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This is a Final draft version of a publication
published by Elsevier
in International Journal of Heat and Mass Transfer

DOI: 10.1016/j.ijheatmasstransfer.2018.10.063

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Please cite the publication as follows:

Kiani Oshtorjani, M., Jalali, P. (2019). Thermal and hydraulic properties of sphere packings using a novel lattice Boltzmann model. International Journal of Heat and Mass Transfer, Vol. 130, p. 98-108. DOI: 10.1016/j.ijheatmasstransfer.2018.10.063

**This is a parallel published version of an original publication.
This version can differ from the original published article.**

Thermal and Hydraulic Properties of Sphere Packings using a Novel Lattice Boltzmann Model

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Abstract

A novel LB model is introduced to deal with the thermal and hydraulic properties in porous media. The pressure and density are separated in the proposed incompressible LB model and solid volume fraction is embedded in the formulations. Moreover, a modified thermal LB model is proposed that is shown to be more accurate than the Gue's type of models. This model treats the energy equation with the second order of accuracy without any additional terms. The model is employed in a packing of monodisperse spherical particles within a rectangular channel. The results of new model have been improved as compared to the analytical solution.

Keywords: Energy Equation, Heat Source Term, Porosity, Pressure Drop

2018 MSC:

1. Introduction

Interaction between solid and fluid widely affects the flow pattern in enormous industrial applications such as in porous media and fluidized beds. The solid volume fraction (SVF) ε_s is defined as the ratio of volume occupied by solid to the container volume. It appears in the governing
5 equations of flow and notably influences on the flow characteristics such as pressure drop, velocity components and temperature distribution.

There are two common strategies for modeling porous media. In the first approach, the solid zone is modeled as no-slip zone with all geometric complexities at surface and the fluid flows within

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Nomenclature		T	Temperature
$\bar{p}$	Averaged pressure	u_α	Fluid velocity
$\bar{u}$	Averaged velocity	δt	Lattice time step
η	Effective thermal conductivity	δx	Lattice length unit
η_s	Solid thermal conductivity	τ	Relaxation time
μ	Fluid dynamic viscosity	τ_T	Thermal relaxation factor
ν	Fluid kinematic viscosity	c_s	Sound speed
ρ_0, ρ_f	Fluid density	d_p	Particle diameter
ρ_s	Solid density	e_i	Discretized fluid particle velocity
σ	Heat capacitance	F_α	Momentum source term
ε	Fluid volume fraction	f_i	Distribution function
ε_s	Solid volume fraction (SVF)	f_i^{eq}	Equilibrium distribution function
C_p	Fluid specific heat	h_i	Thermal distribution functions
C_{ps}	Soild specific heat	h_i^{eq}	Thermal equilibrium distribution func- tions
k_m	Ratio of effective thermal conductivity per fluid heat capacitance	K	Hydraulic permeability of spherical particle packing
L	Length of porous medium		
p	Pressure	Q	Energy source term

the tortuous pore space. This approach provides local details of flow, however, randomly distributed
10 microscopic pores existed in the porous media make the utilization of this approach computationally
expensive. In this context, the exclusive geometry of a sample including all voids is needed which
can be provided by X-ray photo scanning [1].

Due to complexity of porous media in the microscopic (pore) scales, the alternative approach in
the so-called representative elementary volume (REV) scale is usually considered for the classical
15 studies [2]. In this scale, the effects of porosity is constructed by using randomly distributed particles
and by adding the fluid-particle drag relations to the Navier-Stokes equations. From modeling point
of view, an average SVF should be assigned to any REV by proper averaging over the pore network
within the REV. On the other hand, the drag force is added to the Navier-Stokes equations as a
source term that is a function of SVF. There have been proposed many local semiempirical relations
20 in the literature for the drag force [3, 4].

The particle packings can be generated via Lagrangian methods, however, the flow and tem-
perature fields are solved using either computational fluid dynamics (CFD), or any other methods
such as lattice Boltzmann method (LBM), or smooth particle hydrodynamics (SPH). Among them,
LBM has shown a powerful capacity to simulate porous media with either direct and REV scale
25 approaches. It is worth mentioning that there are many LB models proposed for adding source
terms in Navier-Stokes equations[5, 6, 7].

Guo and Zhao [2] have firstly employed LBM for simulation of incompressible flows through
porous media based on the REV scale approach. They added the pre-estimated drag force (Ergun
relation) raised from particle-fluid interaction as a source term. In their model, the static pressure
30 is connected to the density and porosity of medium using the relation $p = c_s^2 \rho / \varepsilon$. Therefore, a
constant density (incompressible flow) implies that pressure is only a function of pre-known porosity.
However, they have assumed that density does not have strictly a constant value $\rho \approx \rho_0$ and it
can vary slightly $\rho = \sum f_i$. The variation of density in incompressible flows affects the momentum
balance which leads to errors in calculation of pressure drop.

35 He and Lue [8] suggested an LB model (HL model) in which pressure is separated from density
and hence it works for incompressible flows. This model is employed [9] to study the solid-liquid
flows by adding a source term into collision operator representing the effects of SVF. However, it
cannot directly add the semiempirical relations as a source term.

On the other hand, the heat transfer equation is usually linked to the momentum equation by

one way coupling in incompressible flows. On the continuation of proposed model for flow simulation in porous media, Gou and Zhao [10] developed a thermal LB model which is applicable for the fluid saturated porous media. In other words, this model is based on the assumption of local thermal equilibrium (LTE) between solid and saturated fluid which makes a combined thermal governing equation as $\sigma \partial_t T + \nabla_\alpha (T u_\alpha) = \nabla_\alpha \kappa_m \nabla_\alpha T$, where, σ is a function of SVF ε_s .

