

Fuzzy Logic and Zadeh Algebra

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Abstract

In this work we create a connection between AFS (Axiomatic Fuzzy Sets) fuzzy logic systems and Zadeh algebra. Beginning with simple concepts we construct fuzzy logic concepts. Simple concepts can be interpreted semantically. The membership functions of fuzzy concepts form chains which satisfy Zadeh algebra axioms. These chains are based on important relationship condition (1) represented in the introduction where the binary relation R_m of a simple concept m is defined more general in Definition 2.10. Then every chain of membership functions form a Zadeh algebra. It demands a lot of preliminaries before we obtain this desired result.

Keywords: AFS fuzzy logic system, Zadeh algebra, simple concepts, membership functions.

1 Introduction

Starting with simple concepts such as "young people" or "tall people" it is possible to form AFS logic system $(EM, \vee, \wedge, ')$. The elements are fuzzy concepts constructed by simple concepts. Notice that (EM, \vee, \wedge) is a completely distributive lattice and is called the *EI* (expanding one set M) algebra over M . So the AFS logic system is a completely distributive lattice equipped with the logical negation $'$. Let X be a non-empty set. For any $\zeta \in EM$, let $\mu_\zeta : X \rightarrow [0, 1]$ be a membership function of the concept ζ . Moreover, we assume that all the elements μ_ζ in the set $\{\mu_\zeta \mid \zeta \in EM\}$ satisfy the three conditions, Definition 2.17. Consider a binary relation $R_\zeta \in X \times X$ of the concept ζ . For example, for any two persons x and y , $(x, y) \in R_\zeta$ if and only if

$$R_\zeta = \{(x, y) \mid (x, y) \in X \times X \text{ age}_x \geq \text{age}_y\}$$

where ζ is a fuzzy concept "old". The exact definition is represented in Definition 2.10. In Section 2 all the results are known and can be found from [5]. Also the used examples are there. For Zadeh algebra axioms we refer to [3]. In Section 3 it is proved new results. But all the preliminaries represented in Section 2 are necessary to know for understanding these results and their proofs. The crucial condition is

$$(x, z) \in R_m \implies (y, x) \in R_m \quad (1)$$

for simple concepts m and for all the pairs (x, z) and (y, x) in $X \times X$. In fact, the condition determines a chain $(\zeta)_{\zeta \in EM}$, Lemma 3.1 (c). Let

$$\{\mu_\zeta \mid \mu_\zeta : X \longrightarrow [0, 1]\}$$

be a set of membership functions of the concept ζ of the AFS fuzzy logic system $(EM, \vee, \wedge, ')$. According to Proposition 3.4 a chain $(\mu_\zeta)_{\zeta \in EM}$ corresponding to the chain $(\zeta)_{\zeta \in EM}$ satisfies the seven Zadeh algebra axioms and then forms some Zadeh algebra, Proposition 3.5. These are the two main results. Observe that by Lemma 3.1 (a) the condition (1) implies the condition (2) needed in Proposition 3.4.

$$(x, z) \in R_m \implies (y, z) \in R_m \quad (2)$$

In the conclusion it is illustrated the research motivation and contribution of this paper.

2 Preliminaries

2.1 Lattices

In this subsection we refer to [1], pages 1, 2, 6, 8, 9, 10, 119 and [5], pages 61-64, 67, 77.

Definition 2.1. A partially ordered set or a poset is a set in which a binary relation \leq is defined satisfying the following conditions (P1) - (P3):

- (P1) For all x , $x \leq x$.
- (P2) If $x \leq y$ and $y \leq x$, then $x = y$.
- (P3) $x \leq y$ and $y \leq z$, then $x \leq z$.

Let

(P4) Given x and y , either $x \leq y$ or $y \leq x$.

A poset which satisfies (P4) is said to be linearly ordered and is called a chain.

Let X be a subset of a poset P . Denote the least upper bound of X by l.u.b. i.e. $\sup X$ and the greatest lower bound of X by g.l.b. i.e. $\inf X$.

Definition 2.2. A lattice L is a poset P where any two of whose elements x and y have g.l.b. or a meet denoted by $x \wedge y$, and l.u.b. or a join denoted by $x \vee y$. A lattice L is complete if each of its subsets has l.u.b. and g.l.b. in L .

It is clear that any nonvoid complete lattice contains a least element 0 and a greatest element 1 .

In any lattice L (or a poset), the operations \wedge and \vee satisfy the following laws, whenever the expressions are referred to exist:

$$(L1) \quad x \wedge x = x, \quad x \vee x = x$$

$$(L2) \quad x \wedge y = y \wedge x, \quad x \vee y = y \vee x$$

$$(L3) \quad x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad x \vee (y \vee z) = (x \vee y) \vee z$$

$$(L4) \quad x \wedge (x \vee y) = x \vee (x \wedge y) = x$$

Conversely, any system L with the two binary operations satisfying (L1) - (L4) is a lattice.

