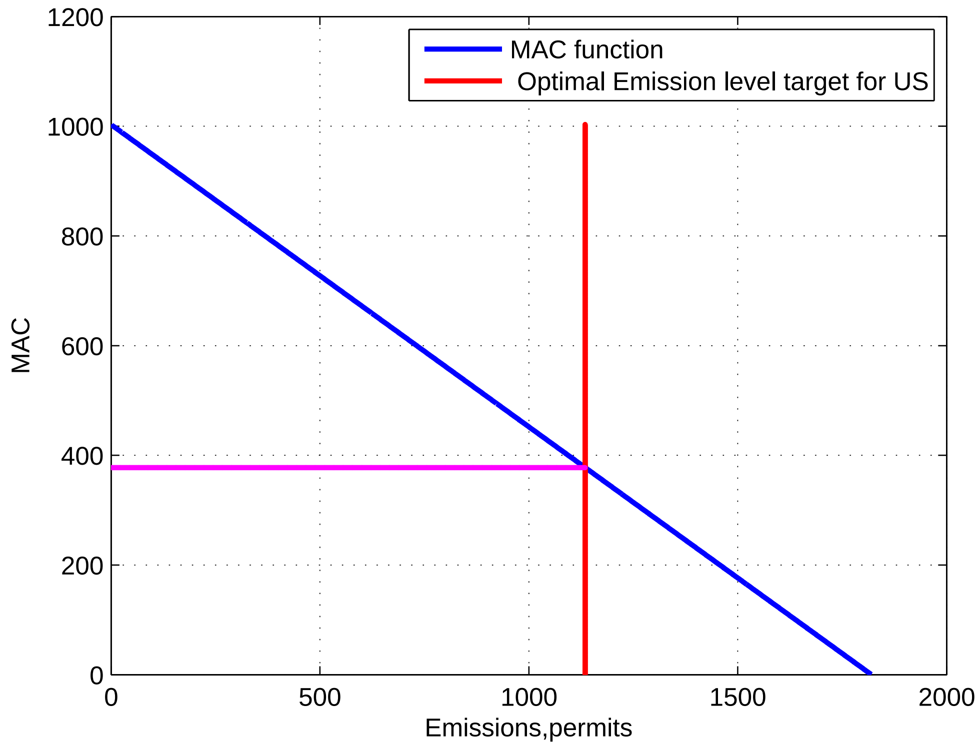

Figure 14: Marginal Abatement Cost (MAC) against Emission for US with reference to data from table 2

It can be shown that there is slight improvement when carbon permits are traded. After the trade of permits the optimal emission for US has increased to 1312 MtC/yr compared to 1135 MtC/yr when no trade was conducted. However MAC has decreased sharply to 278$/tC compared to 377.6$/tC when there were no transaction carried. This is due to the fact that US is a potential buyer of additional permits from the market to pollute more. This is contrast to CANZ and EEFSU where by there is slight decrease in optimal value of emissions and sharp increase in MAC from the situation where no trade of permits is carried to the context where trade is taking place. For illustration compare figure 14 and 15