Some earlier lattice Boltzmann studies without local thermal equilibrium assumption can be found in [11, 12, 13]. Yang et al. [11] investigated the randomness effects of porous media structures composed of either cubes or spheres with different sizes on the flow and heat transfer features. They have used D3Q19 LBM scheme to solve the flow field and D3Q6 scheme to study the thermal behavior of porous media. Wang et al. [12] have used three distribution functions, the one for flow and the two others for the thermal simulations of fluid and solid phases. They have used D2Q9 scheme to validate their model. Gao et al. [13] have extended Guo and Zhao [2] model to simulate porous media under local non-equilibrium thermal conditions. In their work, a three-distribution function LB model has been used to solve the fluid and temperature fields using D2Q9 scheme. In addition, a number of studies have been done in [14, 15, 16] considering local thermal non-equilibrium (LTNE) condition with either compressible or incompressible flows.

Unlike Gue's model, a single relation time (SRT) model, Liu and He [17] proposed a multiple relation time (MRT) model. They have argued that MRT models numerically make the algorithm more stable than the Bhatnagar-Gross-Krook (BGK) models. Considering simplicity and cost efficiency of the BGK model, Wang et al. [18] proposed a BGK model (WMG model) enhancing numerical stability at low viscosity and thermal diffusivity. In their model, the external force appears in both momentum and energy equations which links them.

Zheng et al. [19] proposed a model for covering convection diffusion equation in D3Q5 space. This model covers CDE with the second order of accuracy and without any additional terms. However, Chia and Zhao [20] mentioned that this model is not a local model and needs to use their neighborhood information in the collision process. They suggested to use a local scheme for calculation of gradients. Without considering the heat source term and porosity, this model is a special case of WMG model [18].

Chen et al. [21] have mentioned that one of the shortcomings in Guo's like models is associated with their non-reliability for the domains in which heat capacitance σ varies spatially. To overcome this difficulty, they have recently proposed a model (CYZ model) in which σ_0 , which is a constant

reference value of σ , is introduced to control heat diffusivity. Using this trick, σ can vary spatially in the investigating domain. One of the most serious problems of this model and other Guo's like thermal models is that they cannot cover the thermal governing equation with the second order of accuracy and without any additional terms. For instance, Chen et al. model covers the equation
75 of $\partial_t(\sigma T) + \nabla \cdot (Tu_\alpha) = \nabla \cdot (k_m \nabla T) + \delta \left(\tau_T - \frac{1}{2} \right) \partial_t \nabla_\alpha (Tu_\alpha) + O(\delta^2)$ which has the additional term of $\delta \left(\tau_T - \frac{1}{2} \right) \partial_t \nabla_\alpha (Tu_\alpha)$, or is covered with the first order of accuracy as $\partial_t(\sigma T) + \nabla \cdot (Tu_\alpha) = \nabla \cdot (k_m \nabla T) + O(\delta)$ (see section 4 for more details).

In this paper, an LB approach is presented for incompressible fluid flow interacting with a solid phase. In this model, density has no effects on the pressure and Ergun equation is added to the
80 momentum equation as a source term. Furthermore, a modified version of CYZ model for thermal LB is proposed which covers the energy equation with the second order of accuracy without any additional terms. Moreover, in the thermal model a heat source term is added.

2. Solid-Liquid Flow Problems

Nithiarasu et al. [22] have derived for the first time the revised formulation of momentum and
85 energy balance valid for porous media in the REV scales as [23, 24]:

$$\partial_\beta \bar{u}_\beta = 0 \quad (1)$$

$$\partial_t \bar{u}_\alpha + \bar{u}_\beta \partial_\beta \left(\frac{\bar{u}_\alpha}{\varepsilon} \right) = -\frac{\varepsilon}{\rho_0} \partial_\alpha (\bar{p}/\varepsilon) + \nu \partial_\beta \partial_\beta \bar{u}_\alpha + F_\alpha \quad (2)$$

$$\partial_t(\sigma T) + \partial_\beta (T \bar{u}_\beta) = \partial_\beta (k_m \partial_\beta T) + Q \quad (3)$$

where, $\bar{u}$ and $\bar{p}$ are averaged velocity and pressure, ρ_0 and ν are fluid density and kinematic viscosity, F_α and Q are source terms for momentum and energy equations, respectively. ε is fluid volume fraction, β is subscript for summation, and α denotes the direction.

It should be mentioned that the above-mentioned energy equation is obtained by combining
90 the energy equations corresponding to solid and fluid. Using this combination, we have $\sigma = \varepsilon + (1 - \varepsilon)(\rho C_p)_s / (\rho C_p)_f$ which is a spatial function of ε . In addition, effective heat diffusivity $k_m = \eta / (\rho C_p)_f$ is the ratio of effective thermal conductivity per fluid heat capacitance.

3. LB formulation for Navier-Stokes equations

In order to transfer the mass and momentum equations to lattice Boltzmann space, the modified
 95 standard Boltzmann equation is employed:

$$f_i(x + e_i \delta t, t + \delta t) - f_i(x, t) = \frac{f_i(x, t) - f_i^{eq}(x, t)}{\tau} + T_i \delta t = \Omega_i \quad (4)$$

where, e_i is the discretized fluid particle velocity, τ is the relaxation time, f is the distribution
 function, f_i^{eq} is the equilibrium distribution function, δx is the lattice length unit, and δt is the
 time step. Several relations for $T_i \delta t$ have been proposed for external forces appearing in NS equation.
 It also works as a correction for the pressure gradient (see section 3.1 for more details). During the
 100 derivation of Navier-Stokes equations from the LB equation using ChapmanEnskog expansion, the
 following constraints are used:

$$\sum T_i = 0, \quad \sum T_i e_{i\alpha} = n F_\alpha + n(1 - \varepsilon) \nabla p, \quad \sum T_i e_{i\alpha} e_{i\beta} = 0, \quad (5)$$

which according to them, the following relation have been used:

$$T_i = \frac{w_i n}{c_s^2} e_{id} F_d + \frac{w_i n}{c_s^2} (1 - \varepsilon) e_{id} \partial_d p \quad (6)$$

where, the dummy index d is used for getting summation over three dimensions, and c_s is the sound
 speed. In Eq. (6), $n = 1 - \frac{1}{2\tau}$, p is hydrostatic pressure, ε is the fluid VF, and w_i are the weighing
 105 numbers, which are presented below for three-dimensional grid with nineteen distinct velocities
 $D3Q19$:

$$\omega_i = [12 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]/36 \quad (7)$$