Moreover, $x \leq y$ is equivalent to each of the conditions

$$x \wedge y = x \text{ and } x \vee y = y$$

If a poset P (or a lattice) has an 0 , then $0 \wedge x = 0$ and $0 \vee x = x$ for all $x \in P$. If P has a universal upper bound I , then $x \wedge I = x$ and $x \vee I = I$ for all $x \in P$.

Definition 2.3. A lattice L is distributive if and only if the conditions

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

hold in L . In fact, these conditions are equivalent if they are valid.

Definition 2.4. [5], pages 77, 116 or [1], page 119

Let L be a complete lattice. Then L is called a completely distributive lattice if it satisfies the extended distributive laws: for any family $\{a_{ij} \in L \mid i \in I, j \in J_i\}$ where I and J_i are non-empty indexing sets, the following equations are valid

$$\begin{aligned} \bigwedge_{i \in I} \left(\bigvee_{j \in J_i} a_{ij} \right) &= \bigvee_{f \in \prod_{i \in I} J_i} \left(\bigwedge_{i \in I} a_{if(i)} \right) \\ \bigvee_{i \in I} \left(\bigwedge_{j \in J_i} a_{ij} \right) &= \bigwedge_{f \in \prod_{i \in I} J_i} \left(\bigvee_{i \in I} a_{if(i)} \right) \end{aligned}$$

2.2 A survey to simple concepts and their operations

In this subsection we approach to simple concepts and their operations because it is necessary to form the idea what do simple concepts mean. The exact definition will be represented in Definition 2.13. All these are based on [5], pages 113,114.

Consider the set of four people x_1, x_2, x_3, x_4 and a simple concept "hair colour". By intuition, we may set: x_1 has "hair black" with number 6 and x_2, x_3, x_4 with numbers 4,6,3. So, the numbers imply the order $x_4 > x_2 > x_3 = x_1$ which can be interpreted as follows: Moving from right to left, the relationship states how strongly the hair colour resembles black colour. More exactly, $x_i > x_j$ means that the hair of x_i is closer to the black colour than the colour of the hair which x_j has.

Let M be a set of fuzzy or Boolean concepts on the set X . For each $m \in M$ we associate to a single feature. For example m_1 : "old people" is a fuzzy concept but m_2 : "male" is a Boolean concept. In fact, M is a set of simple concepts. In general let $A \subseteq M$ and denote by $\prod_{m \in A} m$ a conjugation of the concepts m on A . Correspondingly $\sum_{m \in A} m$ means a disjunction.

Example 2.5. Let m_1 : "old people", m_2 : "male", m_3 : "tall people". Then $m_1 m_2$: "old males" and $m_1 + m_3$: "old or tall people". Further, $m_1 m_2 + m_2 m_3 + m_1 m_2 m_3$: "old or tall males". However, $m_1 m_2 + m_2 m_3$ means the same. This is because for any person x the degree of x belonging to the fuzzy concept represented by $m_1 m_2 m_3$ is always less than or equal to the degree of x belonging to the fuzzy concept represented by $m_1 m_2$ or $m_2 m_3$. Therefore the former $m_1 m_2 m_3$ is including in both of the latter ones $m_1 m_2$ or $m_2 m_3$.

2.3 AFS fuzzy logic system

All the definitions and the propositions with their proofs are represented in [5], pages 115-123. For a moment we give up the assumption that M consists only of simple concepts. Let M be a non-empty set. The set EM^* is defined by

$$EM^* = \left\{ \sum_{i \in I} \left(\prod_{m \in A_i} m \right) \mid A_i \subseteq M, i \in I, I \text{ is any non-empty indexing set} \right\}$$

where the elements of EM^* are expressed semantically with "equivalent to", "or" (disjunction) and "and" (conjunction).

Definition 2.6. Let M be a non-empty set. A binary relation R on EM^* is defined as follows: for $\sum_{i \in I} (\prod_{m \in A_i} m), \sum_{j \in J} (\prod_{m \in B_j} m) \in EM^*$,
 $[\sum_{i \in I} (\prod_{m \in A_i} m)] R [\sum_{j \in J} (\prod_{m \in B_j} m)] \iff$

- (i) $\forall A_i, i \in I, \exists B_h, h \in J$ such that $A_i \supseteq B_h$,
- (ii) $\forall B_j, j \in J, \exists A_k, k \in I$ such that $B_j \supseteq A_k$.

R is an equivalence relation and we define EM as the quotient set EM^*/R .

