ABSTRACT

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Ionsonde Measurement Trend

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Time series analysis has gone through different developmental stages before the current modern approaches. These can broadly categorized as the classical time series analysis and modern time series analysis approach. In the classical one, the basic target of the analysis is to describe the major behaviour of the series without necessarily dealing with the underlying structures. On the contrary, the modern approaches strives to summarize the behaviour of the series going through its underlying structure so that the series can be represented explicitly. In other words, such approach of time series analysis tries to study the series structurally. The components of the series that make up the observation such as the trend, seasonality, regression and disturbance terms are modelled explicitly before putting everything together in to a single state space model which give the natural interpretation of the series.

The target of this diploma work is to practically apply the modern approach of time series analysis known as the state space approach, more specifically, the dynamic linear model, to make trend analysis over Ionosonde measurement data. The data is time series of the peak height of F2 layer symbolized by hmF2 which is the height of high electron density. In addition, the work also targets to investigate the connection between solar activity and the peak height of F2 layer.

Based on the result found, the peak height of the F2 layer has shown a decrease during the observation period and also shows a nonlinear positive correlation with solar activity.

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Lappeenranta, November 2, 2015
Dereje Dejene

Contents

List of Symbols and Abbreviations

hmF2	Ionospheric F2 peak height
DLM	Dynamic Linear Model
LSQ	Least Square
MCMC	Markov Chain Monte Carlo
DLM	Dynamic Linear Model

1 INTRODUCTION

In order to get deeper understanding of a certain phenomenon, there are two possible ways to proceed. The first approach is to consider the fundamental processes that are acting and build a model based on the process which is used to make predictions. The other way to go is to analyze available data so as to either find relationships that describe how the system works or make test hypotheses. Such an empirical approach is used in trend analysis, quantifying and explaining changes in the system over a period of time. The statistical tools to do so range from simple to very advanced one. Some of them are: the least squares method, the moving average method, the free hand method and state space method. Among these methods, the state space model approach in general and the dynamic linear models in particular is the focus point and will be discussed in detail.

State space models are models governed by set of equations known as state equation (model equation) that describe the time evolution of the state of the system and observation equation (measurement equation), that tells the connection between the observation and the state of the system underlying the process. This model nowadays is found to be very useful and widely applied in science and engineering disciplines and is also well documented. One of the applications of this model is in time series problems. The fundamental and astonishing quality of the approach in this regard is that, it allows the problem to be analyzed structurally. That is, the components of the series that make up the observation like, trend, seasonality and regression and disturbance terms are modeled explicitly before putting everything together in to a single state space model. This allows the natural interpretation of the series. The approach has also a wider privilege to the effective forecasting and estimation algorithms. In addition, the method can effectively handle a wide range of time series problems and is so flexible than the current analytical time series methods in use such as the Box-Jenkins approach.

To be more specific, the approach together with dynamic regression, strives to minimize the usual problem happening in statistical time series analysis, having a single realization from a not completely understood system that forces one to make assumptions like stationarity in some of the distributional properties of the underlying process responsible for the variability, to do some analysis on the series. This problem can be resolved making regression coefficients vary in time, so that system properties are dynamic, varying in time that in turn create the possibility of analyzing and describing smooth changes of the underlying process behavior. Such

an approach is more appropriate to handle the basic non-stationary time series property attributed to many environmental time series because of external forces that let distributional properties to slowly or suddenly change are there adhered with affecting physical system. The approach better explains the variability and thus avoid correlation of model residuals that could happen as a result of not doing so. For example, one can model the process responsible for the observed variability of seasonality using state allowing some model error to exist [1].

All these ideas suggest, the dynamic approach more advantageous than the static, where regression coefficients stays unchanged in time. Along this, the possibility of making state space representation of dynamic regression together with the fact that defining the process sequentially on conditional dependence just only on previous time step, the kalman filter can be used for estimating the states given observation [2].

The approach has many practical advantages such as to study trend, the change in the statistical properties of underlying process. The focus point of this Diploma work is basically to discuss time series application of state space models more specifically the dynamic linear model, state space model where the operators defining the system equation are linear, and apply it for Ionosonde measurement trend analysis.

2 SOLAR ACTIVITY AND IONOSPHERE

2.1 Solar Activity

The sun, the main source of energy for the planet earth is not quiet at all. There is always different non stationary active processes happening with in it. Broadly speaking, solar activity is such kind of a non stationary usually eruptive process occurring in the sun. In other words, Solar activity is any type of variation in the appearance of energy output of the sun. This output variation occurs in all of its output forms such as light, energetic particles, and it varies in time ranging seconds to centuries and position of the sun. The energy output of the sun basically has two forms, charged particles emission and radiation of electromagnetic waves. Solar cycles, the change of activity in the sun on periodic basis, sunspot (the disturbance of sun's photosphere temporarily), coronal mass ejections (massive ejections from the sun), solar flares (huge explosion happening in the solar atmosphere), coronal holes (cooler and less dense areas relative to the surrounding covering the corona largely) and solar plumes, (feathery jets covering 13 million miles into space emanating from around the pole of the sun) are some of the types of solar activity. Details can be found at [3].

There are different ways of representing solar activity called proxies (indices). Roughly,these indices can be categorized as physical indices or as synthetic indices based on the way they are calculated. The physical proxies are the one which represent the observable, real physical quantity that can be measured. Radio flux emission rate of the sun at a wave length 10.7 cm is one way of representing solar activity lying in the physical indices stream. On the contrary, sunspot numbers are considered as synthesised proxy.

2.2 Layers of Ionosphere

Ionosphere is the layer of the earth's atmosphere that contains a high concentration of ions and free electrons and is able to reflect radio waves. It exists approximately between 50km and 600km. This region consists different layers that are characterized by their own electron density known as D layer, E layer and F layer. The D-layer is found between the altitudes of 50km to 90 km. In this layer, most of Hf signal lost their strength. The E- layer exists above D-layer between 90km and 120km. F-layer

is found above the E layer and has unique property, being dividable further in to sub layer. Figure 1 tries to give some picture about these layers during the day and night.

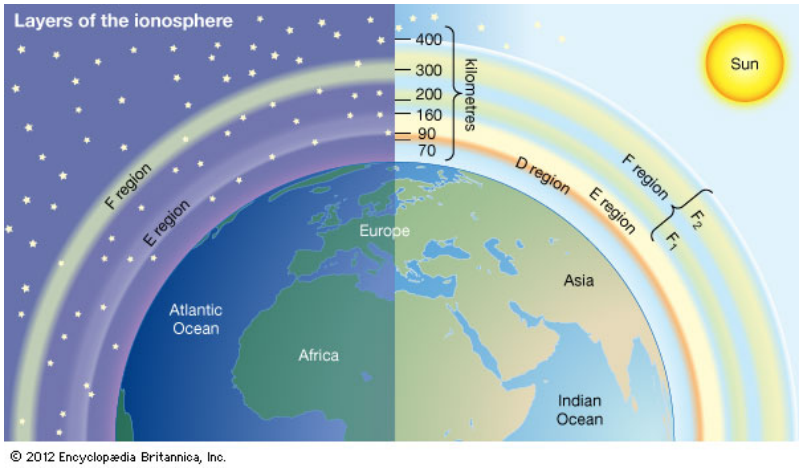

Figure 1: Layers of ionosphere

2.3 Ionosonde

The idea of ionospheric sounding was suggested in 1924 by Briet and Tuve. It uses refractive properties of the ionosphere. The Ionosonde is in principle a high frequency (HF) radar that record the time of flight of a transmitted signal as a measure of ionospheric reflection height. The frequencies used for this purpose runs from 0.5 to 20MHz. The ionosonde gives a record of the reflection height as a function of frequency called an Ionogram. In other words, the ionogram is a trace record of reflected high frequency radio signals that the ionosode generating. The signal overcomes the noises from commercial radio sources as frequency of the sounder sweeps from lower to higher magnitude and gets the signal reflected from the layers of the ionosphere and is recorded that form characteristic patterns of traces comprising the ionogram. The signal travels slowly in the ionosphere as compared to free space, it is therefore a virtual height, not the true height that would be recorded. characteristic values of virtual heights is symbolized by h'E, h'F, h'F2 and so on and critical frequencies,the highest frequency above which the waves penetrates the ionosphere and below it which the waves gets are reflected from the ionosphere, is designated by foE, foF1, and foF2, and so forth. The aforementioned concepts can be shown in figure 2. The Ionogram can be used for different purposes such as for finding electron density distribution as a function of height.

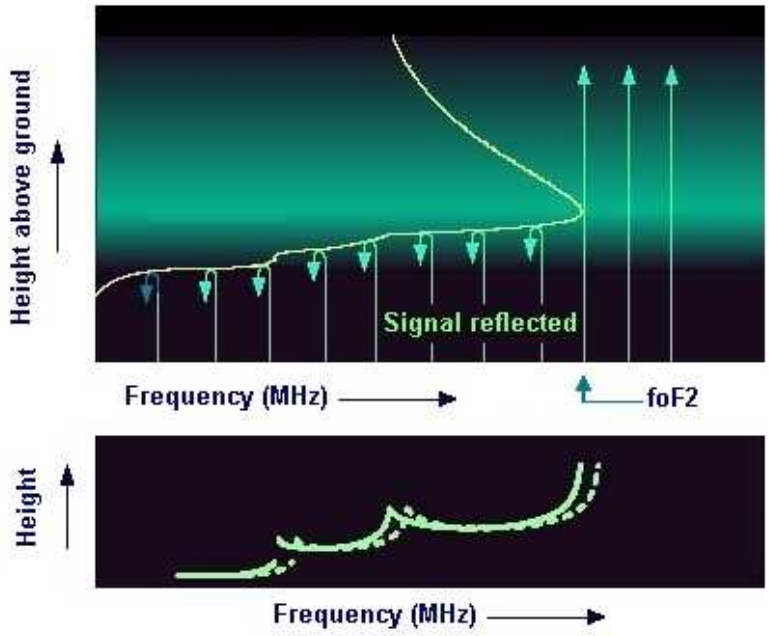

Figure 2: Ionosonde's signal reflection from the ionosphere and Ionogram

2.4 Brief History of the Development of Ionosonde at Sodankyla

The history of Sodankyla Ionosonde development started in 1957. The sounding has been performed in half-hourly at a regular basis. And followed by the second ionospheric sounder, IS-14 type later modified in 1977 sounder vertical soundings of the ionosphere on 1st August 1957. Currently, a new era of ionospheric vertical soundings has been being carried out since 16^{th} of November to early April 2007 on 10 minute basis. A one minute sounding has also been carried out during campaigns such as IPY (International polar year).