The discrete velocity vectors corresponded to the $D3Q19$ models have the following directions:

$$\begin{aligned} e_{ix} &= [0 \ 1 \ -1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ -1 \ -1 \ -1 \ -1 \ 0 \ 0 \ 0 \ 0] \\ e_{iy} &= [0 \ 0 \ 0 \ 1 \ -1 \ 0 \ 0 \ 1 \ -1 \ 0 \ 0 \ 1 \ -1 \ 0 \ 0 \ 1 \ 1 \ -1 \ -1] \\ e_{iz} &= [0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -1 \ 0 \ 0 \ 1 \ -1 \ 0 \ 0 \ 1 \ -1 \ 1 \ -1 \ 1 \ -1] \end{aligned} \quad (8)$$

It is assumed in BGK LBM models that the collisions of the particles will happen in a way that
 makes them to go toward an instantaneous relaxation (instantaneous equilibrium). Therefore, after
 110 the streaming and collision step, the equilibrium distribution function f_i^{eq} will be used to calculate

the macroscopic quantities. The equilibrium distribution function is proposed as:

$$f_i^{eq}(x) = w_i p + w_i c_s^2 \rho_0 \varepsilon \left[3 \frac{e_{i\alpha} \cdot u_\alpha}{c^2} + \frac{9}{2} \frac{(e_{i\alpha} \cdot u_\alpha)^2}{c^4} - \frac{3}{2} \frac{u_\alpha^2}{c^2} \right], \quad (9)$$

in which, ρ_0 is the constant fluid density, u_α is fluid velocity, and $c = \frac{\delta x}{\delta t}$. This relation is proposed to satisfy the following constraints at each time step:

$$p = \sum f_i^{eq} \quad (10)$$

$$c_s^2 \rho_0 \varepsilon u_\alpha = \frac{1}{2} c_s^2 F_\alpha \delta t + \sum e_{i\alpha} f_i^{eq} \quad (11)$$

$$c_s^2 \rho_0 \varepsilon u_\alpha u_\beta + c_s^2 p \delta_{\alpha\beta} = \sum e_{i\alpha} e_{i\beta} f_i^{eq} \quad (12)$$

Using the previous constraints, the velocity and pressure fields are obtained at each time step.

115 Using Chapman-Enskog expansion (see section 3.1), the model finally covers the Navier-Stokes equation as:

$$\begin{aligned} \frac{\partial(\rho_0 \varepsilon u_\alpha)}{\partial t} + \frac{\partial(\rho_0 \varepsilon u_\alpha u_\beta)}{\partial x_\beta} = & -\varepsilon \frac{\partial p}{\partial x_\alpha} + \mu \frac{\partial}{\partial x_\beta} \{ \partial x_\beta (\varepsilon u_\alpha) + \partial x_\alpha (\varepsilon u_\beta) + \partial x_\gamma (\varepsilon u_\gamma) \delta_{\alpha\beta} \} + F_\alpha \\ & - \nabla \cdot \partial x_\gamma (u_\alpha u_\beta u_\gamma) \end{aligned} \quad (13)$$

where, the dynamic viscosity is $\mu = c_s^2 \rho_0 (\tau - \frac{1}{2} \delta)$.

3.1. Chapman-Enskog Expansion

Starting from the LB formulation, the Navier-Stokes equations are derived in this section using
120 Chapman-Enskog expansion. Considering the following equation:

$$f_i(x + e_i \delta t, t + \delta t) - f_i(x, t) = \frac{f_i(x, t) - f_i^{eq}(x, t)}{\tau} + T_i \delta t = \Omega_i \quad (14)$$

and applying Taylor series expansion, Eq. (15) is obtained:

$$\delta D_i f_i + \frac{\delta}{2} D_i^2 f_i + O(\delta^3) = \Omega_i \quad (15)$$

where, $D_i = \partial_t + e_{i\beta} \frac{\partial}{\partial x_\beta}$, and δ is either δ_t or δ_x depended on which term it is multiplied.

In the next step, the distribution function as well as derivative operators are decomposed in the scales of $O(\lambda^0)$, $O(\lambda^1)$, and $O(\lambda^2)$ as:

$$\begin{aligned}
f_i &= f_i^{(0)} + \lambda f_i^{(1)} + \lambda^2 f_i^{(2)} \\
\partial_t &= \lambda \partial_{1t} + \lambda^2 \partial_{2t} \\
\partial_x &= \lambda \partial_{1x} \\
\Omega_i &= \Omega_i^{(0)} + \lambda \Omega_i^{(1)} + \lambda^2 \Omega_i^{(2)} \\
T_{1i} &= \lambda T_i
\end{aligned} \tag{16}$$

125 and accordingly $D_{1i} = \lambda D_i$.

Substituting operators and variables defined in Eq. (16), into Eq. (15), and separating different scales of lambda λ , the following equations are obtained:

$$O(\lambda^0) : \quad f_i^{(0)} = f_i^{(eq)} \tag{17}$$

$$O(\lambda^1) : \quad D_{1i} f_i^{(0)} = -\frac{1}{\tau \delta t} f_i^{(1)} + T_{1i} \tag{18}$$

$$O(\lambda^2) : \quad \partial_{2t} f_i^{(0)} + (1 - \frac{1}{2\tau}) D_{1i} f_i^{(1)} = -\frac{1}{\tau \delta t} f_i^{(2)} + \frac{\delta t}{2} D_{1i} T_{1i} \tag{19}$$

Getting summation over Eqs. (18) and (19) for the discrete directions, i.e., $i = 1 : 19$, we have:

$$\partial_{1t} \sum f_i^{(0)} + \partial_{1x} \sum e_{ix} f_i^{(0)} = -\frac{1}{\tau \delta t} \sum f_i^{(1)} + \sum T_{1i} \tag{20}$$

$$\begin{aligned}
\partial_{2t} \sum f_i^{(0)} + (1 - \frac{1}{2\tau}) (\partial_{1t} \sum f_i^{(1)} + \partial_{1x} \sum e_{ix} f_i^{(1)}) &= -\frac{1}{\tau \delta t} \sum f_i^{(2)} + \frac{\delta t}{2} (\partial_{1t} \sum T_{1i} + \\
&\quad \partial_{1x} \sum e_{ix} T_{1i})
\end{aligned} \tag{21}$$