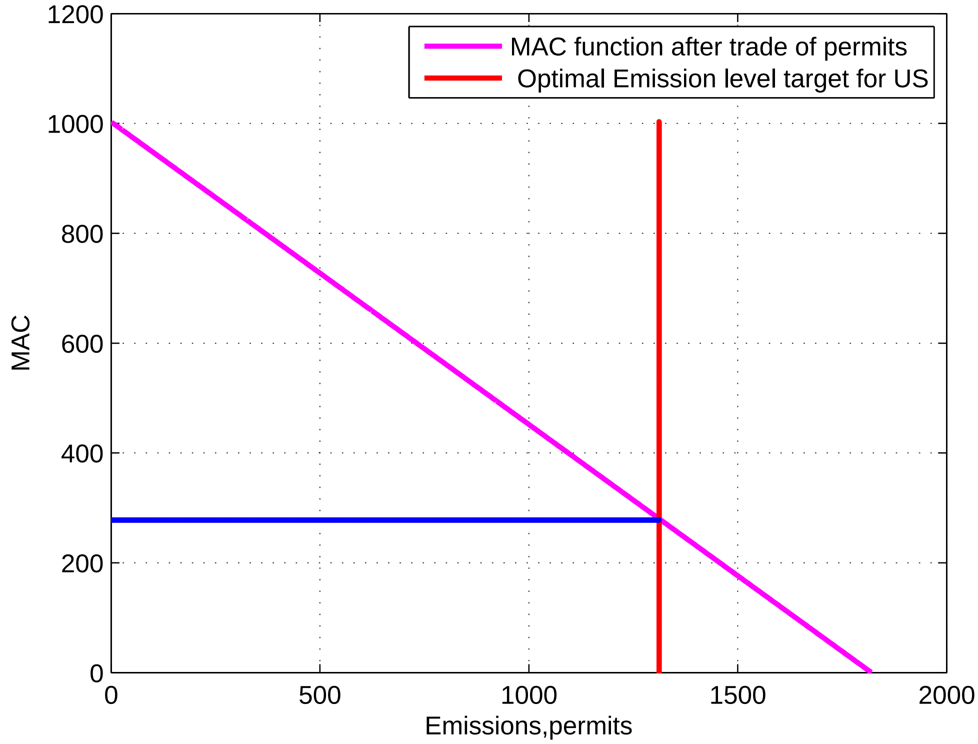

Figure 15: Marginal Abatement Cost (MAC) against Emission for US when permits are traded with reference to data from table 3

The behavior of figures 14 and 15 can be hold together to figure 16. The decrease in MAC from 377.6\$/tC to 278\$/tC indicates that it is cheaper for US to buy additional carbon permits in market to have right to emit more, i.e. from 1135 MtC/yr when permits are not bought to 1312 MtC/yr in the situation permit trade is allowed.

Likewise for CANZ who are potential seller of permit say, optimal level of emission is decreased from 185.6MtC/yr to 181.4MtC/yr and by doing so raising the MAC from 281.6\$/tC to 291\$/tC. This behavior has been shown in figure 17