3 Time Series

A time series is sequence of measurements, observation of the momentary value of the variable in question ordered in time usually at equal intervals. It Mostly cover spans of days up to thousands of years. Temperatures and densities of the plasma, magnetic or electric field vectors can be examples of time series. Mathematically, a time series is defined by the values $Y_1, Y_2, \cdots$ of a variable Y say temperature or density at times $t_1, t_2, \cdots$. Some common features of a time series data includes trend, seasonality or autocorrelation (dependence between successive observations) and so on. The basic model for representing a time series is the additive model given by Equation (2)

$$Y_t = \mu_t + \gamma_t + \varepsilon_t, t = 1, \cdots, n. \tag{1}$$

where, μ_t is a slowly varying component, trend, γ_t is a periodic component of fixed period called the seasonal and ε_t is an irregular component, called the error or disturbance.
A multiplicative approach do also exist in many applications such as in economics. That is,

$$Y_t = \mu_t \gamma_t \varepsilon_t, t = 1, \cdots, n. \tag{2}$$

3.1 The Importance of Time of Series Analysis

Basically, time series analysis is done to find a model describing the pattern the feature, forecasting. Some of the important things to note at the beginning in time series analysis is, to check weather seasonality,(repetition of a pattern in certain fixed period of time), trend, outliers, abrupt changes or long term cycle, constant variance or not via plotting the data. Analysing the graph of the time series plot has to not be void even if there are sophisticated ways for drawing the graph depicts weather the aforementioned characters tics do exist or not in the time series.

3.2 Residual Analysis

A residual is the discrepancy between what is actually observed, say y and the value predicted by the model, $\hat{y}$. It is an important tool to asses weather the selected model is appropriate or not. It can be calculated by differencing these values. That is,

Residual $= y - \hat{y} = \mu$.
In addition, $\sum \mu = 0$ and $\bar{\mu} = 0$
Then with time series plot of the residuals, plot made by making the residual on the y-axis and the independent variable on the x axis, checking the appropriateness of the model can be examined. Adequate models shows a scatter of points on the residual plots about zero with no systematic pattern, randomized around zero as like shown in the Figure 3

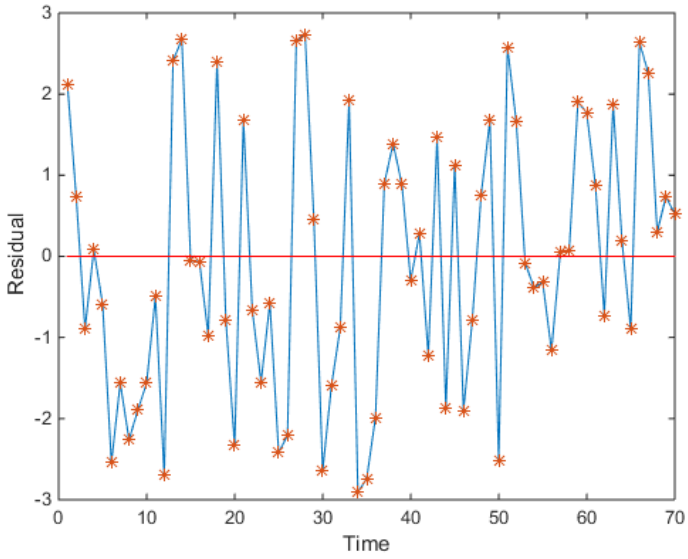

Figure 3: Residual plots

3.3 Time Series Movement Classifications

1. **Long Term Movements:** This movement refers to the general tendency that the time series graph is traversing over long period of time. Such movement is called **secular variation** or **secular trend**. It can be shown by a trend curve or trend line.

2. **Cyclical Movements:** Cyclical movements are long term oscillations or swings about a trend curve or line. The cycles may or may not follow exactly identical patterns after equal interval of time. Business cycles that shows period of wealth, recession, depression and recovery and solar cycles can be examples of this kind of movement.

3. **Seasonal Movements:** This kind of movements refers to almost identical patterns that a time series follow in corresponding times of successive years. This happens as a result of recurring events happening annually, or with periodicity over any interval of time.

4. **Random Movements:** This type of time series movement is spontaneous, erratic motion of time series because of chance events.t short time. Figure 4 summarizes the four kind of time series movements explained above.

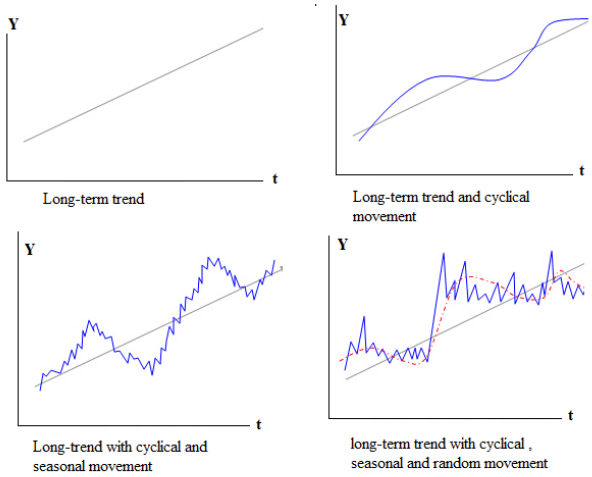

Figure 4: Types of time series movements

3.4 Time Series Analysis Approaches

To go more than giving a simple description about the time series, the time series usually demands the introduction of statistical model. For it complies with requirements that are convenient, specifying the model is arbitrary. The error model and stochastic model types are briefly described as follows

1. **The Error Model:** Here, the time series is defined by certain mathematical function explicitly plus a random error as:

$$\mathrm{x}_t = f(t) + \varepsilon_t,$$

where,
$E(\varepsilon_t) = 0$; $E(\varepsilon_t^2) = \delta^2 < +\infty$; $E(\varepsilon_t \varepsilon_s) = 0, \forall t \neq s$.

Such a model suits for phenomena occurring regularly.

The Stochastic model: In such a case, the time series is defined as a function of stochastic variables:
$x_t = y(\varepsilon_t, \varepsilon_{t-1}, \cdots)$.

3.4.1 Traditional (Classical) Method of Time Series Analysis

The traditional way time series analysis is based on concept of error model explained above and the series is assumed to be a combination of the components, unobserved factors such as the trend, seasonality, irregular fluctuation. This approach is basically targets decomposing the time series in to its components either in additive fashion or multiplicative type so that the function f(t) can be approximated.
Some of the advantages of the classical time series analysis approach includes, being exploitable in spite of the series's length, having an intuitive concept basement and so on. However, it has also disadvantages such as not having unique decomposition and having a stochastic term in the error only.

3.4.2 Modern Time Series Analysis Approach

Here, the series is assumed to be a definite realization of a stochastic process. The approach tries to work over the process mechanism responsible for the generated observation and build a model accordingly. The upcoming sections, section 5 and 6 describes one of the modern approach called the state space approach more specifically the dynamic linear model approach in detail.

3.5 Trend Analysis and its Importance

Trend can be loosely defined as investigation of changes in a system over a period of time. The use of mathematical methods, however, needs an exact expression for the scientific question of interest at hand using numerical terms. Thus, considering a collection of values (data) for variables over time such as that describe the system behaviour is required. The available data for analysis might be a sequence of a regularly spaced observation of a single variable at equal intervals of time: $Y_1, Y_2, \cdots, Y_t$

is the simplest case, and the trend could be defined as "the change in the mean level in the series". Kendall and Ord (1990). Making trend analysis has many advantages. These include, for explaining past behaviours in a process, such as quantifying the change, to create understanding that drives the change, to assessing the possible future picture via extrapolation, for environmental monitory policy efficiencies.

3.6 Classical Trend Estimation Methods

1. **The Least Square Method(LSQ):** Trend analysis by least square method is a matter of finding an appropriate trend line or trend curve that best fits the data by least square regression. A measure of the goodness of the fit of the curve is being the sum of the square of the residual, the difference between the data and the corresponding value determined from the curve, model value.

2. **Moving Average Method:** An appropriate order of moving average, one can remove seasonal, cyclical and irregular patterns and left with trend movement.This method has limitation in that the data at the beginning and end is lost

3. **The Free Hand Method:**This method is simply fitting a trend line or curve just by looking the graph.

4. **The Semi-average Method:** this method by dividing the data in to two equal parts and taking their respective average so that a trend line can be drawn through the two points found from averaging.

3.7 hmF2 Time Series Model

The F2 layer peak height (hmF2) time series can be fitted to a model that consists a linear long-term trend and periodic variation. This multi-parameter hmF2 time series model might be formulated as:

$$hmF2(t) = f(t) + g(t), \tag{3}$$

where $f(t) = X_1 + X_2 t$ and

$$g(t) = X_3 F_{10.7}(t) + X_4 A_p(t) + X_5 \cos(\omega_a t) + X_6 \sin(\omega_a t) + X_7 \cos(\omega_s t).$$

$X_1 ... X_s$ are the parameters to be fitted. The variables used are:

$F_{10.7}$ Solar activity,

A_p Geomagnetic activity,

ω_a Annual variation,

ω_s Semi annual variation.

The parameters are estimated using least square(LSQ)technique. Figure **??** shows its result.