By the following definitions, the above equation will be more simplified. The definitions are:

$$\begin{aligned}
\sum f_i^{(0)} &= p, \quad \sum f_i^{(1)} = 0, \quad \sum f_i^{(2)} = 0 \\
\sum e_{i\alpha} f_i^{(0)} &= c_s^2 \rho_0 \varepsilon u_\alpha, \quad \sum e_{i\alpha} f_i^{(1)} = m F_{1\alpha} \delta t + m(1 - \varepsilon) \partial_\alpha p \delta t, \quad \sum e_{i\alpha} f_i^{(2)} = 0 \\
\sum e_{i\alpha} e_{i\beta} f_i^{(0)} &= c_s^2 \rho_0 \varepsilon u_\alpha u_\beta + c_s^2 p \delta_{\alpha\beta}, \quad \sum e_{i\alpha} e_{i\beta} f_i^{(1)} = \sum e_{i\alpha} e_{i\beta} f_i^{(2)} = 0 \\
\sum T_{1i} &= 0, \quad \sum e_{i\alpha} T_{1i} = n F_{1\alpha} + n(1 - \varepsilon) \partial_\alpha p, \quad \sum e_{i\alpha} e_{i\beta} T_{1i} = 0
\end{aligned} \tag{22}$$

130 Therefore, Eq. (20) is simplified to $\partial_{1t} \sum f_i^{(0)} + \partial_{1x} \sum e_{ix} f_i^{(0)} = 0$ and Eq. (21) to $\partial_{2t} \sum f_i^{(0)} = (\frac{1}{2}n - (1 - \frac{1}{2\tau})m)(\partial_{1x} F_{1x} + (1 - \varepsilon)\partial_\alpha p)$. Thus, assuming $n = 2(1 - \frac{1}{2\tau})m$ and adding two simplified equations with their coefficients of λ and λ^2 yields the following equation:

$$(\lambda\partial_{1t} + \lambda^2\partial_{2t}) \sum f_i^{(0)} + \lambda\partial_{1x} \sum e_{ix} f_i^{(0)} = 0 \quad (23)$$

Consequently, using the definitions of (16) and (22), Eq. (23) is simplified to:

$$\frac{1}{c_s^2} \partial_t(p) + \partial_x(\rho_0 \varepsilon u_\alpha) = 0 \Rightarrow \partial_x(\rho_0 \varepsilon u_\alpha) \approx 0 \quad (24)$$

135 On the other hand, by multiplying Eqs. (18) and (19) by $e_{i\alpha}$ and getting summation over the discrete directions, the following equations are derived:

$$\partial_{1t} \sum e_{i\alpha} f_i^{(0)} + \partial_{1\beta} \sum e_{i\alpha} e_{i\beta} f_i^{(0)} = -\frac{1}{\tau\delta t} \sum e_{i\alpha} f_i^{(1)} + \sum e_{i\alpha} T_{1i} \quad (25)$$

$$\begin{aligned} \partial_{2t} \sum e_{i\alpha} f_i^{(0)} + (1 - \frac{1}{2\tau})(\partial_{1t} \sum e_{i\alpha} f_i^{(1)} + \partial_{1\beta} \sum e_{i\alpha} e_{i\beta} f_i^{(1)}) = & -\frac{1}{\tau\delta t} \sum e_{i\alpha} f_i^{(2)} + \\ & \frac{\delta t}{2} (\partial_{1t} \sum e_{i\alpha} T_{1i} + \partial_{1\beta} \sum e_{i\alpha} e_{i\beta} T_{1i}) \end{aligned} \quad (26)$$

By taking $n = 2(1 - \frac{1}{2\tau})m$ as it is already assumed, $(1 - \frac{1}{2\tau}) \sum e_{i\alpha} f_i^{(1)} = \frac{\delta t}{2} \sum e_{i\alpha} T_{1i}$. Then, using this relation and the definitions presented in Eq. (22), Eq. (26) is simplified as:

$$\partial_{2t} \sum e_{i\alpha} f_i^{(0)} + (1 - \frac{1}{2\tau}) \partial_{1\beta} \sum e_{i\alpha} e_{i\beta} f_i^{(1)} = \frac{\delta t}{2} \partial_{1\beta} \sum e_{i\alpha} e_{i\beta} T_{1i} \quad (27)$$

Considering Eq. (18) and multiplying it by $e_{i\alpha} e_{i\beta}$ the following relation is obtained:

$$-\frac{1}{\tau} \partial_{1\beta} \sum e_{i\alpha} e_{i\beta} f_i^{(1)} + \delta t \partial_{1\beta} \sum e_{i\alpha} e_{i\beta} T_{1i} = \delta t \partial_{1\beta} \left(\partial_{1t} \sum e_{i\alpha} e_{i\beta} f_i^{(0)} + \partial_{1\gamma} \sum e_{i\alpha} e_{i\beta} e_{i\gamma} f_i^{(0)} \right) \quad (28)$$

Using definitions of Eq. (22), in which $\sum e_{i\alpha} e_{i\beta} T_{1i} = 0$, and combining Eq. (27) and (28) yield:

$$\partial_{2t} \sum e_{i\alpha} f_i^{(0)} = \tau(1 - \frac{1}{2\tau}) \delta t \partial_{1\beta} \left(\partial_{1t} \sum e_{i\alpha} e_{i\beta} f_i^{(0)} + \partial_{1\gamma} \sum e_{i\alpha} e_{i\beta} e_{i\gamma} f_i^{(0)} \right) \quad (29)$$