Proposition 2.7. Let M be a non-empty set. Then (EM, \vee, \wedge) forms a completely distributive lattice under the binary compositions \vee and \wedge defined as follows: for any $\sum_{i \in I} (\prod_{m \in A_i} m), \sum_{j \in J} (\prod_{m \in B_j} m) \in EM$,

$$\begin{aligned} \sum_{i \in I} \left(\prod_{m \in A_i} m \right) \vee \sum_{j \in J} \left(\prod_{m \in B_j} m \right) &= \sum_{k \in I \sqcup J} \left(\prod_{m \in C_k} m \right) \triangleq \sum_{i \in I} \left(\prod_{m \in A_i} m \right) + \sum_{j \in J} \left(\prod_{m \in B_j} m \right) \\ \sum_{i \in I} \left(\prod_{m \in A_i} m \right) \wedge \sum_{j \in J} \left(\prod_{m \in B_j} m \right) &= \sum_{i \in I, j \in J} \left(\prod_{m \in A_i \cup B_j} m \right) \end{aligned}$$

where the disjoint union $I \sqcup J$ means that every element in I and every element in J are always regarded as different elements in $I \sqcup J$. Therefore for any $k \in I \sqcup J$, $C_k = A_k$ if $k \in I$, and $C_k = B_k$ if $k \in J$.

The proof of the proposition can be found from [5].

To be a distributive lattice means that for any $\gamma, \zeta, \eta \in EM$

$$\begin{aligned} \gamma \wedge (\zeta \vee \eta) &= (\gamma \wedge \zeta) \vee (\gamma \wedge \eta) \\ \gamma \vee (\zeta \wedge \eta) &= (\gamma \vee \zeta) \wedge (\gamma \vee \eta) \end{aligned}$$

A completely distributive lattice is defined in Definition 2.4. Because (EM, \vee, \wedge) is such a lattice it guarantees the existence of the EM elements $\sum_{i \in I} (\prod_{m \in A_i} m)$ and $\sum_{j \in J} (\prod_{m \in B_j} m)$. We can also define the order in (EM, \vee, \wedge) as follows:

$$\sum_{i \in I} (\prod_{m \in A_i} m) \geq \sum_{j \in J} (\prod_{m \in B_j} m) \iff \forall B_j, j \in J, \exists A_k, k \in I \text{ such that } B_j \supseteq A_k.$$

Further, as a (distributive) completely lattice (EM, \vee, \wedge) is also a complete lattice.

The lattice (EM, \vee, \wedge) is called the EI (expanding one set M) algebra over M .

Proposition 2.8. Let M be a set and $g : M \rightarrow M$ be a map satisfying $g(g(m)) = m$ for all $m \in M$. If the operator $^g : EM \rightarrow EM$ is defined as follows

$$\left(\sum_{i \in I} (\prod_{m \in A_i} m) \right)^g = \bigwedge_{i \in I} \left(\bigvee_{m \in A_i} g(m) \right) = \bigwedge_{i \in I} \left(\sum_{m \in A_i} g(m) \right)$$

Then for any $\alpha, \beta \in EM$, g has the following properties:

- (1) $(\alpha^g)^g = \alpha$,
- (2) $(\alpha \vee \beta)^g = \alpha^g \wedge \beta^g$, $(\alpha \wedge \beta)^g = \alpha^g \vee \beta^g$,
- (3) $\alpha \leq \beta \implies \alpha^g \geq \beta^g$

Therefore the operator g is an order reversing involution in the EI algebra (EM, \vee, \wedge) .

The operator g defines the negation m' of the concept m : $m' = g(m)$. Then $m'' = (g(m))' = g(g(m)) = m$.

Let $\zeta = \sum_{i \in I} (\prod_{m \in A_i} m) \in EM$. Then

$$\zeta' = \bigwedge_{i \in I} \left(\bigvee_{m \in A_i} m' \right) = \bigwedge_{i \in I} \left(\sum_{m \in A_i} m' \right)$$

stands for the logical negation of ζ . $(EM, \vee, \wedge, ')$ is called an AFS fuzzy logic system.

Example 2.9. Let m_1 : "old people", m_2 : "tall people", m_3 : "males". Then

$$\begin{aligned}
\zeta &= m_1 m_3 + m_2 = m_1 m_3 \vee m_2 \\
\zeta' &= (m_1 m_3 + m_2)' = (m_1 \wedge m_3 \vee m_2)' = (m_1' \vee m_3') \wedge m_2' \\
&= (m_1' \wedge m_2') \vee (m_3' \wedge m_2') = m_1' m_2' + m_3' m_2'.
\end{aligned}$$

where ζ : "old males or tall people" and ζ' : "not old and not tall people or not tall males".

The AFS fuzzy logic system $(EM, \vee, \wedge, ')$ can be regarded as a completely distributive lattice. It is also a complete lattice. But this lattice is equipped with the logical negation.

We conclude that the complexity of human concepts is a direct result of the combinations of a few relatively simple concepts. In fact, some suitable simple concepts play the same role as used in linear vector spaces and we can regard them as a "basis".

2.4 Relations, simple and complex concepts

For this subsection we refer to [5], pages 124,125.

Definition 2.10. Let ζ be any concept on the universe of discourse X . $R_\zeta \subset X \times X$ is called the binary relation of the concept ζ if R_ζ satisfies: $x, y \in X$, $(x, y) \in R_\zeta$ if and only if x belongs to concept ζ at some extent or x is a member of ζ and the degree of x belonging to ζ is larger or equal to that of y , or x belongs to concept ζ at some degree and y does not at all.