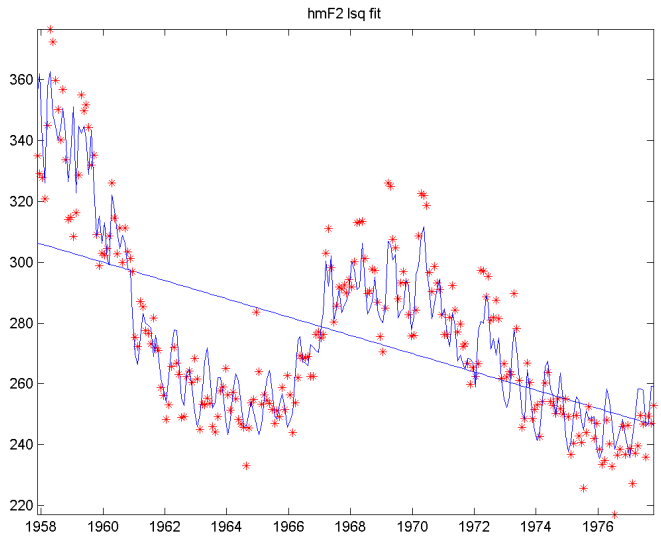

Figure 5: hmF2 trend obtained by LSQ

4 Time Series Analysis by State Space approach

4.1 Introduction

Classical time series analysis are mainly descriptive that tries to be means of summarizing the behaviour of a series without necessarily wondering its underlying structure. On the contrary,time series analysis using state space modelling approach tries to represent these structures explicitly. This modelling approach gives a unified for handling a wide range time series analysis problems. The method is basically based on an assumption that the development over time of the system under study is determined by an unobserved series of vectors $X_1, ..., X_n$, with which are associated a series of observations $Y_1, ..., Y_n$. where the connection of X_t and the Y_t is specified by the state space model. Mainly the state space analysis is to infer the relevant properties of the X_t from a knowledge of the observations $X_1, ..., X_n$.

4.2 State Space Models

State-space models are models based on an assumption that an observation is noisy function of certain unobservable variables that underlie the process called state. The state might be seen as some physical system that give the output in engineering sector or like a random auxiliary process that help to specify the probability law of of the observation. To illustrate this, consider for example a time series Y_t where, $t = 1, 2, 3....$ and Y_t is a random vector of the observable. specifying the probability law of the process, to give the dependence structure among the variables of the Y_t's is required in order to make inference on the time series specially for predicting Y_{t+1} when observations are given. Figure 6 summarizes the overall idea.

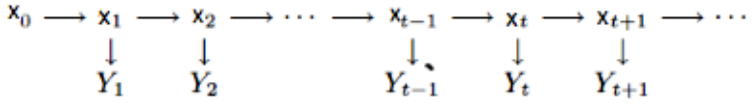

Figure 6: structure that shows the existing dependence among the the variables in state-space model with X_t representing the states and Y_t is the observations

State space models bases two assumptions. The first one is, the X_t where $t = 0, 1, 2...$

is a Markov chain, there is no dependency on history, what is observed previously. That means X_t depends only on X_{t-1} and therefore the process,$(X_t, t = 0, 1, 2, \cdots)$ probability law can be specified by giving the initial density p_0 for X_0 and $p(X_t \mid X_{t-1})$, transition$X_1, \cdots, X_n$ have joint conditional density $\prod f(Y_t \mid X_t)$ for any $n \geq 1$

These assumption together with relevant densities specified enables to write the random process probability law, $(X_t, Y_t), t = 0, 1, 2, \cdots$ which create the possibility of deducing the all the variable dependence among them.

For any $n \geq 1$,

$$\begin{aligned}(X_0, X_1, \cdots, X_n, Y_1, \cdots, Y_n) &\sim p_0(X_0) \prod_{t=1}^{n} p(X_t, Y_t \mid X_0, X_1, \cdots, X_{t-1}, Y_1, \cdots Y_{t-1}) \\ &= p_0(X_0) \prod_{t=1}^{n} f(Y_t \mid X_0 \cdots X_t, Y_1, \cdots, Y_{t-1}) p(X_t, \cdots X_{t-1}, Y_1, \cdots, Y_{t-1}) \\ &= p_0(X_0) \prod_{t=1}^{n} f(Y_t \mid X_t) p(X_t \mid X_{t-1})\end{aligned}$$

By integrating out all X variables from the joint density given above, one can get the density $Y_1, \cdots, Y_n$ but generally the density $Y_1, \cdots, Y_n$ does not exist in a closed form.

4.3 Dynamic Linear Model(DLM)

Dynamical linear model is a dynamic regression analysis based on state space, It is a special case of general state space model, when the operators involved on the system are linear. It is defined by two sets of equation called the observation equation and the model equation as shown in Equation (4).

$$\begin{aligned} Y_t &= F_t X_t + v_t, \qquad V_t \sim N(0, V_t), \\ X_t &= G_t X_{t-1} + w_t, \qquad W_t \sim N(0, W_t). \end{aligned} \tag{4}$$

.

where the first and the second respectively represent the observation equation and model state equation. F_t and G_t are known matrices called observation operator and model operator. In addition, V_t and w_t represent observation and model error and are independent between them and between time steps. That is, $W_t \perp V_t$ and $W_t \perp W_k$, for $t \neq k$.

Dynamic linear models can also be seen as hierarchical statistical model involving

three levels, data, process, and parameter. In terms of statistical distributions, the observations uncertainty $p(y_t \mid x_t, \theta)$ described by the observation equation, the process uncertainty of the unknown states x_t and their evolution given by the process equations as $p(x_t \mid \theta)$, and lastly, the uncertainty related to model parameters $p(\theta)$. These conditional formulations provides efficient description of the system and computational tools to estimate its components [5]. Many environmental and time series problems can be solved by such formulation as it is flexible and general. The upcoming sections will go through the details of the approach for a time series problem.

4.4 Important DLM's for Time Series Application

4.4.1 Local level model

Assume a series y_t with a trend but no cyclical or seasonal variation. The series can be represented as:

$$Y_t = \mu_t + \varepsilon_t. \tag{5}$$

where μ_t is the underlying general level (signal) at time t and ε_t is the noise, random variation. If an assumption is made to ε_t normally distributed and μ_t is random and not changing significantly overtime instead of being deterministic, one can produce a more specific model. This can be achieved by modelling μ_t using the random walk concept. That is μ_t will be a scalar series governed by $\mu_{t+1} = \mu_t + \eta_t$ where $\eta_t's$ are independent and identically distributed random variable with a mean of zero and variance ${\delta_r}^2$. This together with Equation (23) forms a model called local level model also known as random walk plus noise model as shown in Equation (6.)

$$\begin{aligned} Y_t &= \mu_t + \varepsilon_t, \qquad \sim N(0, {\delta_\epsilon}^2) \\ \mu_{t+1} &= \mu_t + \eta_t, \qquad \sim N(0, {\delta_\eta}^2) \end{aligned} \tag{6}$$

where, ε_t's and η_t's are uncorrelated for $t = 1, ..., n$.

The model model is used to represent a series that has no trend or seasonal variations but its level varies in time. In addition to this, even if this model looks simple, it plays an important role in forming the basis for treating time series problems in practice.

4.4.2 Local Linear Trend Model

The local linear trend model can be obtained by extending the local level model in such a way that there is a trend with a slope ν_t that both are allowed to change in time. This model can be given by

$$\begin{aligned} Y_t &= \mu_t + \varepsilon_t \qquad \sim N(0, {\delta_\zeta}^2) \\ \mu_{t+1} &= \mu_t + \nu_t + \xi_t \qquad \sim N(0, \delta_\varepsilon^2) \\ \nu_{t+1} &= \nu_t + \zeta_t \qquad \sim N(0, \delta_\epsilon^2) \end{aligned} \tag{7}$$

where the noises ξ_t, ζ_t and ε_t are mutually uncorrelated with mean all zero and variances δ_ε^2, δ_ϵ^2 and ${\delta_\zeta}^2$ respectively. In this model, using the respective disturbance terms with variances greater than zero one can vary the slope and level to change in time.

4.5 Structural Time Series Model

Structural time series model is a model in which the trend, seasonal and error terms and other components, are modeled explicitly. Despite the classical time series methods briefed in sections 3.2 that describe the general behaviour of the time series without necessarily considering its underlying structure, this approach provides a means to summarize the behaviour of series with its underlying structure that attempt to represent the series explicitly.
The local level model is the simplest and important example of a structural time series model is the local level model. The handling of trend component and seasonal components in this regard is described as follows

4.5.1 Trend Component

Consider the local level model given in Equation (6). If one takes slope, ν_t generated by the random walk in to account, another model called local linear trend can be formulated as given in Equation (8).

$$\begin{aligned} \mu_{t+1} &= \mu_t + \nu_t + \xi_t \qquad \sim N(0, \delta_\varepsilon^2), \\ \nu_{t+1} &= \nu_t + \zeta_t \qquad \sim N(0, \delta_\epsilon^2), \\ y_t &= \mu_t + \varepsilon_t \qquad \sim N(0, {\delta_\zeta}^2). \end{aligned} \tag{8}$$

where ε_t and ζ_t are mutually uncorrelated noises.

If $\zeta_t = \xi_t = 0$ in the Equation 8, $\nu_{t+1} = \nu_1 = \nu$,
$\Rightarrow \nu_{t+1} = \nu_t = \nu$,
$\Rightarrow \mu_{t+1} = \mu_t + \nu$, a linear trend. That is, Equation (8) is reducible to deterministic linear trend plus noise model. It can also be written in matrix form as:

$$y_t = (1 \quad 0)\begin{pmatrix}\mu_t \\ \nu_t\end{pmatrix} + \varepsilon_t, \tag{9}$$

$$\begin{pmatrix}\mu_{t+1} \\ \nu_{t+1}\end{pmatrix} = \begin{bmatrix}1 & 1 \\ 0 & 1\end{bmatrix}\begin{pmatrix}\mu_t \\ \nu_t\end{pmatrix} + \begin{pmatrix}\xi_t \\ \zeta_t\end{pmatrix}.$$

4.5.2 Seasonal Component

Many environmental and economic time series shows seasonal variability. This seasonal behaviour of the series can be modelled in different ways. One way to do so is to use harmonic functions, to use trigonometric functions at the seasonal, $\lambda_j = 2\pi j/s$ where s is the number of cyclic components and $j = 1, ..., [s/2]$
This gives the seasonal effect of time t to be:

$$\psi_t = \sum_{j=1}^{s/2}(\psi_j \cos\lambda_j t + \psi_j^\star \sin\lambda_j t). \tag{10}$$