140 Multiplying Eq. (25) by λ and Eq. (29) by λ^2 and adding them results to below equation:

$$(\lambda\partial_{1t} + \lambda^2\partial_{2t}) \sum e_{i\alpha} f_i^{(0)} + \lambda\partial_{1\beta} \sum e_{i\alpha} e_{i\beta} f_i^{(0)} = \lambda^2\tau(1 - \frac{1}{2\tau})\delta t \partial_{1\beta} \left(\partial_{1t} \sum e_{i\alpha} e_{i\beta} f_i^{(0)} + \partial_{1\gamma} \sum e_{i\alpha} e_{i\beta} e_{i\gamma} f_i^{(0)} \right) + \lambda(-\frac{m}{\tau\delta t} + n)(F_{1\alpha} + (1 - \varepsilon)\partial_\alpha p) \quad (30)$$

By finding the values of $m = c_s^2\delta t/2$ and $n = c_s^2(1 - 1/2\tau)$ which satisfy both relations of $n = 2(1 - \frac{1}{2\tau})m$ and $-\frac{m}{\tau\delta t} + n = c_s^2$, and by referring to the definitions of Eq. (16), Eq. (30) is then simplified to:

$$\partial_t \sum e_{i\alpha} f_i^{(0)} + \partial_\beta \sum e_{i\alpha} e_{i\beta} f_i^{(0)} = (\tau - \frac{1}{2})\delta t \partial_{1\beta} \left(\partial_{1t} \sum e_{i\alpha} e_{i\beta} f_i^{(0)} + \partial_{1\gamma} \sum e_{i\alpha} e_{i\beta} e_{i\gamma} f_i^{(0)} \right) + c_s^2 F_\alpha + c_s^2(1 - \varepsilon)\partial_\alpha p \quad (31)$$

145 Considering the definitions presented in Eq. (22) and the equality of $\partial_{1t} \sum e_{i\alpha} e_{i\beta} f_i^{(0)} = -\partial_{1\gamma} u_\alpha u_\beta u_\gamma$, we have:

$$\begin{aligned} \partial_t(c_s^2\rho_0\varepsilon u_\alpha) + \partial_\beta(c_s^2\rho_0\varepsilon u_\alpha u_\beta) &= -\varepsilon c_s^2\partial_\alpha p + \nu\partial_\beta \left(\partial_\beta(c_s^2\rho_0\varepsilon u_\alpha) + \partial_\alpha(c_s^2\rho_0\varepsilon u_\beta) \right. \\ &\quad \left. + \partial_\gamma(c_s^2\rho_0\varepsilon u_\gamma)\delta_{\alpha\beta} \right) + c_s^2 F_\alpha - \partial_\beta(\partial_\gamma u_\alpha u_\beta u_\gamma) \end{aligned} \quad (32)$$

where, $\nu = c_s^2(\tau - \frac{1}{2})\delta t$. By dividing the above equation by c_s^2 and neglecting the last right hand side term, which is very small, the momentum equations are derived as:

$$\partial_t(\rho_0\varepsilon u_\alpha) + \partial_\beta(\rho_0\varepsilon u_\alpha u_\beta) = -\varepsilon\partial_\alpha p + \nu\partial_\beta \left(\partial_\beta(\rho_0\varepsilon u_\alpha) + \partial_\alpha(\rho_0\varepsilon u_\beta) + \partial_\gamma(\rho_0\varepsilon u_\gamma)\delta_{\alpha\beta} \right) + F_\alpha \quad (33)$$

150 It should be highlighted here that u_α and p are fluid velocity and pressure which are related to averaged velocity and pressure as $\bar{u}_\alpha = \varepsilon u_\alpha$, and $\bar{p} = \varepsilon p$. Using these definitions and considering continuity equation, Eq. (33) will be alternatively reformed as:

$$\partial_t \bar{u}_\alpha + \bar{u}_\beta \partial_\beta \left(\frac{\bar{u}_\alpha}{\varepsilon} \right) = -\frac{\varepsilon}{\rho_0} \partial_\alpha (\bar{p}/\varepsilon) + \nu \partial_\beta \left(\partial_\beta(\rho_0 \bar{u}_\alpha) + \partial_\alpha(\rho_0 \bar{u}_\beta) + \partial_\gamma(\rho_0 \bar{u}_\gamma)\delta_{\alpha\beta} \right) + F_\alpha \quad (34)$$

4. LB formulation for energy equation

For covering the energy equation of (3), the following LB equation is proposed:

$$h_i(x + e_i\delta t, t + \delta t) = \chi h_i(x, t) + (1 - \chi)h_i(x + e_i\delta t, t) + \frac{h_i(x, t) - h_i^{eq}(x, t)}{\tau_T} + Q_i\delta t \quad (35)$$

in which, h_i^{eq} and h_i denote equilibrium and non-equilibrium distribution functions, respectively, τ_T is thermal relaxation factor, and Q_i is the heat source term. The equilibrium distribution function
155 h_i^{eq} will be calculated using the following relation at each timer step:

$$h_i^{eq}(x) = \begin{cases} T(\sigma - \sigma_0) + w_i T(\sigma_0 + \frac{e_{i\alpha} u_\alpha}{c_s^2 \chi}) & i = 0 \\ w_i T(\sigma_0 + \frac{e_{i\alpha} u_\alpha}{c_s^2 \chi}) & i \neq 0 \end{cases} \quad (36)$$

Moreover, for having the thermal source term in the $D3Q19$ framework, the following equation (Eq. 37) is proposed. This relation is obtained by considering a time independent thermal source term and assuming $\sum Q_i = Q$ which guarantees the full achievement of the energy equation Eq. (3) by using Chapman-Enskog expansion as presented in section 4.1.

$$Q_i = Q \frac{10w_i - c_s^2}{10 - 19c_s^2} \quad (37)$$

160 4.1. Chapman-Enskog Expansion

Considering the following equation:

$$h_i(x + e_i \delta t, t + \delta t) = \chi h_i(x, t) + (1 - \chi) h_i(x + e_i \delta t, t) + \frac{h_i(x, t) - h_i^{eq}(x, t)}{\tau_T} + Q_i \delta t \quad (38)$$

and applying Taylor series expansion in it, we have:

$$\delta D_i h_i + \frac{\delta^2}{2} D_i^2 h_i = (1 - \chi) (\delta(e_i \cdot \nabla) h_i + \frac{\delta^2}{2} (e_i \cdot \nabla)^2 h_i) + \frac{h_i(x, t) - h_i^{eq}(x, t)}{\tau_T} + Q_i \delta t \quad (39)$$