Example 2.11. Let fuzzy concept ζ : "old" and

$$R_\zeta = \{(x, y) \mid (x, y) \in X \times X \text{ age}_x \geq \text{age}_y\}$$

Therefore $(x, x) \in R_\zeta$ means that x belongs to ζ at some degree and that $(x, x) \notin R_\zeta$ means that x does not belong to ζ at all. If for the two persons x and y , $\text{age}_x = 30$ and $\text{age}_y = 20$ then $(x, y) \in R_\zeta$ but $(y, x) \notin R_\zeta$.

Example 2.12. Let fuzzy concept ζ : "hair black" and define R_ζ in the corresponding way as above. By human intuition, we assume that for the three persons x_i , $i = 1, 2, 3$ the degree of ζ is the following: $\text{black}_{x_1} > \text{black}_{x_2} > \text{black}_{x_3}$ but the fourth person x_4 has no hairs. Then $(x_1, x_2), (x_1, x_3), (x_2, x_3) \in R_\zeta$ and $(x_4, x_4) \notin R_\zeta$ but $(x_i, x_4) \in R_\zeta$. See Definition 2.13 (2).

Definition 2.13. Let X be a set and R be a binary relation on X . R is called a sub-preference relation on X if for $x, y, z \in X$, $x \neq y$, R satisfies the following conditions:

- (1) if $(x, y) \in R$, then $(x, x) \in R$,
- (2) if $(x, x) \in R$ and $(y, y) \notin R$, then $(x, y) \in R$,
- (3) if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$,
- (4) if $(x, x) \in R$ and $(y, y) \in R$, then either $(x, y) \in R$ or $(y, x) \in R$.

We define that a concept ζ on X is simple if R_ζ is a sub-preference relation on X . Otherwise ζ is called a complex concept on X .

Example 2.14. Let ζ : "old people". The concept is simple. For example if for the persons x_1, x_2, x_3 we have $age_{x_1} > age_{x_2} > age_{x_3}$ R_ζ is a sub-preference relation on the set $\{x_1, x_2, x_3\}$ where R_ζ is the binary relation defined in Example 2.11. Observe that the latter of the assumptions of (2) in Definition 2.13 is not valid and so the condition (2) is valid. In general it is known that all elements belonging to a simple concept at some degree are comparable and are arranged in a linear order, that is, they form a chain. In above we can think shortly that $x_1 > x_2 > x_3$.

Further, there exists a pair of different elements belonging to a complex concept at some degree such that their degrees in this complex concept are incomparable.

Example 2.15. The set X consists of disjoint sets Y : "males" and Z : "females". The concept ζ : "beautiful" is simple on Y and on Z :

$$\begin{aligned} R_\zeta &= \{(x_1, x_2) \mid (x_1, x_2) \in Y \times Y, b_{x_1} \geq b_{x_2}\} \\ R_\zeta &= \{(y_1, y_2) \mid (y_1, y_2) \in Z \times Z, b_{y_1} \geq b_{y_2}\} \end{aligned}$$

However, if we apply ζ to the whole set X it is a complex concept because the degrees of the elements $x \in X$ and $y \in X$ may be incomparable:

If $x \in Y$ and $y \in Z$, then $(x, y) \notin R_\zeta$ and If $y \in Y$ and $x \in Z$, then $(y, x) \notin R_\zeta$. In this case $(x, x) \in R_\zeta$ and $(y, y) \in R_\zeta$ implies that both $(x, y) \notin R_\zeta$ and $(y, x) \notin R_\zeta$. The condition (4) in Definition 2.13 is not satisfied and so ζ is a complex concept.

2.5 The AFS fuzzy logic and coherence membership functions

For introduction to characteristic and membership functions we refer to [2], page 255, and [4], pages 12-18. Definitions 2.16 and 2.17 can be found from [5], pages 128,130. We first become acquainted with concepts fuzzy sets and membership functions.

Let $X \neq \emptyset$ and $x \in X$, $A \subset X$. Define a characteristic function for the set A as follows:

$$f_A : X \longrightarrow \{0, 1\} \quad f_A = \begin{cases} 1, & x \in A \\ 0, & \text{otherwise} \end{cases}$$

Consider an extended case $0 \leq f_A(x) \leq 1$, that is, $0 < f_A(x) < 1$ is also possible. We call for the set $A \subset X$

- (a) a crisp set, if its characteristic function is $f_A : X \longrightarrow \{0, 1\}$,
- (b) a fuzzy set, if its extended characteristic function or a membership function is $\mu_A : X \longrightarrow [0, 1]$.

Therefore for every element $x \in X$ there is a membership degree $\mu_A(x) \in [0, 1]$. The set of pairs

$$A = \{(x, \mu_A(x)) \mid x \in X\}$$

determines completely the fuzzy set A . The characteristic function of a crisp set A is a special case of a membership function $\mu_A : X \longrightarrow [0, 1]$.