The cycle can also be recursively built up to a model comprising stochastcity as:

$$\begin{pmatrix}\psi_{j,t} \\ \psi_{j,t}^\star\end{pmatrix} = \begin{pmatrix}\cos\lambda_j & \sin\lambda_j \\ -\sin\lambda_j & \cos\lambda_j\end{pmatrix}\begin{pmatrix}\psi_{j,t-1} \\ \psi_{t-1}^\star\end{pmatrix} + \begin{pmatrix}w_{j,t} \\ w_{j,t}^\star\end{pmatrix}. \tag{11}$$

where $w_{j,t}$ and $w_{j,t}^\star, j = 1, ..., s/2$ are uncorrelated white noises and

$$G_{\text{seas(j)}} = \begin{pmatrix}\cos(\lambda_j) & \sin(\lambda_j) \\ -\sin(\lambda_j) & \cos(\lambda_j)\end{pmatrix}. \tag{12}$$

Equation (12) defines the harmonic matrix which is the model matrix.
Once the model matrix is known, the observation matrix can be deduced from it with the help of the model equation for the seasonality as follows:

$$X_t = \begin{pmatrix}\psi_{j,t} \\ \psi_{j,t}^\star\end{pmatrix} = \begin{pmatrix}\cos\lambda_j & \sin\lambda_j \\ -\sin\lambda_j & \cos\lambda_j\end{pmatrix}\begin{pmatrix}\psi_{j,t-1} \\ \psi_{t-1}^\star\end{pmatrix} + \begin{pmatrix}w_{j,t} \\ w_{j,t}^\star\end{pmatrix}, \tag{13}$$

$$= \begin{pmatrix}\psi_j \cos\lambda_j t + \psi_j^\star \sin\lambda_j t\lambda_j \\ -\psi_j \sin\lambda_j t + \psi_j^\star \cos\lambda_j t\end{pmatrix} + \begin{pmatrix}w_{j,t} \\ w_{j,t}^\star\end{pmatrix}. \tag{14}$$

Now looking at Equation (11) gives the the seasonal observation matrix to be

$$F_{\text{seas(j)}} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{15}$$

Generally, any periodic behaviour of an observation can be captured by a periodic function that can be written as a sum of sinusoidal waves at frequencies called harmonics, which are an integer multiple of the annual cycle.

4.6 Fundamental Theories and Tools For State Space Analysis

4.6.1 Dynamical State Estimation

The term 'state' of a system refers to a collection of dynamical variables like velocity, position, concentration and so on that can describe the system completely. Many problems are adhered with the fact that the state of the system is not known and is observable only partially. Dynamical state estimation is estimating these changing states of a given system. The method is applied for many kinds of tasks including weather prediction, target tracking, and inverse problems and more. The formulation of state estimation is done as follows. Consider the model given by Equation (4)

At discrete time t, the state of a system, X_t is estimated using previous observation $Y_{1:t} = (Y_1...Y_t)$. The observation model F maps the state to the observation and the state in turn evolves in time based on the evolution model G. In dynamical state estimation, observation are taken in real time, the estimated states demands an update by the measurements taken by applying Bayes' formula sequentially. The prior is obtained by moving the posterior of the model state from the previous time steps according to the model G. This is prediction stage. It will then be updated with the likelihood of the measurement, updating stage, to get the posterior that in turn evolved to be used as the next time step prior. Continuing this in similar way for the next time step realizes an on line estimation of states. This sequential estimation method is known as filtering. Such methods targets estimating the marginal distribution of the states $P(X_t \mid Y_{1:t})$ given observations up to the current time. Then the whole distribution of states will be moved with the dynamical model to the next time step for the prediction step. Figure 7 shows the estimation procedures.

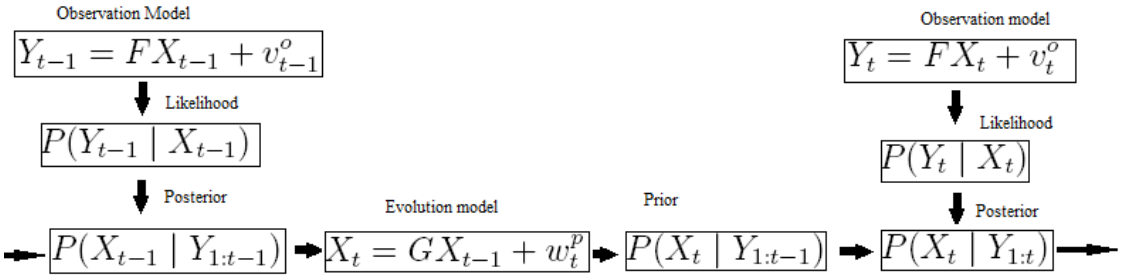

Figure 7: Two iteration of state estimation at time, t and $t-1$

4.6.2 Kalman Filter

Kalman filter is considered as an optimal solution to many data prediction and tracking analysis. The objective of filtering is mainly to compute the state vectors X_t for time steps $t = 1, 2, ...$ using Baye's rule sequentially so that the prediction from the previous time step is used as prior which is updated with new measurements that become available. That is, if X^{est}_{t-1} is a state with covariance matrix C^{est}_{t-1} at time $t-1$, he prior center point for the next time step t is given by the model the model prediction, $X^p_t = G_t X^{\text{est}}_{t-1}$.
Using the assumption that the state vector and model error are statistically independent, the covariance of the prediction (prior), C_t is.

$$C^p_t = \text{cov}(G_t X^{\text{est}}_{t-1} + w^p_t). \tag{16}$$

Applying the covariance property, $\text{cov}(AX) = \text{cov}(A)\text{cov}(x)A^T$ results in

$$\begin{aligned} C^p_t &= G_t \text{cov}(X^{\text{est}}_{t-1}) G^T_t + \text{cov} w^p_t, \\ G^T_t C^{est}_{t-1} G_t &= V_t, \end{aligned}$$

Where, $\text{cov} w^p_t = V_t$.
This Gaussian with mean,X^p_t and covariance C^p_t is used as a prior that is to be updated with new measurement vector Y_t..
After non trivial matrix computation the usual Kalman filter formulas are given as;

$$K_t = C^p_t F^T_t (F_t C^p_t F^T_t + R_t)^{-1}, \tag{17}$$

$$X^{est}_t = X^p_t + K_t(Y_t - F_t X^p_t), \tag{18}$$

$$C^{est}_t = C^p_t - K_t G_t C^P_t. \tag{19}$$

where, K_t is called the Kalman gain matrix.
The algorithm used for implementing Kalman filter can be summarized as follows

1. Predict the state estimate X_{t-1}^{est} and its covariance matrix C_{t-1}^{est} in the time:

   - Compute $X_t^p = G_t X_{t-1}^{est}$.
   - Compute $C_t^p = G_t C_{t-1}^{est} + V_t$.

2. Updating the prior with observation Y_t:

   - Computing the Kalman gain.
     $K_t = C_t^p F_t^T (F_t C_t^p F_t^T + R_t)^{-1}$.

   - Compute the state estimate.
     $X_t^{est} = X_t^p + G_t(Y_t - F_t X_t^p)$.

   - Compute the covariance estimate.
     $C_t^{est} = C_t^p - K_t G_t C_t^P$.

3. Repeating the steps for next time step.

Figure 8 summarizes the kalman filter algorithm diagrammatically as shown below.

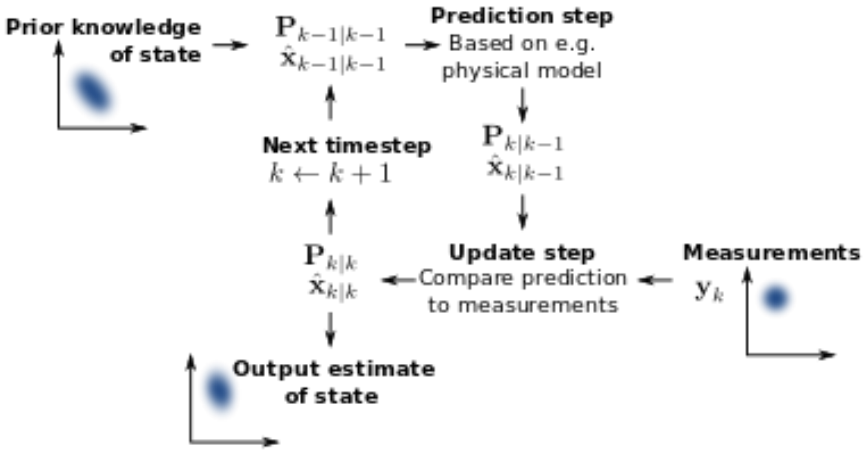

Figure 8: Algorithm of Kalman filter

4.7 State and Parameter Estimation

If x_t state of the system for $t = 1, ..., n$, y_t are the observations and θ is the model parameter contains auxiliary parameters needed to define the model, W_t is observation errors and V_t and the system matrices G_t and F_t. For dynamic linear models

we have efficient and well founded computational tools for all the relevant statistical distributions.

- $p(x_{t+1} \mid x_t, y_{1:t}, \theta)$ by Kalman filter
- $p(x_t \mid y_{1:t}, \theta)$ by Kalman filter
- $p(x_t \mid y_{1:n}, \theta)$by Kalman smoother
- $p(x_{1:n} \mid y_{1:n}, \theta)$ by simulation smoother
- $p(y_{1:t} \mid \theta)$ by Kalman filter likelihood
- $p(x_{1:n,\theta} \mid y_{1:n})$ by MCMC
- $p(x_{1:n} \mid y_{1:n})$ by MCMC

4.8 Recursive Kalman Formula

As mentioned earlier, the state of a system is which is the basic interest of state space models is not observable directly. Therefore, it needs to be estimated through use of an observation. The kalman filter is the main mathematical tool to do this calculation. Prediction, filtering and smoothing are the the problems involved, By prediction it means, estimating X_t from $y_{t-1}, y_{t-2}, \cdots$ whereas filtering is estimating X_t from $y_t, y_{t-1}, \cdots$) and smoothing is the estimation of X_t from $y_{t-1}, y_{t-2}, \cdots$ for$n > t$).
The kalman recursive formulas for Kalman filter and smoother for estimating the marginal distributions of DLM states given the observations are given below. On the assumption that the initial distributions at $t = 1$ are known. First Kalman filter forward recursion for the predicted states is done
$p(x_{t+1} \mid x_t, y_{1:t}, \theta) = N(\hat{x}_{t+1}, \hat{C}_{t+1}), t = 1, 2, ..., n-1.$

. $v_t = y_t - F_t\hat{x}_t$ prediction error.