Therefore, the following relations are obtained using the definitions in (16):

$$O(\lambda^0) : \quad h_i^{(0)} = h_i^{(eq)} \quad (40)$$

$$O(\lambda^1) : \quad (\partial_{1t} + \chi e_i \cdot \nabla_1) h_i^{(0)} = -\frac{1}{\tau \delta} h_i^{(1)} + Q_{1i} \quad (41)$$

$$O(\lambda^2) : \quad \partial_{2t} h_i^{(0)} + (1 - \frac{1}{2\tau}) D_{1i} h_i^{(1)} + (\chi - 1) (\delta(e_i \cdot \nabla_1) h_i^{(1)} + \frac{\delta^2}{2} (e_i \cdot \nabla_1)^2 h_i^{(1)}) = -\frac{1}{\tau \delta} h_i^{(2)} \quad (42)$$

Substituting $h_i^{(1)}$ from Eq. (41) into Eq. (42) and considering a time independent thermal
 165 source term, the following relation is achieved:

$$O(\lambda^2) : \quad \partial_{2t} h_i^{(0)} + \delta \left((-\tau_T + \frac{1}{2}) \partial_{1t}^2 + (\frac{\chi}{2} - \tau_T \chi^2) (e_i \cdot \nabla_1)^2 + \right. \\ \left. (-2\chi\tau_T + 1) \partial_{1t} e_i \cdot \nabla_1 \right) h_i^{(0)} = -\frac{1}{\tau \delta t} h_i^{(2)} \quad (43)$$

Getting summation over the summation of Eqs. (40) and (43) as $\sum (\lambda O(\lambda) + \lambda^2 O(\lambda^2))$ and using the definitions in (16), the final form of equation is obtained as:

$$\partial_t \sum h_i^{(0)} + \chi \nabla \cdot \sum e_{ix} h_i^{(0)} + \delta \left((-\tau_T + \frac{1}{2}) \partial_t (\partial_t \sum h_i^{(0)}) + (\frac{\chi}{2} - \tau_T \chi^2) \nabla \cdot \nabla (\sum e_i e_i h_i^{(0)}) \right. \\ \left. + (-2\chi\tau_T + 1) \partial_t \nabla \cdot (\sum e_i h_i^{(0)}) \right) + O(\delta^2) = \sum Q_i \quad (44)$$

By making the following definitions:

$$\sum h_i^{(0)} = \sigma T, \quad \sum h_i^{(0)} e_{i\alpha} = \frac{T u_\alpha}{\chi} \quad \sum h_i^{(0)} e_{i\alpha} e_{i\beta} = c_s^2 \sigma_0 T \delta_{\alpha\beta} \quad (45)$$

we obtain:

$$\partial_t (\sigma T) + \nabla_\alpha (T u_\alpha) + \delta \left((-\tau_T + \frac{1}{2}) \partial_t (\partial_t (\sigma T) + (\frac{\chi}{2} - \tau_T \chi^2) \nabla \cdot \nabla (c_s^2 \sigma_0 T \delta_{\alpha\beta})) \right. \\ \left. + \frac{(-2\chi\tau_T + 1)}{\chi} \partial_t \nabla_\alpha (T u_\alpha) \right) + O(\delta^2) = Q \quad (46)$$

170 which can be rewritten as:

$$\partial_t (\sigma T) + \nabla_\alpha (T u_\alpha) + \delta \left((-\tau_T + \frac{1}{2}) \partial_t (\partial_t (\sigma T) + \nabla_\alpha (T u_\alpha)) + (\frac{\chi}{2} - \tau_T \chi^2) \nabla \cdot \nabla (c_s^2 \sigma_0 T \delta_{\alpha\beta}) \right. \\ \left. + (\frac{(-2\chi\tau_T + 1)}{\chi} + (\tau_T - \frac{1}{2})) \partial_t \nabla_\alpha (T u_\alpha) \right) + O(\delta^2) = Q \quad (47)$$

According to Eq. (47), the following relations are valid:

$$\partial_t (\sigma T) + \nabla_\alpha (T u_\alpha) + O(\delta) = Q \quad (48)$$

$$\partial_t (\sigma T) + \nabla_\alpha (T u_\alpha) = \delta (\tau_T \chi^2 - \frac{\chi}{2}) c_s^2 \nabla \cdot (\sigma_0 \nabla T) + \delta \left(\frac{2\chi\tau_T - 1}{\chi} - (\tau_T - \frac{1}{2}) \right) \partial_t \nabla_\alpha (T u_\alpha) + O(\delta^2) = Q \quad (49)$$

It should be mentioned here that by setting $\beta = 1$ the CYZ model is obtained in which there is an additional term of $\delta \left(\frac{2\chi\tau_T-1}{\chi} - (\tau_T - \frac{1}{2}) \right) \partial_t \nabla_\alpha (Tu_\alpha)$ which makes the equation to be covered with the first order of accuracy $O(\delta)$. However, by equating $\frac{2\chi\tau_T-1}{\chi} - \tau_T + \frac{1}{2} = 0$, χ is obtained as: $\chi = 1/(0.5 + \tau_T)$ which covers the equation with the second order of accuracy $O(\delta^2)$ and without any additional term as:

$$\partial_t(\sigma T) + \nabla \cdot (Tu_\alpha) = \nabla \cdot (k_m \nabla T) + Q \quad (50)$$

where, $k_m = \delta c_s^2 (\tau_T \beta^2 - \frac{\beta}{2}) \sigma_0$.