Definition 2.16. [5] Let X, M be sets and 2^M be the power set of M . Let $\tau : X \times X \longrightarrow 2^M$. (M, τ, X) is called an *AFS* structure if τ satisfies the following axioms:

- (1) $\forall (x_1, x_2) \in X \times X, \tau(x_1, x_2) \subseteq \tau(x_1, x_1)$,
- (2) $\forall (x_1, x_2), (x_2, x_3) \in X \times X, \tau(x_1, x_2) \cap \tau(x_2, x_3) \subseteq \tau(x_1, x_3)$.

We again return to the case that M is a set of simple concepts.

Let X be a set of objects and M be a set of simple concepts on X . $\tau : X \times X \longrightarrow 2^M$ is defined as follows: for any $(x, y) \in X \times X$

$$\tau(x, y) = \{m \mid m \in M, (x, y) \in R_m\} \in 2^M$$

where R_m is the binary relation of simple concepts $m \in M$ defined in Definition 2.10 (it was defined more general than for simple concepts).

It is proved in [5] that (M, τ, X) is an *AFS* structure.

Definition 2.17. [5] Let (M, τ, X) be an *AFS* structure of a data set X . For $x \in X, A \subseteq M$, the set $A^\tau(x) \subseteq X$ is defined as follows:

$$A^\tau(x) = \{y \mid y \in X, \tau(x, y) \supseteq A\}$$

For $\zeta \in EM$, let $\mu_\zeta : X \rightarrow [0, 1]$ be the membership function of the concept ζ . $\{\mu_\zeta(x) \mid \zeta \in EM\}$ is called a set of coherence membership functions of the *AFS* fuzzy logic system $(EM, \vee, \wedge, ')$ and the *AFS* structure (M, τ, X) , if the following conditions are satisfied:

- (1) For $\alpha, \beta \in EM$, if $\alpha \leq \beta$ in lattice $(EM, \vee, \wedge, ')$, then $\mu_\alpha(x) \leq \mu_\beta(x)$ for any $x \in X$.
- (2) For $x \in X, \eta = \sum_{i \in I} (\prod_{m \in A_i} m) \in EM$, if $A_i^\tau(x) = \emptyset$ for all $i \in I$ then $\mu_\eta(x) = 0$.
- (3) For $x, y \in X, A \subseteq M, \eta = \prod_{m \in A} m \in EM$, if $A^\tau(x) \subseteq A^\tau(y)$, then $\mu_\eta(x) \leq \mu_\eta(y)$; if $A^\tau(x) = X$ then $\mu_\eta(x) = 1$.

Remark: It is important to see that M consists of simple elements m .

2.6 Zadeh algebra

We refer to [3].

Definition 2.18. Suppose that (L, \vee, \wedge) is a complete distributive lattice. Let $L^X = \{\mu \mid \mu : X \rightarrow L\}$ be the set of all functions from X to L . Assume that the lattice operations the least upper bound \vee and the greatest lower bound \wedge on L are extended pointwise for the functions on L^X . Further, define the extreme constant functions $X \rightarrow L, \mathbf{0} : x \mapsto \perp$ and $\mathbf{1} : x \mapsto \top$ for all $x \in X$, where \perp and \top are the least and the greatest elements of L , respectively. A unary operation η on L satisfies the involution property for any $a \in L$, and η is extended pointwise for the functions on L^X , i.e., $\eta(\eta(\mu)) = \mu$ for any $\mu \in L^X$. Then $Z = (L^X, \vee, \wedge, \eta, \mathbf{0}, \mathbf{1})$ is called Zadeh algebra if it satisfies the following conditions:

- (Z1) The operations \vee and \wedge are commutative on L^X .
- (Z2) The operations \vee and \wedge are associative on L^X .
- (Z3) The operations \vee and \wedge are distributive on L^X .
- (Z4) The neutral elements of the operations \vee and \wedge are $\mathbf{0}$ and $\mathbf{1}$, respectively, i.e., for all $\mu \in L^X$ and for all $x \in X$, $(\mu \vee \mathbf{0})(x) = \mu(x)$ and $(\mu \wedge \mathbf{1})(x) = \mu(x)$.

(Z5) For any function $\mu \in L^X$ and for all $x \in X$, there exists $\eta(\mu) \in L^X$ such that $\eta(\mathbf{1}(x)) = \mathbf{0}(x) \leq \eta(\mu(x)) \leq \eta(\mathbf{0}(x)) = \mathbf{1}(x)$, i.e., η is order reversing.

(Z6) $\mathbf{0} \neq \mathbf{1}$.

(Z7) Zadeh algebra fulfils the Kleene condition: for any function $\mu, \nu \in L^X$ and for any $x, y \in X$, $\mu(x) \wedge \neg\mu(x) \leq \nu(y) \vee \neg\nu(y)$, where $\neg\mu$ and $\neg\nu$ are the logical negations of μ and ν , respectively.