. $C_t^y = F_t\hat{C}_tF_T + V_t$ prediction error covariance.

. $K_t = G_tF_T\hat{C}_T^{y-1}$ Kalman gain.

. $\hat{x}_{t+1} = G\hat{x}_t + K_{tv_t}$ next state prior mean.

. $\hat{C}_{t+1} = G_t\hat{C}_t(G_t - K_tF_t)^T + W$ next state prior covariance.

Then, apply Kalman smoother backward recursion to obtain the smoothed states. $p(x_t \mid y_{1:n}, \theta) = N(\hat{x}_t, \hat{C}_t), t = 1, 2, ..., n-1$

. $L = G_t - K_tF_t$

. $r_t = F_t^T C_t^{y-1} v_t + L^T r_{t+1}$

. $N = F_t^T C_t^{y-1} F_t + L^T NL$

. $\tilde{x}_t = \hat{x}_t + \hat{C}r$, smoothed state mean.

. $\hat{C}_t = \hat{C}_t - \hat{C}_t N\hat{C}_t$, smoothed state covariance.

4.9 Simulation Smoother

The Kalman smoother gives a Gaussian marginal distribution $p(x_t \mid y_{1:n}, \theta)$ for every t. In the study of dynamic features of the system such as the trend the joint distribution which span the whole time, $p(x_{t:n} \mid y_{1:n}, \theta)$ and this high dimensional distribution is neither Gaussian nor closed form representation exist. As usual, drawing realization from the distribution is more useful than the analytical expression. To do so, the system equation can be used to recursively produce the realization of states $x_{1:n}$ and observations,$y_{1:n}$ But the generated states is independent of the original observation. The residual process of generated and, smoothed state is independent of the $x_{1:n}$ and $y_{1:n}$. This means that adding these residual on the smoothed state $x_{1:n}$ can brought new realization which is conditional on $y_{1:n}$, the original observations. Therefore, to produce $x^*_{1:n} \sim p(x_{1:n} \mid y_{1:n}, \theta)$,

1. get $\tilde{x}_{1:n}$ and $\tilde{y}_{1:n}$ by sampling from the system equations.
2. Smooth $\tilde{y}_{1:n}$ to get $\tilde{\tilde{y}}_{1:n}$.
3. Add the residuals to the original smoothed. state,$\star x_{1:n} = \tilde{x}_{1:n} - \tilde{\tilde{x}}_{1:n} + x_{1:n}$.

In trend trend analysis this simulation smoother is used as a part of more general simulation algorithm that will sample from the joint posterior distribution $p(x_{1:n}, \theta \mid y_{1:n})$, and by marginalization argument also from $p(x_{1:n} \mid y_{1:n})$ where the uncertainty

in θ has been integrated out. A better description for sections (6.8), (6.9) and (6.10) is found at [2]

4.10 The Maximum Likelihood and Bayesian Model Fitting Methods

Besides to the least square approach where finding model parameter values in a sense that give good prediction of data at hand, the maximum likelihood and Bayesian methods are also other mostly used alternatives to do statistical model fitting.
In case of likelihood estimation, it is the value of the parameter estimates for which the observation probability density is maximized. This has some sense of intuition in that the values found agrees more than others and of course is known for its optimality [10], page 79.
For example, consider the linear trend given by:

$$y_t = \phi_0 + \phi_1 x_t + \varepsilon_t.$$
$$\text{where } t = 1, \cdots, n \text{ and } x_t = t$$

Let the observation y_t is from a normal distribution of variance σ^2 and mean given as $\mu_t = \phi_0 + \phi_1 x_t, x_t = t$,
Now, the probability density function of the distribution gives the density density of the observation as:

$$f_t(y_t; \phi, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} exp[-\frac{(y_t - \mu_t)^2}{2\sigma^2}].$$

In this case $\phi = (\phi_1, \phi_2)'$ is the regression coefficient vector.

For the observation are all assumed to be independent, the product of individual marginal density yield joint density:

$$f(y; \phi, \sigma^2) = \prod_{t=1}^{n} f_t(y_t; \phi, \sigma^2) = \prod_{t=1}^{n} (2\pi\sigma^2)^2 exp[-\frac{1}{2\sigma^2} \sum_{t=1}^{n} (y_t - \mu_t)^2]. \tag{20}$$

The likelihood function for the parameters can be obtained from regarding the joint density as a function of ϕ and σ^2 and the maximum likelihood estimates are values of the parameter maximizing 20. Log-likelihood function can be used for these values

are able to maximize its logarithm.

$$L(\phi, \sigma^2; y) = \log f(y; \phi, \sigma^2) = \sum_{t=1}^{n} \log f_t(y_t; \phi, \sigma^2) = -\frac{n}{2} \log 2\pi\sigma^2 - \frac{1}{2\sigma^2} \sum_{t=1}^{n} (y_t - \mu_t)^2.$$

Maximizing is done by differentiating and equating to zero. Thus, for the variance for example it would be:

$$\frac{\partial L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{t=1}^{n} (y_t - \mu_t)^2 = 0. \tag{21}$$

This results, $\sigma^2 = \frac{1}{n} \sum_{t=1}^{n} (y_t - \mu_t)^2$.

It is important to note that, this estimate is different from the common unbiased least square estimate in that $n - k$ is the divisor instead of n where K here is the number of elements involving in ϕ. However, the difference between them will be smaller for T is considerably large.

Generally, the maximum likelihood estimation targets learning about parameter values of the model which are taken as unknown and fixed using the data. In Bayesian estimation on the contrary, the data is used to update the foreknowledge about the parameters. That is, because either from understanding the system under study or past experience of similar data one might have some idea about the parameter before the actual observation, y and the Bayesian method needs this information in the form of joint probability distribution for the parameters known as prior distribution. The density of prior distribution is mostly denoted as $\pi(\phi)$ whenever the vector ϕ contains the parameters. Such a way of handling ϕ in probabilistic sense helps to represent the uncertainty of the analyst and treatment of the parameters as fixed quantities.

After observation for y is made, it then be explained by the joint conditional probability distribution of ϕ given $Y = y$ known as posterior distribution as $\pi(\phi \mid y)$. Using Bayes' theorem, $\pi(\phi \mid y$ can be written as:

$$\pi(\phi \mid y) = \frac{f(y \mid \phi)\pi(\phi)}{f(y)}.$$

where, $f(y \mid \phi)$ is the density of y given ϕ. In other words, since $f(y \mid \phi) = L(\phi \mid y)$ it is the likelihood of ϕ and $f(y)$ is unconditional density of y that does not change with ϕ. This implies that:

$$\pi(\phi \mid y) \propto L(\phi \mid y)\pi(\phi),$$
$$\text{posterior} \propto \text{Likelihood} \times \text{prior}. \tag{22}$$

The posterior distribution summarizes the available information about ϕ and the relationship, Equation (22) is known to be at heart of all Bayesian inference.

4.11 Markov Chain Monte Carlo (MCMC)

Parameter estimation can be done in different methods ranging from those which threats parameters as some constant value estimated based on measurement to the range where the parameter itself is seen as a random variable as in the case of Bayesian approach in which its goal is finding the posterior distribution $\pi(\theta \mid y)$ of the parameters which give the probability density for value of the parameters given the measurements, y and θ is the parameter. The MCMC in this regard, targets the generation of sequence of random samples $\theta_1, \theta_2, \cdots \theta_N$ sequentially that its distribution approaches the the posterior distribution asymptotically as the sample size increases. Here, sampling from the posterior distribution is done not from the posterior density directly and the method is purely based on generation of the random numbers. The term 'Monte carlo' is used to elite this. In addition, the samples generated at each point only depends on the previous point and not on the history. These samples produce a chain called Markov Chain which is used to show the resulting sample distribution approaches the target, the posterior. The MCMC methods is a powerful mathematical tool applied in many fields. For example it can be used in DLM's that contain unknown parameters with prior distribution to get posterior distribution.

4.12 Adaptive MCMC

The adaptive MCMC computation targets tuning proposal during the run while the sampling proceeds through use of the information of previously sampled points. This can be done by computing the empirical covariance matrix of the points already sampled and make it proposal covariance matrix. The algorithm gives correct result if the adaptation is bases the increasing history of the chain so that the number of previous points used in the computation of empirical covariance matrix

increases constantly during the sampling. These approach plays has minimized the basic burden of MCMC computation of finding a proposal distribution matching target distribution. The adaptive metropolis algorithm, delayed rejection adaptive metropolis algorithm are the versions of the adaptive MCMC. The former will here also be briefly explained and applied in this diploma work to estimate model parameters.

Among the different Class of MCMC algorithms, the Metropolis algorithm and from the adaptive types , the adaptive metropolis MCMC algoritms will be briefly discussed here.

4.12.1 Metropolis Algorithm

The metropolis algorithm is a simple, widely applied and influential MCMC algorithm. The working principle is based accept and reject technique of proposed values generated for the candidate parameter. The algorithm assumes a symmetric proposal distribution, equal probability density of traversing from current point to proposed point or the reverse, do exist. That is,

$$q(\hat{\theta} \mid \theta) = q(\theta \mid \hat{\theta}).$$

The Metropolis algorithm can be given as follows:

First Choose a point to start and initialize

Second Based on the the previous point in the chain, select new candidate hat θ from a suitable proposal distribution,$q(. \mid \theta_n)$

Third Accept the candidate with probability
$\alpha(\theta_n, \hat{\theta}) = \min\left(1, \frac{\pi(\hat{\theta})}{\pi(\theta_n)}\right)$.
Repeat the previous point in the chain if rejected.

Forth Move on to step 2.