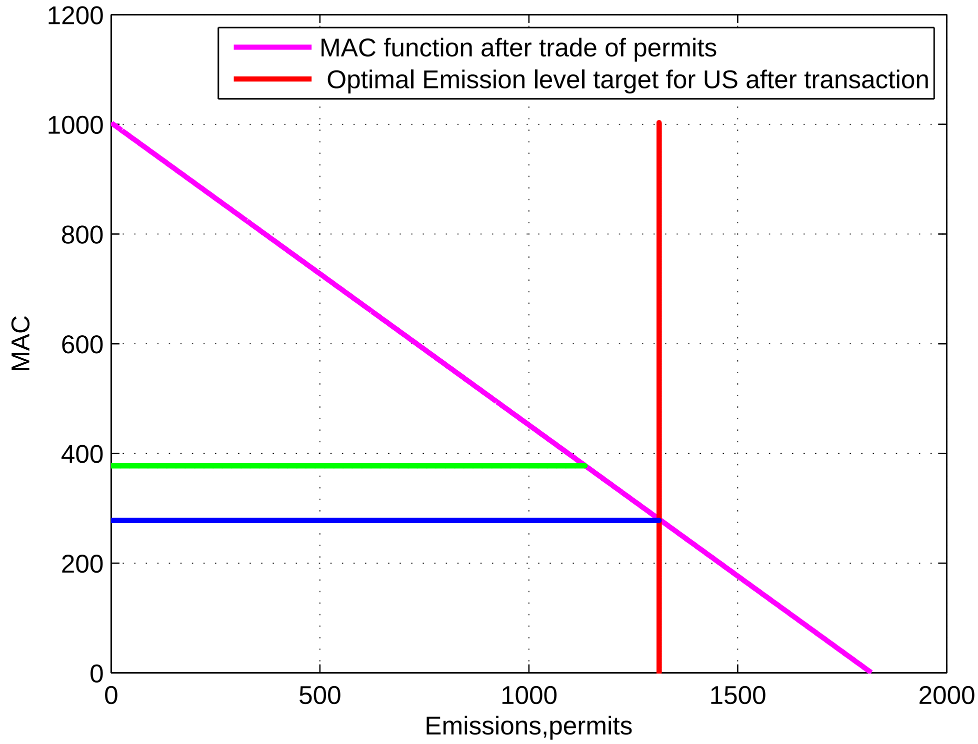

Figure 16: Marginal Abatement Costs (MACs) against Emissions for US for situations when no permit trade and when permit trade is allowed with reference to data from tables 2 and 3

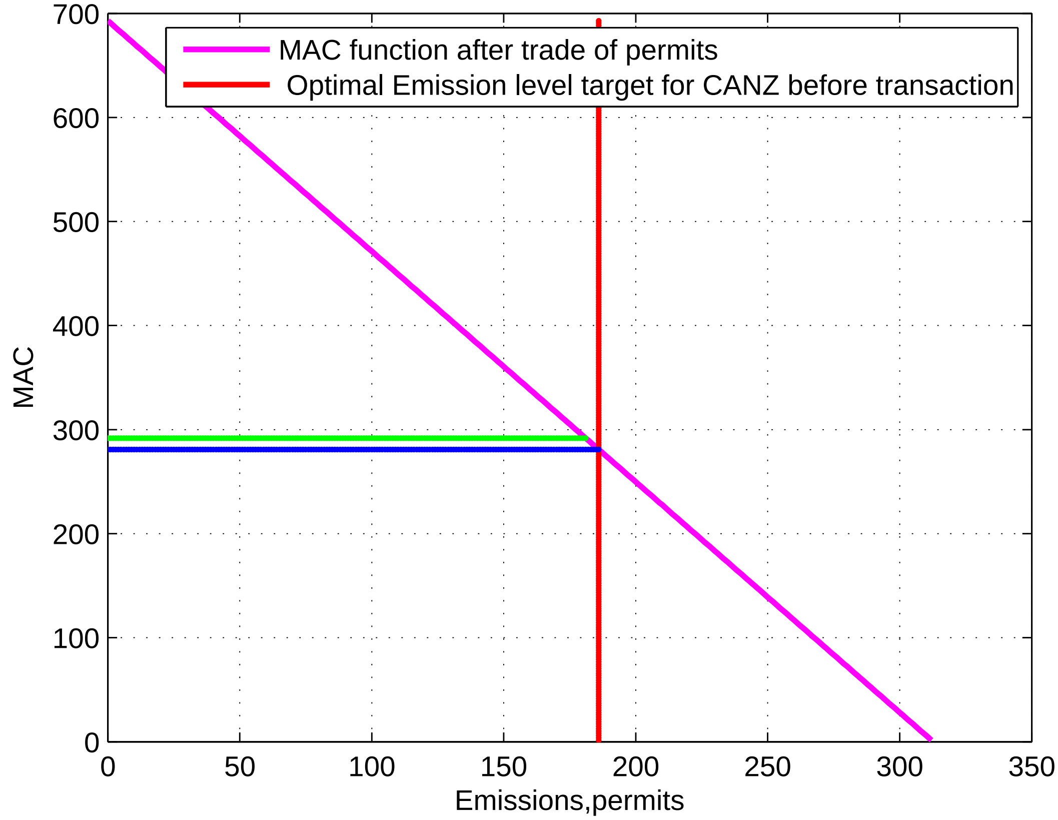

Figure 17: Marginal Abatement Costs (MACs) against Emissions for CANZ for situations when no permit trade and when permit trade is allowed with reference to data from tables 2 and 3

The relationship of all three variables,i.e. Emissions, uncertainty and total costs for reducing both emissions an d relative uncertainty can be shown in 3-D surface of figure 18