3 Connection between coherence membership functions of the AFS fuzzy logic system $(EM, \vee, \wedge, ')$ and Zadeh algebra

Lemma 3.1. Let $(\sum_{i \in I} (\prod_{m \in A_i} m))$ and $\sum_{j \in J} (\prod_{m \in B_j} m)$ be elements in EM where concepts m are simple and let X be a non-empty set. If every relation $R_m \subset X \times X$ satisfies the condition

$$(x, z) \in R_m \implies (y, x) \in R_m \quad (1)$$

for pairs (x, z) and $(y, x) \in X \times X$, then

(a) $(x, z) \in R_m \implies (y, z) \in R_m$

(b) there exists A_k in the set $\{A_i \mid i \in I\}$ such that $A_k \subseteq B_j$ for every B_j , that is,

$$\sum_{i \in I} (\prod_{m \in A_i} m) \geq \sum_{j \in J} (\prod_{m \in B_j} m)$$

(c) Let $\alpha = (\sum_{i \in I} (\prod_{m \in A_i} m))$, $\beta = (\sum_{j \in J} (\prod_{m \in B_j} m))$, $\delta = (\sum_{\nu \in V} (\prod_{m \in C_\nu} m))$, \dots be the elements in EM such that the condition (a) is satisfied for all pairs (α, β) , (β, δ) , \dots . Then $\alpha \geq \beta \geq \delta \geq \dots$.

Proof. Assume that the condition (1) holds.

(a) Let $(x, z) \in R_m$. Then $(y, x) \in R_m$. Because m is simple, R_m is a sub-preference relation and by Definition 2.13 (3) it is transitive. This implies that $(y, z) \in R_m$ and (a) is valid.

(b) Let

$$\begin{aligned}\tau(x, z) &= \{m \in A_i \mid (x, z) \in R_m\} \\ \tau(y, x) &= \{m \in B_j \mid (y, x) \in R_m\}\end{aligned}$$

Because m is simple $\tau(x, y)$ and $\tau(y, z)$ are defined.

Because $(x, z) \in R_m \implies (y, x) \in R_m$ we conclude that $\tau(x, z) \subseteq \tau(y, x)$ and so there exists A_k in the set $\{A_i \mid i \in I\}$ such that for every B_j is $A_k \subseteq B_j$. If there exists B_r which does not contain A_k then $B_r \supseteq \cup A_i$ does not hold. This is a contradiction. According to discussion after Proposition 2.7 we have

$$\sum_{i \in I} (\prod_{m \in A_i} m) \geq (\sum_{j \in J} (\prod_{m \in B_j} m))$$

(c) Consider a pair (β, δ) , where $\beta = \sum_{j \in J} (\prod_{m \in B_j} m)$ and $\delta = \sum_{v \in V} (\prod_{m \in C_v} m)$. We will prove that $\beta \geq \delta$, that is,

$$\sum_{j \in J} (\prod_{m \in B_j} m) \geq \sum_{v \in V} (\prod_{m \in C_v} m)$$

if $(y, w) \in R_m \implies (v, y) \in R_m$ (condition (1)).

Repeating the proof of (b) we obtain the following: Let $\tau(y, w) = \{m \in B_j \mid (y, w) \in R_m\}$, $\tau(v, y) = \{m \in C_v \mid (v, y) \in R_m\}$. Then $\tau(y, w) \subseteq \tau(v, y)$ and there exists B_l in the set $\{B_j \mid j \in J\}$ such that $B_l \subseteq C_v$ for all C_v . This proves $\beta \geq \delta$. In the same way we can prove that $\alpha \geq \beta$ for a pair (α, β) and then we conclude that $\alpha \geq \beta \geq \delta \geq \dots$

□

Lemma 3.2. Let $\zeta = \sum_{i \in I} (\prod_{m \in A_i} m) \in EM$. If the condition

$$(x, z) \in R_m \implies (y, x) \in R_m$$

holds for every simple concept $m \in A_i$ then

$$\mu_\zeta(x) \leq \mu_\zeta(y)$$

where $\mu_\zeta : X \longrightarrow [0, 1]$ is the membership function of the concept ζ .

Proof. Let $m \in M$ be simple concepts and

$$\begin{aligned}\tau(x, z) &= \{m \in A_i \mid (x, z) \in R_m\} \\ A_i^\tau(x) &= \{z \in X \mid \tau(x, z) \supseteq A_i\}\end{aligned}$$

Assume that the condition

$$(x, z) \in R_m \implies (y, x) \in R_m$$

holds for every $m \in A_i$. By Lemma 3.1 (a)

$$(x, z) \in R_m \implies (y, z) \in R_m$$

also holds. Then $\tau(x, z) \subseteq \tau(y, z)$ and so

$$A_i \subseteq \tau(x, z) \implies A_i \subseteq \tau(y, z)$$

It follows that $A_i^\tau(x) \subseteq A_i^\tau(y)$.