4.12.2 Adaptive Metropolis

The Adaptive metropolis algorithm, Haario et al. 2001 is basically does updating a the proposal distribution during a run using the full knowledge collected so far. That is, tuning the proposal distribution with the based on process history. This idea plays a great advantage in alleviating the usual difficulty of making a right choice of proposal distribution for target density is unknown which inturn determines the MCMC performance.
In adaptive metropolis algorithm, the empirical covariance matrix is computed from the history and the proposal distribution is assumed to be Gaussian with the current point at the center and making the adaptation is possible by setting
C_n=$S_d Cov(\theta_0, \cdots, \theta_{n-1}) + \epsilon I_d$, where
C_n is next candidate of the proposal distribution.
S_d is scaling factor
$\theta_0, \cdots, \theta_{n-1}$, sampled points, history and
ϵ is a regularization parameter to control the positive definiteness of the proposal covariance matrix.
Based on prior knowledge, a random strictly positive definite covariance C_0 is required to begin the adaptation and the length of the initial non adaptation period so called burn-in is defined by the time index $n_0 > 0$. That is,

$$C_n = \begin{cases} C_0 & \text{if } n \geq n_0 \\ S_d Cov(\theta_0, \cdots, \theta_{n-1}) & \text{if } n > n_0 \end{cases}.$$

Approximate error analysis obtained from model linearization can be a good start for the initial proposal covariance matrix, C_n= $S_d \sigma^2 (J^T J)$, scaled Jacobian or depending on the case,other the choice to specify initial proposal covariance matrix be made. Details of this adaptive Metropolis algorithm can be found [4, 6] and its pseudo code given below briefs the algorithm.

First

Set length of the chain and starting θ_1 and C_1.

Second For $t = 1, 2, \cdots N$,

1. do metropolis step by using the proposal $N(\theta_t, C_t)$.
2. update $C_{t+1} = Cov(\theta_1, \cdots, \theta_t)$.

4.13 Analysing Trends

As described in previous sections, trend is the change in distributional properties that generates the observation. It can be the change in the mean of the process. There are many ways to get the trend. Fitting a smoother, like moving average one method.However, most of them has no statistical ways to estimate smoothness parameters or the uncertainty adhered with it. Trend analysis with dynamic linear model on the contrary, can solve the problem. In this method, the slowly varying background of the system can be modelled using random walk concept with variance parameter that controls time wise smoothness of the level and is variance parameter is to be estimated. The data could give information about how smooth the trend component is. In some cases prior information can be used for deciding the time scale of changes that one want to extract. DLM models can also provide qualitative prior information in the form of the model equations and quantitative information by prior distributions on variance parameters.

If $x_{level,t}$ is the model state that defines the background level of the process. Estimating the whole state, as either $p(x_{1:n} \mid y_{1:n,\hat{\theta}})$, where some estimates of auxiliary parameters θ is plugged in using maximum likelihood or by $p(x_{1:n} \mid y_{1:n}) = \int p(x_{1:n}, \theta \mid y_{1:n}) d\theta$ where the uncertainty of auxiliary parameters θ are integrated out,Bayesian approach by MCMC. Some more explanations and examples can be found at [2]

4.14 Combining the Model

A more general models can be obtained or formed by combing a few basic models. These models might stand for describing some of the nature of the observation process like periodic component,trend, and more that produce the actual observation when they are added.
This notion tells that a more comprehensive dlm model can be obtained by adding the individual models through defining the state of the system and the matrices involved.
To illustrate this consider Equation (4) once again with k independent DLM's for m dimensional observation which results:
$F_t = (F_t^{(1)}, \ldots, F_t^k)$,

$$G_t = \begin{bmatrix} G_t^{(1)} & & \\ & \ddots & \\ & & G_t^{(k)} \end{bmatrix}, \quad W_t = \begin{bmatrix} W_t^{(1)} & & \\ & \ddots & \\ & & W_t^{(k)} \end{bmatrix},$$

$$m_0' = \begin{bmatrix} m_0^1 & \dots & m_0^k \end{bmatrix}, \quad C_0 = \begin{bmatrix} C_0^{(1)} & & \\ & \ddots & \\ & & C_0^{(k)} \end{bmatrix}.$$

where G_t and W_t are block diagonal matrices.

4.15 Checking The Model

Before making any kind of analysis of the result of a given model, its assumption needs to be checked and this done in mostly bases on the measure of the departure between the model values and the data, that is lying on the residual process and is applied in many statistical models. There are different methods to carry out this diagnosis among these methods, the empirical autocorrlation function of the the residual and the quantile-quantile plot (QQ plot) are very important.

4.15.1 Checking with Empirical Auto-correlation Function

With the autocorrelation function, one can see the deviation from uncorrelatedness. A correlogram, time lag plot of sample autocorrlations, of the residual should closely or more behave like for white noise sequence correlogram.

4.15.2 Checking with QQ plots

For a uni-variate observations the standardized innovations, forecast errors can be defined as independent identically distributed zero-mean normal random variables sequence, that is as sequence of Gaussian white noise. Checking model assumption can in turn be done using the idea that if the model is adequate, the sequence obtained from the data should behave like a standard normal distribution. The QQ

plot is a statistical method to test this normality. It is a plot of expected values against the ordered residuals and if the points lie on straight line, then one can assume normality is achieved. Figure 9 shown below illustrates a normal QQ plot. Section 2.9 of [8] presents the details of the treatment in case of DLM.

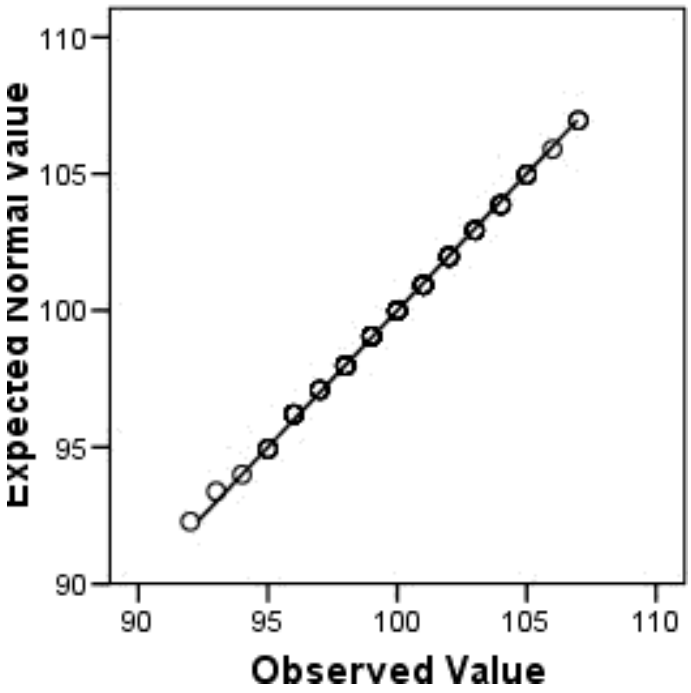

Figure 9: Normal QQ plot

5 DLM APPLICATION FOR IONOSOND TREND ANALYSIS

So far, the basics of dynamic linear model were discussed. In this section, the model will be applied for practical application for the purpose of ionosonde trend analysis. The details of the modeling procedures will be presented as follows.
As presented in earlier section, a dynamic linear model can be described by set of equations as given in Equation (4). The Ionosonde time series model can be constructed from considering local level and trend, seasonal components and from proxy time series in additive fashion as:

$$y_t = \mu_t + \gamma_t + \phi_t x_t + \varepsilon_t. \tag{23}$$

where $\mu_t, \gamma_t, \phi_t, x_t$ and ε_t are the level, seasonal components, regression coefficients, proxy time series values and the error term respectively. The state X_t in this case can be written as:
$X_t = \left(\mu_t, \alpha_t, \psi_{t,1}, \psi^{\star}_{t,1}, \psi_{t,2}, \psi^{\star}_{t,2}, \phi_t\right)^T$

5.1 Modelling the Background level

The background level can be modeled using the concept of random walk by local level and local linear trend with two hidden states, the mean level μ_t and successive level changes in time, α_t which can be written as $x_t = [\mu_t, \alpha_t]^T$. To let changes happen to the trend and level and observation, the addition of stochastic term to the respective parts is required as shown in the Equations (24).

$$\begin{aligned} y_t &= \mu_t + \epsilon_{\text{obs}} \quad \epsilon_{\text{obs}} \sim N(0, \sigma^2_{\text{obs}}), \\ \mu_t &= \mu_{t-1} + \alpha_t + \epsilon_{\text{level}} \quad \epsilon_{\text{level}} \sim N(0, \sigma^2_{\text{level}}), \\ \alpha_t &= \alpha_{t-1} + \epsilon_{\text{trend}} \quad \epsilon_{\text{trend}} \sim N(0, \sigma^2_{\text{trend}}). \end{aligned} \tag{24}$$

In state space form these equations will be given as:

$$G_{\text{trend}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad F_{\text{trend}} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad W_{\text{trend}} = \begin{bmatrix} \delta^2_{\text{level}} & 0 \\ 0 & \delta^2_{\text{level}} \end{bmatrix}$$

5.2 Modelling Seasonality

Like most atmospheric time series do, ionosonde time series shows seasonal variations and this variations can be modelled by using cycles of annual and semi annual through use of harmonic functions. Just as described in subsection[3.6.2], if s is the number of cyclic components, s/2 harmonics will there be in the seasonal model, $k = 1, ...k/2$ and In order to represent the annual and semiannual variations, two harmonics k= 1 and k=2 would be enough to explain the seasonality. For s = 12, monthly data, the model operator and observation matrices of the seasonality are given by:

$$G_{\text{seas(k)}} = \begin{pmatrix} \cos(\pi/6) & \sin(\pi/6) \\ -\sin(\pi/6) & \cos(\pi/6) \end{pmatrix} \tag{25}$$

$\text{F}_{\text{seas}} = \begin{pmatrix} 1 & 0 \end{pmatrix}$

Two state variables for each harmonics needs to be included to represent its state and in total four state variables are required and the the respective operators for the observation and process operator of the seasonality can be rewritten as partitioned form as:

$$G_{\text{seas}} = \begin{pmatrix} \cos(\pi/6) & \sin(\pi/6) & 0 & 0 \\ -\sin(\pi/6) & \cos(\pi/6) & 0 & 0 \\ 0 & 0 & \cos(\pi/3) & \sin(\pi/3) \\ 0 & 0 & -\sin(\pi/3) & \cos(\pi/3) \end{pmatrix}$$