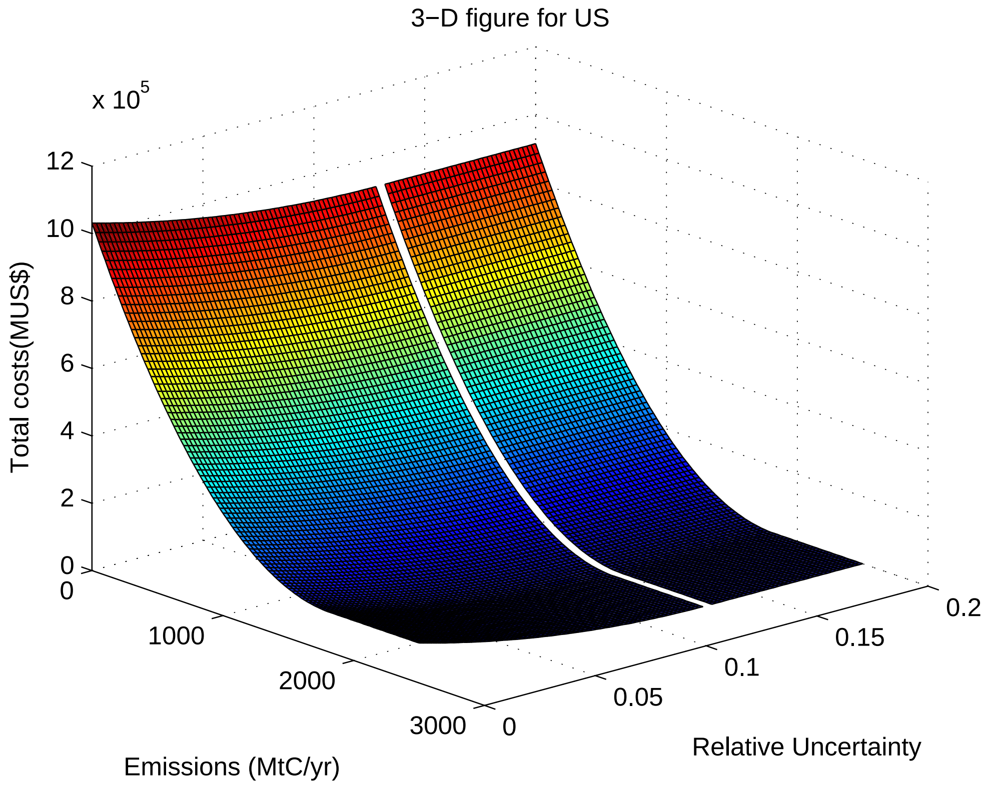

Figure 18: 3-D surface shown how Emissions, uncertainty and total cost for reducing emission are related in US region with reference to data from table 1

It is clearly noticed that as total costs for cutting down both emissions and relative uncertainty increase then volume of emissions and relative uncertainty decrease.

6 Stochastic Emission Uncertainty

Emission data in this paper are reported over a certain period of time, i.e, yearly by committing small error. This kind of error can not be neglected for better results of modeling, that is why we have to switch from deterministic of assessing emission uncertainty to probabilistic one. In this section we are going to analyze emission uncertainty in stochastic framework.

Let us assume that optimal emission levels, x_i^*, reported in table 2 are estimated averages of emissions taken over a year for each Party or Region. Relative uncertainties, R_i^*, are fractions of unreported emissions due to emission factors, emission source and lack of general knowledge about emission generating process and so on. The actual emission will be given by:

$$X = x_i^* + x_i^* R_i \tag{6.1}$$

Suppose that emission level, X is the random variable and is normally distributed with mean, $E(X) = \mu$ and $Var(X) = \sigma^2$, where by X is real emission levels.

X can be generated by considering the relationship:

$$X = \mu_i + \xi_i \tag{6.2}$$

where by error in emission ξ_i is normally distributed with mean 0 and variance, $Var(\xi_i) = \sigma^2$, i.e $\xi_i \sim N(0, \sigma^2)$. In this case for each Region or Party, i, the newly generated emission levels will be:

$$X_i = \mu_i + N(0, \sigma^2), \text{for} \quad i = 1, ..., 5. \tag{6.3}$$

For better result we are going to simulate emission levels by using the random generator r from normal distribution with sample size of one million (1000000) defined by:

$$r = \mu_i + \sigma * randn(nsample, 1) \tag{6.4}$$

where r will be newly generated emission level, μ_i is estimated mean or average of emission levels and σ is its corresponding standard deviation.