Let $\zeta_i = \prod_{m \in A_i} m$. These ζ_i exist because EM is a completely lattice. By Definition 2.17 (3) $\mu_{\zeta_i}(x) \leq \mu_{\zeta_i}(y)$ for every $i \in I$ where $\mu_{\zeta_i} : X \rightarrow [0, 1]$ is a membership function. Let $\zeta = \sum_{i \in I} (\prod_{m \in A_i} m) = \text{sup}_{i \in I} \zeta_i$. Also ζ exists and we obtain $\mu_\zeta(x) \leq \mu_\zeta(y)$ \square

Lemma 3.3. Assume that the binary relations $R_m \subset X \times X$ of simple concepts m satisfy the condition

$$(x, z) \in R_m \implies (y, x) \in R_m$$

for pairs $(x, z), (y, x) \in X \times X$.

Let $\alpha = \sum_{j \in J} (\prod_{m \in B_j} m)$ and $\beta = \sum_{i \in I} (\prod_{m \in A_i} m)$ be elements in EM . Then $\alpha \leq \beta$ and membership functions μ_α and μ_β satisfy the Kleene condition

$$\mu_\alpha(x) \wedge \neg \mu_\alpha(x) \leq \mu_\beta(y) \vee \neg \mu_\beta(y)$$

Proof. By Lemma 3.1 (b), $\alpha \leq \beta$. On the other hand, by Lemma 3.2

$$\mu_\beta(x) \leq \mu_\beta(y)$$

Because $\alpha \leq \beta$, by Definition 2.17 (1), $\mu_\alpha(x) \leq \mu_\beta(x)$. We obtain

$$\mu_\beta(y) \vee \neg \mu_\beta(y) \geq \mu_\beta(y) \geq \mu_\beta(x) \geq \mu_\alpha(x) \geq \mu_\alpha(x) \wedge \neg \mu_\alpha(x)$$

\square

Proposition 3.4. Let

$$[0, 1]^X = \{\mu_\zeta \mid \mu_\zeta : X \longrightarrow [0, 1]\}$$

be a set of membership functions of the AFS fuzzy logic system $(EM, \vee, \wedge, ')$. The EM elements ζ are of the form

$$\zeta = \sum_{i \in I} \left(\prod_{m \in A_i} m \right)$$

Let R_m be binary relations of simple concepts m . If the condition

$$(x, z) \in R_m \implies (y, z) \in R_m$$

is valid for every simple concept $m \in A_i$, $i \in I$ and $x, y, z \in X$ then the membership functions μ_ζ satisfy the conditions (Z1) - (Z7) of Zadeh algebra in Definition 2.18.

Proof. We verify the Zadeh algebra axioms: The first three axioms (Z1) - (Z3) are clear.

(Z1) The operations \vee and \wedge are commutative on $[0, 1]^X$.

$$\begin{aligned} \mu_{\zeta_1}(x) \vee \mu_{\zeta_2}(x) &= \mu_{\zeta_2}(x) \vee \mu_{\zeta_1}(x) \\ \mu_{\zeta_1}(x) \wedge \mu_{\zeta_2}(x) &= \mu_{\zeta_2}(x) \wedge \mu_{\zeta_1}(x) \end{aligned}$$

(Z2) The operations \vee and \wedge are associative on $[0, 1]^X$.

(Z3) The operations \vee and \wedge are distributive on $[0, 1]^X$.

(Z4) The neutral elements of the operations \vee and \wedge are $\mu_0 : X \longrightarrow [0, 1]$, $\mu_0(x) \equiv 0$ and $\mu_1 : X \longrightarrow [0, 1]$, $\mu_1(x) \equiv 1$.

$$\begin{aligned} \mu_\zeta(x) &= 0 \vee \mu_\zeta(x) = \mu_0(x) \vee \mu_\zeta(x) = (\mu_0 \vee \mu_\zeta)(x) \\ \mu_\zeta(x) &= 1 \wedge \mu_\zeta(x) = \mu_1(x) \wedge \mu_\zeta(x) = (\mu_1 \wedge \mu_\zeta)(x) \end{aligned}$$

(Z5) For any $\mu_\zeta : X \longrightarrow [0, 1]$ and for all $x \in X$ there exists a unary operation

$$\eta : [0, 1]^X \longrightarrow [0, 1]^X, \eta(\mu_\zeta) = \mu_{g(\zeta)}$$

where the operation $g : EM \longrightarrow EM$ is defined by

$$\left(\sum_{i \in I} \left(\prod_{m \in A_i} m \right) \right)^g = \bigwedge_{i \in I} \left(\sum_{m \in A_i} g(m) \right)$$

Let $m \in M$ be simple concepts. Observe that we need this assumption for Definition 2.17 used below: In Definition 2.16 and Definition 2.17 the definition of $\tau(x, y)$ demands m to be simple. According to Proposition 2.8 g is an order reversing involution and $g(g(m)) = m$ but in this case m need not be simple. We obtain

$$\eta(\eta(\mu_\zeta)) = \eta(\mu_{g(\zeta)}) = \mu_{g(g(\zeta))} = \mu_\zeta$$

Therefore η is an involution. Here ζ is not necessary simple.