,

$$F_{\text{seas}} = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}$$

5.3 Solar Proxy

If $F10.7(t)$ denotes the value of the solar proxy variable at time t. A proxy covariate for the solar proxy can be defined as:

$G_{\text{proxy}} = [1], F_{\text{proxy}(t)} = [F10.7(t)], \text{and} W_{\text{proxy}} = [\sigma^2_{\text{proxy}}]$

5.4 Forming the combined model

Finally the aforementioned component models are combined as diagonal block matrices.

$$G = \begin{bmatrix} G_{\text{trend}} & 0 & 0 \\ 0 & G_{\text{seas}} & 0 \\ 0 & 0 & G_{\text{proxy}} \end{bmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\pi/6) & \sin(\pi/6) & 0 & 0 & 0 \\ 0 & 0 & -\sin(\pi/6) & \cos(\pi/6) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\pi/3) & \sin(\pi/3) & 0 \\ 0 & 0 & 0 & 0 & \sin(\pi/3) & \cos(\pi/3) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{F}_t = (F_{\text{trend}} \quad F_{\text{seas}} \quad F_{\text{prox}}(t))$$
$$= \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & F_{10.7}(t) \end{pmatrix}$$

$$W = \begin{bmatrix} W_{\text{trend}} & 0 & 0 \\ 0 & W_{\text{seas}} & 0 \\ 0 & 0 & W_{\text{proxy}} \end{bmatrix}$$

5.5 Computational Procedure

The approach of the computation is based on Bayesian statistics and starts from space equations and the elements involving, the model states X_t, model matrices G_t, observation operator matrix, F_t, model error covariance matrix W_t and covariance of observation error V_t. Here the target is the joint posterior uncertainty distribution of the unknown parameters in the system matrices that define it and the state, $X_{1:N}$ and given the observation $y_{1:N}$. The static parameters involving is to be collected as a vector $\theta = [\sigma_{\text{trend}}, \sigma_{\text{seas}}, \sigma_{\text{proxy}}]$.

The initial step is to assume that at time t=0, the distribution is known and with

Kalman filter forward recursion the distribution of the state vector X_t is calculated when observation up to time t is given, $p(x_t \mid y_{1:t}, \theta) = N(\bar{x}_t, \bar{C}_t)$ by first calculating it as prior, mean and covariance of one step prediction of the states $p(X_t \mid X_{t-1}, y_{1:t-1}, \theta) = N(\hat{X}_t, \hat{C}_t)$ and predicted covariance as follows:

- Prior mean for X_t, as $\hat{X}_t = G_t \bar{X_{t-1}}$ and calculate
- Prior covariance for X_t, $\hat{C}_t = G_t \bar{C}_{t-1}(G_t G_t)^T + W_t$,

- Covariance matrix for predicted y_t, as $C_{y,t} = F_t \hat{C}_t F^T + V_t$,

The posterior state and its covariance matrix are computed as follows:

- The prediction residual as $v_t = y_t - F_t \hat{x}_t$ and
- Posterior mean for X_t, as $\bar{X}_t = \hat{X}_t + K_t \hat{X}_t$, where $K_t = G_t F_T \hat{C}_{y,t}^{-1}$ is the Kalman gain.

- Posterior covariance for X_t, as $\bar{C}_t = \hat{C}_t - K_t F_t \hat{C}_t$.

By assigning initial values for $\bar{x}_t$ and $\bar{C}_0$, the iteration of these equation is then carried out for $t = 1, \cdots, N$.
The next step is to use the kalman smoother backward recursion to get smoothed state for $t = N, N-1, \cdots, 1$ by using the generated matrices from the forward recursion. Here L_t, r_t and N_t are used as auxiliary variables with zero assigned to r_{N+1} and N_{t+1}. The computation is as follows:

- $L_t = G_t - G_t K_t F_t$, $r_t = F_t^T \hat{C}_{y,t}^{-1} v_t = L_t^T r_{t+1} L_t$ and $N_t = F_t^T \hat{C}_{y,t}^{-1} F_t + L_t^T N_{t+1} L_t$.
- The smoothed state of X_t is calculated as $\tilde{X}_t = \hat{X}_t + \hat{C}_t r_t$ and the smoothed covariance matrix as $\tilde{C}_t = \hat{C}_t - \hat{C}_t N_t \hat{C}_t$
  Therefore the the distribution of the state for each time t is:
  $p(x_t \mid y_{1:N}, \theta) = N(\tilde{X}_t, \tilde{C}_t)$.

As a third step, the full joint distribution of all the states given all observations and parameters, $p(x_{1:N} \mid y_{1:N}, \theta)$ to study trends. For this distribution has no closed form, simulation of its realization with simulation smoother needs to be done. Even if it is possible to recursively produce realizations using system state space equations directly but it will be independent of the original observation. Instead there is a way to produce samples from $p(x_{1:N} \mid y_{1:N}, \theta)$ by first sample from state space equations to get to get the states and observation via sampling from state space equation then

to use kalman smoother with new observation and get smoothed states and finally by adding the residuals of the states on the original smoothed states as described section 4.9.
The next step is to estimate the parameters, θ that were assumed to be fixed so far. To do so, the the distribution marginal likelihood function $p(y_{1:N} \mid \theta)$ is required. This likelihoods can be obtained as a by-product kalman filter calculation for each fixed θ. The likelihood function is evaluated sequentially as product of time wise marginal likelihood using the Markov property of the state space equation. For linear Gaussian models, the likelihood is given as:

$$-2\log p(y_{1:N} \mid \theta) = constant + \sum_{t=1}^{n}[(y_t - F_t\hat{X}_t)^T\hat{C}_{y,t}^{-1}(y_t - F_t\hat{X}_t + \log(\mid \hat{C}_{y,t} \mid)].$$

Although this likelihood could have been used to estimate the parameter by maximum likelihood method, MCMC approach has been preferred in order to integrated out the uncertainty in the parameter. This is done by using the likelihood function and setting the prior densities for each element of the parameter, θ to get the posterior. For this task, efficient adaptive MCMC [4] was employed.

Finally, combining MCMC outputs and simulation smoother in order to produce samples from the joint posterior distribution $P(x_{1:N}, \theta \mid y_{1:N})$ and from marginal distribution of the state, $P(x_{1:N} \mid y_{1:N})$ in which uncertainty in the parameter θ was taken in to account.Using this, distribution Ionsonde measurement trend was performed.

5.6 Statistics of the Computation

Computing the aforementioned procedures with a matlab code results the following statistics. MCMC analysis of the model parameter for $(\sigma_{\text{trend}}, \sigma_{\text{seas}}, \sigma_{\text{proxy}}]$ are given in table 1.

Table 1: MCMC result for model parameters.

σ_{trend}	σ_{seas}	σ_{proxy}
0.0012	1.1646	2.3787

The MCMC chain plot of the five thousand simulation steps for the model parameters is also given in Figure 10. It should indicates the parameters are identified well by the data.

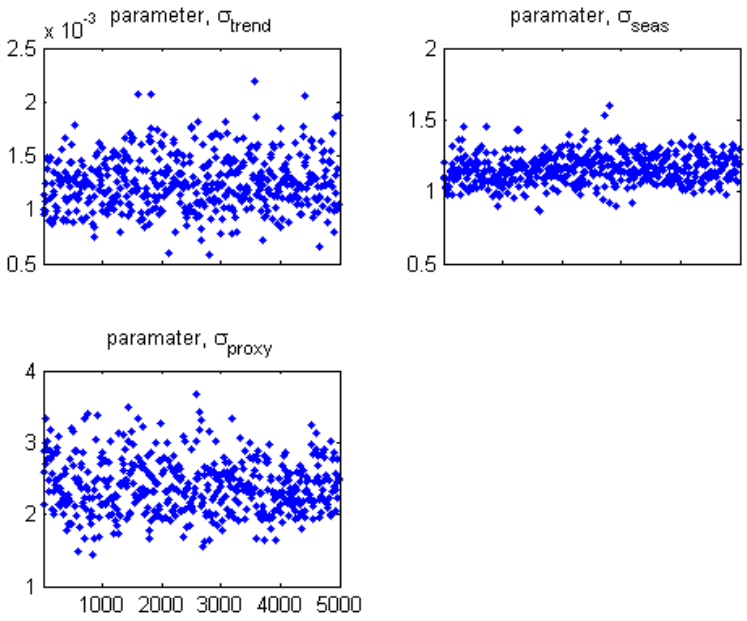

Figure 10: MCMC chain plot for the model parameters

5.7 Description of the DLM Toolbox Used

For dynamic linear model calculations, MATLAB dlm toolbox available at [2] was used. The main components of the toolbox and the way it was used in Ionosonde trend measurement will be discussed respectively as follows. The DLM toolbox is composed of many functions. However, the very basic ones includes, dlmsmo, dlmfit, dlmsmosam, dlmgensys and plotting funcctions. The function 'dlmsmo' is the one of main function to do dlm calculation. It takes defined Dlm model, the observations, observation operators matrix, model operator matrix and the uncertainty of the initial states as an input and gives the estimates of the states by kalman filter and kalman smoother and sample of state and observation.

The function 'dlmfit' on the other hand plays the role of estimation of some of model

parameters and building the system matrices (model structures) via use of the another function called 'dlmgensys'. It does also optimization or MCMC sampling analysis tasks for parameters in the diagonal of model error matrix optionally. This function uses the function 'dlmsmo' for fitting univariate time series DLM model to observations. It uses the observation and its uncertainties as a standard deviation, square root of the diagonal of the model error, the the model states initial values and covariance matrix for its uncertainty. In return, it gives the output of the function 'dlmsmo'.

Another basic function found in the DLM tool box is the 'dlmsmosam'. It does sampling task, samples from the joint distribution of the states at at all times when given all the observations.

The Plotting functions such as 'dlmplotfit' and 'dlmplotdiag' are used to plot the observations with the model level component estimated and for plot of residual diagnostics for the DLM fit obtained by 'dlmfit' or 'dlmsmo' respectively.