After the random generation of one million sample size of emission levels and having calculating the standard deviation, σ from the relationship:

$$Z = \frac{X - \mu}{\sigma} \qquad (6.5)$$

where by $Z = 1.645$ so for probability that the actual emissions in table 2 should not exceed Kyoto target, i.e $\mathcal{P}(X_i \leq K_i)$ is brought to 95%. Then the histograms of each Party or Regions are drawn as follows:

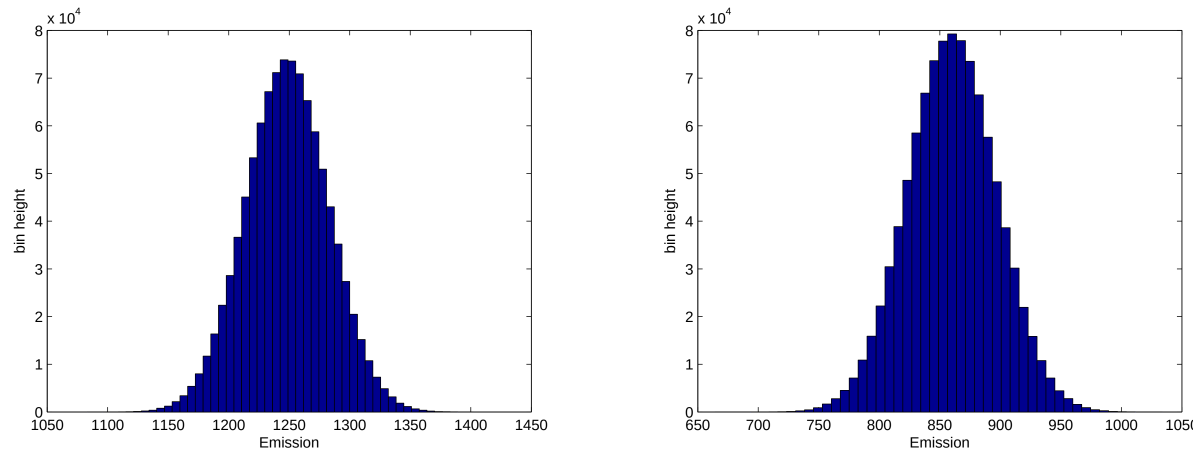

Figure 19: The histogram showing normal distribution for emissions in US and OECDE