Let $\alpha \leq \beta$ be elements in EM , and since g is order reversing, $g(\alpha) \geq g(\beta)$. Using Definition 2.17 (1) it is $\mu_{g(\alpha)}(x) \leq \mu_{g(\beta)}(x)$. Therefore

$$\eta(\mu_\alpha) = \mu_{g(\alpha)} \geq \mu_{g(\beta)} = \eta(\mu_\beta)$$

and η is order reversing.

(Z6) $\mu_0(x) \neq \mu_1(x)$ for all $x \in X$

(Z7) The Kleene condition. For any function $\beta, \alpha \in [0, 1]^X$ and for any $x, y \in X$ we have

$$\mu_{\beta(y)} \vee \neg \mu_{\beta(y)} \geq \mu_{\beta(y)} \geq \mu_{\beta(x)} \geq \mu_{\alpha(x)} \geq \mu_{\alpha(x)} \wedge \neg \mu_{\alpha(x)}$$

which is proved in Lemma 3.3.

□

Proposition 3.5. Let

$$[0, 1]^X = \{\mu_\zeta \mid \mu_\zeta : X \longrightarrow [0, 1]\}$$

be a set of membership functions μ_ξ of the AFS fuzzy logic system $(EM, \vee, \wedge, ')$. The EM elements ζ are of the form

$$\zeta = \sum_{i \in I} (\prod_{m \in A_i} m)$$

Let R_m be binary relations of simple concepts m . If the condition

$$(x, z) \in R_m \implies (y, x) \in R_m$$

is valid for every simple concept $m \in A_i$, $i \in I$ and $x, y, z \in X$ then

(a) Functions μ_ζ form a chain corresponding to the chain $(\zeta)_{\zeta \in EM}$.

(b) Any chain $(\mu_\zeta)_{\zeta \in EM}$ constitutes some Zadeh algebra $([0, 1]^X, \vee, \wedge, \eta, \mu_0, \mu_1)$.

Proof. We conclude

(a) By Lemma 3.1 the elements ζ forms a chain $(\zeta)_{\zeta \in EM}$. By Definition 2.17 (1) $\zeta_k \leq \zeta_l \implies \mu_{\zeta_k} \leq \mu_{\zeta_l}$.

(b) Lemma 3.1 (a) implies that

$$(x, z) \in R_m \implies (y, z) \in R_m$$

and in Proposition 3.4 it is proved that every chain $(\mu_\zeta)_{\zeta \in EM}$ satisfies (Z1) – (Z7)

□

4 Conclutions

Simple concepts form chains. The elements of any chain form a "basis" in AFS fuzzy logic system $(EM, \vee, \wedge, ')$ with operations disjunction \vee , conjunction \wedge and the logical negation $'$. The elements are of the form $\sum_{i \in I} (\prod_{m \in A_i} m)$ where simple concepts $m \in A_i$ are defined in Definition 2.13 and operations in $(EM, \vee, \wedge, ')$ are defined in Proposition 2.7. (EM, \vee, \wedge) is a completely distributive lattice. By means of the binary relations $R_m \in X \times X$ defined in Definition 2.10 we construct the condition $(x, z) \in R_m \implies (y, x) \in R_m$ which implies the condition $(x, z) \in R_m \implies (y, z) \in R_m$. Here X is a non-empty set and x, y, z on X and m are simple concepts. Then the conditions constitute the two things: First, the membership functions $\mu_\zeta : X \longrightarrow [0, 1]$ of the fuzzy concepts ζ form chains $(\mu_\zeta)_{\zeta \in EM}$ in $(EM, \vee, \wedge, ')$. Second, every chain $(\mu_\zeta)_{\zeta \in EM}$ forms a Zadeh algebra. These results are represented in Propositions 3.4 and 3.5 and we can use them as starting points to continue theoretical considerations. The other way to continue the investigations is to utilize direct the conditions above: there are two kinds of successive events. The first one implies the second one or they have no connection. In the latter case the second event only follows the first one although they are independent of each other. In the first case it is possible to apply to the conditions (above) (1) or (2) represented in the introduction.

In Example 2.1, [3], the set $\{\mu \mid \mu : X \longrightarrow [0, 1]\}$ of membership functions forms a Zadeh algebra. More exactly, if $I = [0, 1]$ then (I, \vee, \wedge) is a complete distributive lattice. Further, $I^X = \{\mu \mid \mu : X \longrightarrow [0, 1]\}$ is the set of membership functions. The operations \vee and \wedge are extended pointwise on I^X . Now

$Z = \{I^X, \vee, \wedge, \neg, \mathbf{0}, \mathbf{1}\}$ is a Zadeh algebra with $\mathbf{0}(x) = 0$, $\mathbf{1}(x) = 1$, and the logical negation of μ is $\neg\mu(x) = (1 - \mu)(x)$. In this paper we have considered more general membership functions and constructed Zadeh algebras.

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