In the actual practice of the toolbox for ionosonde trend measurement work, some utility functions of the toolbox for scaling data and removing data rows for unobserved data points as in the case of proxy data purpose were used. With the option available in the function 'dlmfit', the structure of the DLM was defined. This includes, deciding the number of seasonal components involved, in this case two (annual and semiannual), the diagonal model error matrix involving variance parameters to be estimated by 'dlmfit'. This can be done with 'winds' option that map diagonal indexes to parameters. After feeding the required initial values such as for the model states and error covariance choose. Then with dlmfit, the model is fitted with dlmfit. As an output the estimated model states and model error matrix were obtained.

Finally using the plotting functions available in the toolbox, plotfit and dlmplotdiag, the fitted trend over an Ionosonde measurement data with 95% confidence limit, and model residual autocorrelation and normal probability plots were produced respectively.

6 RESULT AND DISCUSSION

As previously mentioned in earlier section, the main target of this diploma work is to make trend analysis over an Ionosonde data using dynamic linear model and also to study the effect of solar activity on the the peak height of the ionosphere, hmF2 of the ionosphere. This section strives to present and discuss the results foun.
As outlined in the introductory section of time series analysis, state space approach of time series analysis can explain the underlying structures of the time series explicitly. The trend analysis made for F2 layer peak height shown in Figure (11) below for example is presented by removing the some effects such as the solar effect contributing to the data. The result found by defining trend as a change in the

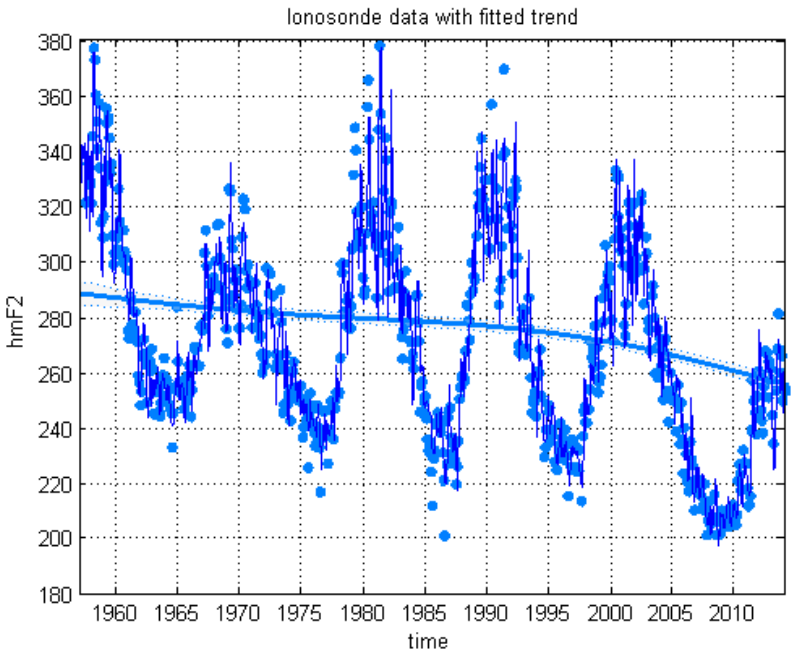

Figure 11: Trend obtained through dlm approach fitted over Ionosonde data

background level could be evident to think that allowing natural systems to evolve the way it goes in time preferable than the one made by approximating the trend with linear trend equation as the shown in Figure (5). Furthermore, in many cases, phenomenons occurring in some physical system could not be revealed by assuming some kind of increase or decrease in monotonic sense. Unlike this, dlm trend analysis can allows the smooth change of trend curve. This would help to make extensive studies to answer relevant questions about the physical system under study. With this regard, the peak height of F2 layer, shows a considerable decrease in the given observation period. Furthermore, a rapid decrease of it is observed in late years,

1990 - 2010. In addition, it is clearly visible that the trend is time varying. Generally, this result is in agreement with atmospheric model prediction and theories in that the increase of greenhouse gases such as methane and carbon dioxide has an effect of cooling the thermosphere and therefore lowers peak height of F2 layer. A detailed explanatory paper about the decrease of peak height of F2 layer made by Thomas Ulich and Esa Turunen at sodankyla [13] and [16].

In order to examine the connection between the F2 layer peak height and the solar activity, they are plotted together as shown in Figure 12. As explained in section

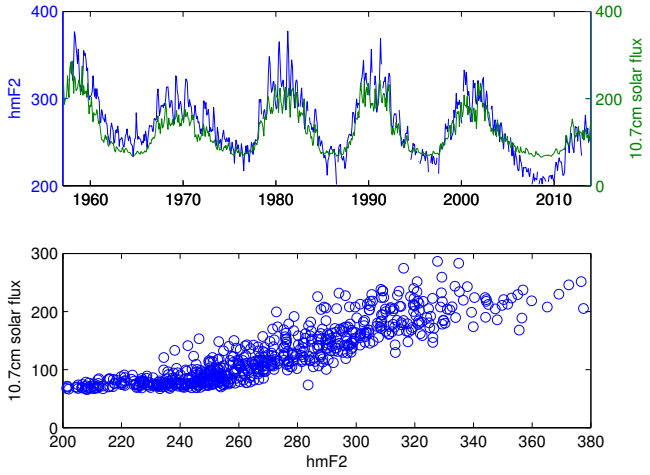

Figure 12: Top to down the panel, time series plot of the hmF2 peak height versus solar flux and scatter plot of solar radio flux together with hmF2

2.1, among the different ways of representing solar activity, here, solar radio flux at 10.7cm wavelength was used to do so. The time series plot of hmF2 indicates the solar variation of the peak height of the F2 layer during all the observation period. Both quantities traverse inphase. In addition, the scatter plot reveals the presence of positive correlation between F2 layer peak height and solar activity. quantities more or less positive linear correlation seems to be inferred. Generally speaking, it is found that, solar activity and the height of solar activity and the peak height of the F2 layer are positively correlated. And it seems this correlation is much more stronger than one think of it as just a simple coincidence.
The time series of the seasonal effect, the solar proxy and its coefficient are also plotted as shown in Figure 13. It clearly shows that, the coefficient of the solar proxy and the amplitude of the seasonal has strong dynamic property and thus dynamic regression indeed is needed for better fitting of the model (23)

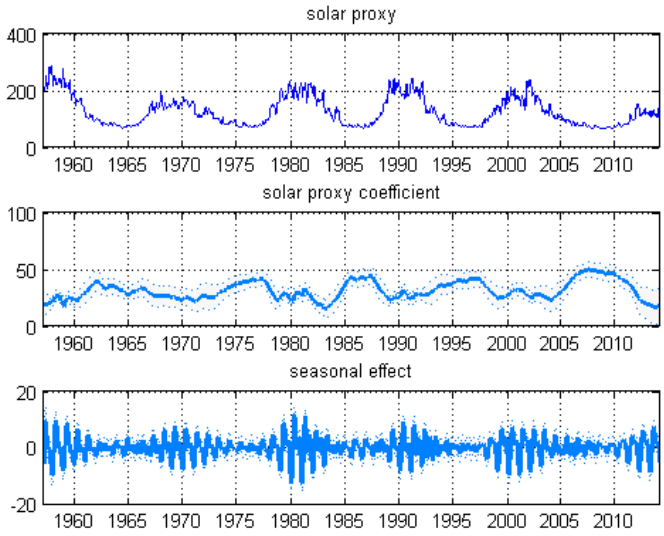

Figure 13: Top to down the panel respectively presents the value of solar proxy, the value of solar proxy coefficient and the seasonal effect against the observation period

Finally, to check how well the model employed works, model checking was done. For this purpose, the autocorrelation plot of the residuals, the QQ plot are used and the results obtained are given in Figure 14.

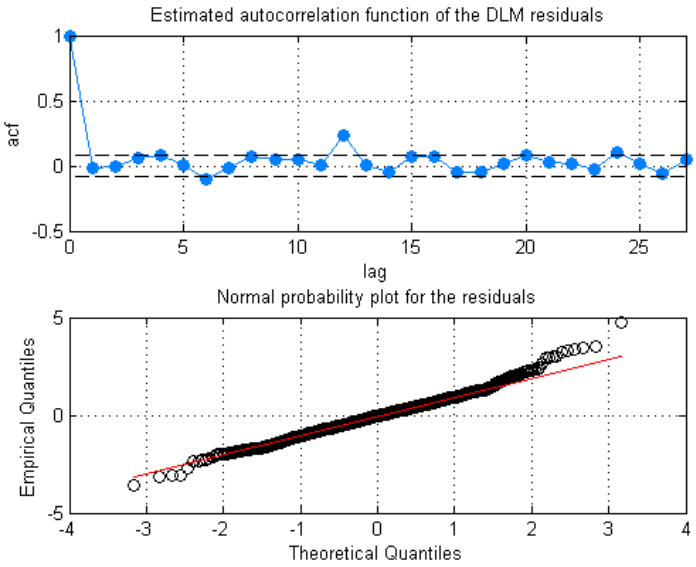

Figure 14: Top to down the panel Estimated autocorrelation function of DLM residuals, the QQ plot of residuals

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As detailed in the top panel of figure 14, plot of estimated autocorrelation, the autocorrelation function is generally inside the confidence interval except at lag 15 and very few lags at the boundary and this seems good result for it conforms that the residuals. This suggests that the model applied is reasonably good. The bottom panel plot is the qq plot and as described in section 4.15.2, this statistical method is used to test normality. The plots shows the quantile more or less lie on the line which imply the achievement of the normality condition.

7 CONCLUSION

In summary, this diploma work tried to employ one of the modern approach of time series analysis called the state space approach, more specifically dynamic linear model. The method is applied for trend analysis task over an Ionsonde measurement data. Based on the analysed trend found, one can conclude that the peak height of the F2 layer shows a decrease during the observation period and this is in harmony with what is predicted by atmospheric models in that the increase of green house gases has an effect of coolling the atmosphere and lower peak height of the F2 layer. In addition to this, it is found that solar activity and the peak height of the F2 layer shows a strong correlation. Besides, as classical trend estimation method for hmF2 time series gives a monotonic decrease of the peak height, the dynamic linear model approach has resulted in smoothly varying trend that helps to make further analysis and diagnostics about the physical problem in point instead. On top of this, the different components of the series can be analysed easily as opposed to the classical trend estimation approaches.

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