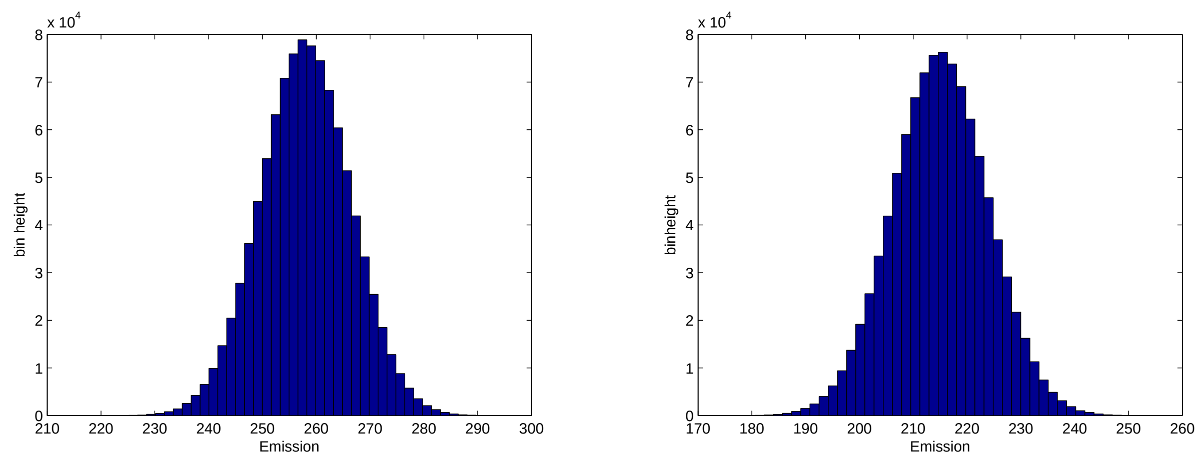

Figure 20: The histogram showing normal distribution for emissions in JAPAN and CANZ

The standard deviation, σ, of each region was calculated by using Equation 6.5 and we found that $\sigma = 34.5, 37.0, 8.4, 8.9$ and 98.1 for US, OECDE, Japan, CANZ

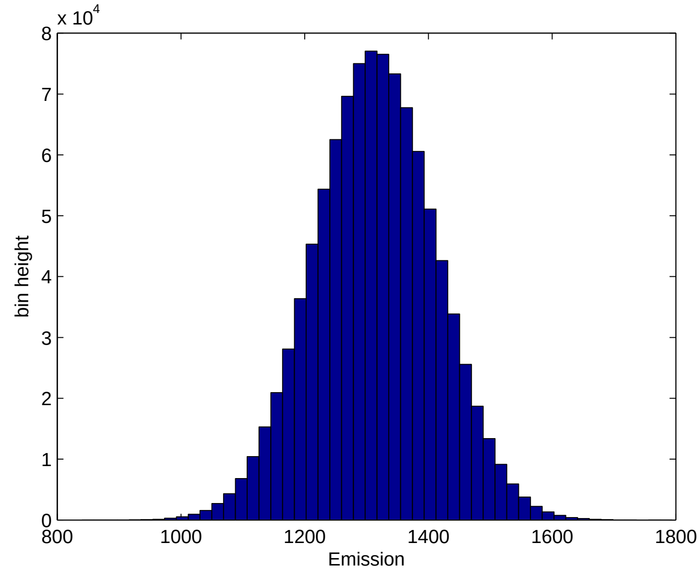

Figure 21: The histogram showing normal distribution for emissions in EEFSU

and EEFSU respectively. EEFSU has widest distribution of all, having the highest measure of variability, $\sigma = 98.1$. This is due to the fact that its uncertainty emission relative to the reported one is too high, i.e. its $R_i^* = 0.3257$, that means about 33% of emission, $x_i^* = 991.2MtC/yr$ is not reported (refer table 2). Japan has least measure of dispersion, i.e. $\sigma = 8.4$. The data points are close to the mean emission ,x_i^* and its uncertainty emission, $R_i^* = 0.1199$ is too small compared to reported emission, which is about 12% of the reported emission.

However, we can assume that total emissions, $\sum X_i$ is normally distributed with mean, $\sum \mu_i$ and variance, $\sum \sigma_i^2$, i.e. $\sum X_i \sim N\left(\sum \mu_i, \sum \sigma_i^2\right)$.

The probability that the total emissions in table 2 do not exceed the total Kyoto target, i.e. $\mathcal{P}\left(\sum_{i=1}^{5} X_i < \sum_{i=1}^{5} K_i\right)$ in table 1 is brought to 95% and it is shown in the following histogram:

We are now interested to shift our focus to theoretical approach of how to introduce stochastic modeling and probability distribution to describe uncertainty. Let us assume that the aggregate (total) emission, X denoted by $\sum_{i=1}^{5} X_i$ is normally distributed with mean, μ_i and variance, $Var(X) = {\sigma_i}^2$, i.e. $X_i \sim N(\mu_i, \sigma_i^2)$. By using Equation (6.5) we can build the following relationship:

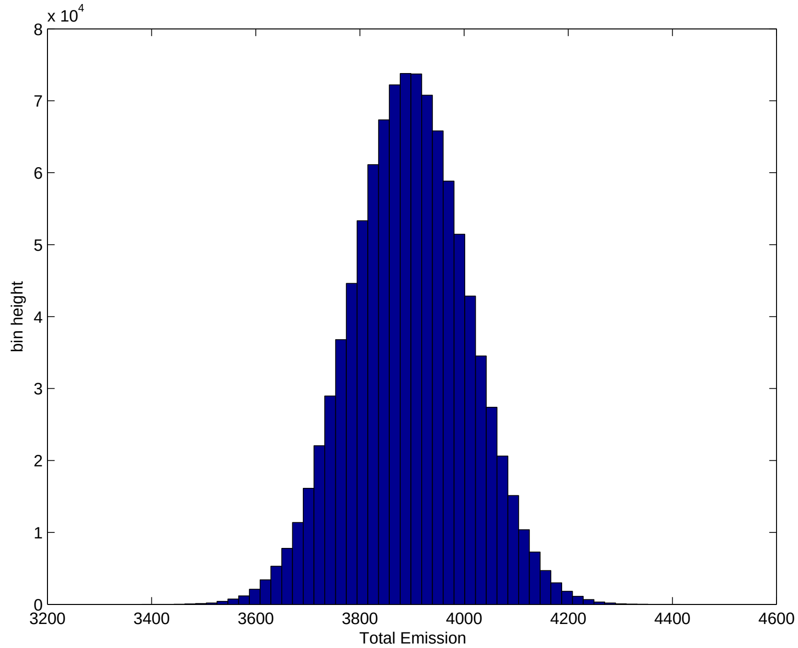

Figure 22: The histogram showing normal distribution for total emissions in all Regions described in table 2

Let us call the upper α-level of emissions, E_α, for Kyoto regions, $i = 1, ..., 5$ is given by:

$$E_\alpha = \mu(X) + Z_\alpha \sigma(X) \tag{6.6}$$

$$= \sum_{i=1}^{5} \mu_i + Z_\alpha \sqrt{\sum_{i=1}^{5} \sigma_i^2} \tag{6.7}$$

A cost function for keeping emission levels and uncertainty levels at $\{\mu_i, \sigma_i\}$ can be formulated as follows:

$$W(\mu, \sigma) = \sum_{i=1}^{5} \left[\alpha_i(\mu_i - a_i)^2 + \beta_i/\sigma_i^2\right] \tag{6.8}$$

Now let t be a target level for total emissions which is strictly bigger than total Kyoto target, i.e. $t > \sum_{i=1}^{5} K_i$, then we define our optimization task as follows:

We are required to find values for $\{\mu_i, \sigma_i\}$ that minimize the cost of bringing the upper α-level (for instance 95% level) of emissions down to the target value t.
This leads to the following optimization task:

$$\min_{\mu_i,\sigma_i} \quad \sum_{i=1}^{5} \left[\alpha_i(\mu_i - a_i)^2 + \beta_i/\sigma_i^2\right]$$

$$\text{s.t} \ \sum_{i=1}^{5} \mu_i + Z_\alpha \sqrt{\sum_{i=1}^{5} \sigma_i^2} < t$$

We suggest this approach to be the best way to introduce stochastic modeling and probability distributions to describe the uncertainty.

The optimum solution naturally depends on the chosen target level t and the α-level. Using other non-gaussian distribution to model the uncertainty would also give variants to this method.

7 Conclusion and Future work

The challenges posed by global warming effects have motivated most of researchers to turn around it, so as to get the way to alleviate them.

In this Master's Thesis, the model for solving optimal level of emissions and relative uncertainty was well developed. It is convex, continuous and differentiable at every point. The smoothness of this model makes it easy for classical optimization methods to give best results [17]. In the part of analysis, the data extracted from [3] (confer table 1) were useful and consistent in describing some interesting characteristics in the context of carbon trading as shown in figure 8.

Exact and heuristic optimization methods have been used for solving optimal values and analyzing real and practical situations in carbon trading context. These methods are Lagrangian and Differential Evolution (DE) respectively. The results were presented in table 2- 5. MATLAB software was used to plot features of our interest such as parameter values showing in figure 6 and 7. The relationship (correlation) between parameter values and objective function values were shown in figure 8 and 9.