Finally, based on the definitions presented in Eq. (45), the equilibrium distribution function is proposed as:

$$h_i^{eq}(x) = \begin{cases} T(\sigma - \sigma_0) + w_i T(\sigma_0 + \frac{e_{i\alpha} u_\alpha}{c_s^2 \chi}) & i = 0 \\ w_i T(\sigma_0 + \frac{e_{i\alpha} u_\alpha}{c_s^2 \chi}) & i \neq 0 \end{cases} \quad (51)$$

5. Porous Media

There are generally three different flow regimes occurring in the porous media, namely the Darcy, Forchheimer, and turbulent regimes. Darcy regime refers to the regime of creeping flow, and the pressure gradient is proportional to dynamic viscosity and flow velocity. The coefficient of proportionality is the inverse of hydraulic permeability. The pressure drop across a porous medium made of spherical particles is calculated as:

$$\frac{\Delta P}{Lu} = \frac{\mu}{K} \quad (52)$$

where, L is the length of porous medium, u is velocity, μ is dynamic viscosity, $K = \frac{\varepsilon^3 d_p^2}{36\kappa(1-\varepsilon)^2}$ denotes the hydraulic permeability of spherical particle packing, ε is the porosity of the packing, and d_p is particle diameter.

On the other hand, the flow is still laminar in Forchheimer flow regime, however, not creeping. It is considered as the transient regime from the Darcy to turbulent regime in which the pressure drop is proposed by the formula given in Eq. (53).

$$\frac{\Delta P}{Lu} = \frac{\mu}{K} + \frac{\rho F}{\sqrt{K}} |u| \quad (53)$$

in which, $F = 1.75/\sqrt{150\varepsilon^3}$ represents the form drag.

5.1. Results and Discussion

The domain of study is shown in Fig. (1) which its dimensions are $35 \times 35 \times 210 \text{ mm}^3$. The velocity boundary condition is applied for the right boundary (inlet), where the left side is set to pressure outlet boundary condition. Other boundaries are considered as walls with no-slip boundary condition. The flowing fluid is air with density and kinematic viscosity of $\rho_f = 1.225 \text{ kg/m}^3$ and $\nu_f = 1.84 \times 10^{-5} \text{ m}^2/\text{s}$. The porous material is chosen as copper with density, specific heat, and thermal conductivity of $\rho_s = 8978$, $C_{ps} = 381 \text{ J/(kg.k)}$, and $\eta_s = 387.6 \text{ W/(m.k)}$, respectively.

The effects of porosity is usually modeled by considering particles randomly distributed in a container. The drag force acting on particles has a local semiempirical equation which includes the VF of solid and fluid. For any number of generated particles (with a certain SVF), the size of averaging control volume for SVF is taken as 10-15 LB nodes.

For the sake of validation of LBM results, they are compared to those obtained from theoretical approach as well as the results of commercial package. In this concern, ANSYS-FLUENT v.17.0 is used which works based on finite volume method (FVM). The settings of the model are made to handle laminar flow coupled with energy equation. The domain is defined as a porous medium through cell zone conditions. Then the porous zone permeability is introduced in the model. A heat source term in energy equation is also introduced. Note that the geometry and all other settings, including the air properties, the porous media properties, the value of heat source term, the initial and boundary conditions are set the same as in the LBM.

The particles embedded in the domain are depicted in Fig. (1) which is sketched for SVF of $\varepsilon_s = 0.2$. In addition, the slices across solid zone at the middle of domain are illustrated in Figs. (2a-2d) for different SVFs.

5.2. Pressure drop

In order to verify the suitability of the grid used in this study, we have examined different grid resolutions. The physical conditions are considered to be same for all grids. The length of porous medium is set to $L = 0.21 \text{ m}$, inlet velocity as $u = 0.3 \text{ m/s}$, and its porosity was varied between 0 and 0.25 in different cases. In order to have a reasonable comparison, the pressures drop through the porous media of different grids are plotted in Fig. (3) versus dimensionless length at the same time step. According to this figure, the grid corresponding to $40 \times 40 \times 210$ is appropriate for taking pressure drop values.

Figure 1: Illustration of porous media with $\varepsilon_s=0.2$

(a)

(b)

(c)

(d)

Figure 2: Slice planes at the middle of domain for different solid VFs (a) $\varepsilon=0.05$, (b) $\varepsilon=0.1$ (c) $\varepsilon=0.2$ (d) $\varepsilon=0.25$.

As the first validation case, the calculated pressure drops are compared to Forchheimer equation in Fig. (4). The validation study is performed for the SVF values of 0.05, 0.1, 0.15, 0.2 and 0.25. The inlet velocity is set to $u = 0.3m/s$ and gravity is activated. Figure (4) shows that the LB has predicted the pressure drop values closely to the Forchheimer equation. The smallest difference between the LB solution and the Forchheimer equation for pressure drop corresponds to the highest value of SVF, that is $\varepsilon_s = 0.25$. This is due to the implementation of Ergun equation in the LB code which predicts pressure drop more accurately in higher packing fractions. LB results slightly shift above the Forchheimer prediction over the porosity of 0.8 such that the LB prediction becomes about 10 Pa, which is twice as the prediction of Eq. (53) at the porosity of 0.05. It should be noted that the predictions of FVM and LB for total pressure drop nearly coincide. Thus the data related to the FVM results are not shown in Fig. 4. Instead, the differences of the FVM and LB can be observed in the profiles of pressure, which is discussed in Fig. 5 as follows.

The pressure drop profiles from the LB and FVM are compared in two different SVFs and velocities in Figs. (5a-5d), which display good agreement with each other. As anticipated, higher velocity and SVF lead to higher pressure drop through porous medium. It can be noted that the LB profile is slightly shifted upward, while the total pressure drop remains the same for the two approaches. The amount of the shift hardly reaches to 2 Pa. At the SVF of 0.2, increasing the velocity from 0.1 to 0.3 m/s boosts the shift from 0.5 to 2 Pa though the total pressure drop rises about 4 times. At the SVF of 0.25, similar ratio of pressure drop can be observed between the two velocities, while the shift remains below 2 Pa.

5.3. Temperature Distribution

Considering a one way coupling between momentum and energy equations, temperature field is solved. The materials are chosen as air and copper for the fluid and solid phases, respectively. The initial temperature of porous media is set to $400K$, whereas the inlet fluid temperature is $300K$. The walls are insulated and there is not any temperature gradient at the outlet. The inlet velocity and SVF are set as $u = 0.3m/s$ and $\varepsilon_s = 0.2$, respectively. The heat source term is activated and its value is $5 \times 10^8 w/m^3$, which is applied in two possible ways as follows.