For the sake to avoid neglecting errors in reported emissions, stochastic emission uncertainty were explicitly analyzed in stochastic framework. Emission, X was randomly generated by assuming that it follows the gaussian (normal) distribution with mean, μ and variance, $Var(X) = \sigma^2$. Errors, ξ_i in reported emission were normally distributed with mean, $\mu = 0$ and variance, $Var(\xi_i) = \sigma^2$. Results and interpretations were presented in figure 19 up to figure 22.

In spite of the fact that we got some fruitful results, this work should be taken as a base for future researches. Data used in this thesis were too deterministic , so we recommend in future work, the application of Time series data, i.e. carbon permit prices collected daily for past five years to the following areas:

1. Relationship between electricity and carbon markets in analyzing equilibrium prices of the two markets, seasonality, mean reversion, volatility and jumps (spikes). Analysis of data can be extended to examine the effects of recent global economic crisis to both markets.

2. Econometric analysis of carbon prices using regression models, ARMA (Auto Regressive Moving Average), ARCH (Auto Regressive Conditional Heteroscedasticity), GARCH (Generalized Auto Regressive Conditional Heteroscedasticity) and statistical tool, MCMC (Markov Chain Monte Carlo) which includes random walk.

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Appendix: The MATLAB Codes for SQP results presented in table 4

```
% Function files
%1.Function file for non-linear constraints

function [c,ceq] = NonLinearConstraints(X,data)

x   = X(1:5);
y   = X(6:10);
R   = X(11:15);
c   = x.*(1+R)-y-data.K';
ceq = [];

%2.Function file for objective function value

function out = ObjfFunCon(X, data)

x   = X(1:5)';
y   = X(6:10)';
R   = X(11:15)';

out = sum( data.b.*(x - data.a).^2 + data.d.*(R - data.R0).^2 );
%%Working file
clc; clear all; close all
disp('Solution using FminCon: READ MATLAB HELP FOR REFERENCES');

%define data
data.K  = [1251.0     860.0    258.0    215.0   1314.0    ];
data.R0 = [   0.13      0.20     0.15     0.20     0.30   ];
data.a  = [1820.3    1038.0    350.0    312.7    898.6    ];
data.b  = [   0.2755    0.9065   2.4665   1.1080   0.7845 ];
data.d  = (data.a.^2.*data.b)./data.R0;

%define options for optimizer
optimset('fmincon');
options.MaxFunEvals      = 1e6;
options.MaxIter          = 1e6;
```

```
options.ShowStatusWindow = 'on';

%%this turns the medium scale on.
options = optimset('LargeScale','off');
options.Display = 'iter';

%%initialize
init = 10*rand(3,length(data.K)); init = init(:);

%%sum constraint ( sum(y)=0 )
Aeq  = [0 0 0 0 0 1 1 1 1 1 0 0 0 0 0]; %matrix for sum constraint
beq  = 0; %sum must be zero

%%bounds [x1 x2 x3 x4 x5  y1   y1   y3   y4   y5  R1 R2 R3 R4 R5]
%%lower bound
lb    = [0  0  0  0  0 -Inf -Inf -Inf -Inf -Inf  0  0  0  0  0];
%%upper bound
ub    = [data.a         Inf  Inf  Inf  Inf  Inf  data.R0];

%%call the optimizer
[out cost]  =  fmincon(@(X) ObjFunCon(X,data),init,[],[],Aeq,beq,lb,ub,...
 @(X) NonLinearConstraints(X,data),options);

%%give out results

test = out;
test(6:10) = round(test(6:10));
disp('    x         y         R')
disp(vpa([out(1:5) out(6:10) out(11:15)],5))
out = vpa([out(1:5) out(6:10) out(11:15)],5);
disp(' sum of y')
disp(sum(out(:,2)))
%val = ObjFunCon(test,data)
```