In the first way of applying heat source, the corresponding term is applied over the entire domain. For the sake of comparison of CYZ and modified models, the same problem is analytically solved as presented in Appendix A. Therefore, both models are compared to analytical solution with the

Figure 3: Grid study for $20 \times 20 \times 120$ (long dashed line), $30 \times 30 \times 180$ (dot dashed line), $40 \times 40 \times 240$ (solid line), $50 \times 50 \times 300$ (dashed line).

Figure 4: Comparison of pressure drop obtained with LB (circular symbols) and Forchheimer equation, Eq 53 (solid line) for the velocity of $u=0.3\text{m/s}$.

(a)

(b)

(c)

(d)

Figure 5: Comparison of pressure drop obtained with LB model (solid line) and FVM (dashed line), (a) $\varepsilon_s=0.2$, $u=0.1\text{m/s}$ (b) $\varepsilon_s=0.25$, $u=0.1\text{m/s}$ (c) $\varepsilon_s=0.2$, $u=0.3\text{m/s}$ (d) $\varepsilon_s=0.25$, $u=0.3\text{m/s}$.

relative error defined as:

$$Error = 100 \frac{|T_{LB} - T_{Theroy}|}{T_{Theroy}} \quad (54)$$

The temperature distribution errors of CYZ and modified LB models are compared for different SVF values in Figs. (6a-6d). As these figures indicate, the modified model has smaller error than the CYZ model especially at the front side of porous domain where the temperature rapidly increases. In the rest of porous domain, the temperature remains approximately uniform as Fig. (7) reveals and thus the errors coincide in that region. Note that higher SVF yields smaller relative errors.

In the second way of applying heat source, the related term is only applied in the cells located within the thin layer from $X = 0.23L$ to $X = 0.27L$. The temperature distribution obtained by the modified model is depicted in Fig. (8) in various times. Moreover, the temperature distribution within porous medium is compared to the results of the CYZ model as well as the finite volume method (FVM) in Fig. (9a). In addition, the differences between the results of either CYZ model or FVM and the modified LB model are defined as the errors depicted in Fig. (9b).

According to Fig. (9a), FVM could not capture the temperature gradient existed in the entrance region of porous medium. Its relative error with respect to the modified LB model is about 29 percent in this region. As Fig. (9b) indicates, the maximum difference between the CYZ and the modified LB models is about 2 percent at the location of the heat source. In contrast, the FVM relative error at the same position is about 5 percent. In the rest of the porous domain, the temperature is predicted to be constant equal to the initial temperature by all three methods.

(a)

(b)

(c)

(d)

Figure 6: The temperature error defined as a difference between CYZ (dot-dashed line) or modified model (solid line) and theoretical solution for different VFs (a) $\varepsilon_s=0.05$ (b) $\varepsilon_s=0.1$ (c) $\varepsilon_s=0.2$ (d) $\varepsilon_s=0.25$.

Figure 7: The cross-sectional mean temperature variation from the LB solution along the porous medium for the case in which source term is applied over the entire domain for different times as 0, 98, 196, 294, 392, 490, 589, 687, 785, 883, 981 μs . The lowest line with $T = 400K$ corresponds to $t = 0$ and the successive lines upward represent the other times.

Figure 8: The cross-sectional mean temperature variation from the LB solution along the porous medium for the case in which source term is applied in the cells located within the thin layer from $X = 0.23L$ to $X = 0.27L$. Different lines correspond to the times 0, 98, 196, 294, 392, 490, 589, 687, 785, 883, 981 μs . The lowest line with $T = 400K$ corresponds to $t = 0$ and the successive lines upward represent the other times.

Figure 9: Solution and relative errors corresponded with case 2 for different methods (a) Comparison of CYZ model (dot-dashed line), new LB model (solid line), and FVM (dashed line) (b) Relative error between CYZ and new LB models (solid line), and between FVM and new LB model (dashed line).

6. Conclusion

The pressure drop calculated by a proposed LB model is validated by analytical formulas and FVM method. In addition, the thermal LB model is introduced to cover the energy equation with the second order of accuracy and without any additional terms. The higher accuracy of the proposed model with respect to other Gue's like models is verified by comparing the relative errors of the new model and the CYZ model. Meanwhile, the proposed heat source term is tested and validated in two different ways. The results presented in this paper confirms that the modified LB model introduced here is capable of capturing the hydrodynamics and thermal features of porous media made of monodisperse spheres. In principle, this method can be extended to any other particle packings made of particles of different shape, size and packing properties, which will constitute future extension of this study.

Appendix A: Analytical Solution of Energy Equation

Here, the energy equation Eq. (3) is analytically solved. The heat conductivity of particles is high in our studied porous media that is consistent with that of copper. Therefore, the cross-

sectional variation of temperature can be considered negligible. Moreover, the mean fluid velocity component within any cross-section perpendicular to the flow direction is negligible due to the side boundary conditions and the fact that the solid phase particles are uniformly distributed. Therefore, noting that the temperature is not varying within any cross-section, the temperature gradient is considered to be negligible in all directions perpendicular to the flow direction. Consequently, the energy equation can be rewritten as:

$$\hat{\sigma} \partial_t T + \hat{u}_x \partial_x T = k_m \partial_x^2 T + Q \quad (\text{A1})$$

where, $\hat{u}$ and $\hat{\sigma}$ are averaged velocity and sigma over cross-section area of porous media. The initial and boundary conditions of the problem are as below:

$$\begin{aligned} T(x, 0) &= T_0 \\ T(0, t) &= T_i \\ \partial_x T(\infty, t) &= 0 \end{aligned} \quad (\text{A2})$$

Using dimensionless parameters of $\xi = \hat{u}x/k_m$, $\zeta = \hat{u}^2 t / (\hat{\sigma} k_m)$, and $\theta = \frac{T - T_0}{T_i - T_0}$, Eq. (A1) reforms as:

$$\partial_\zeta \theta + \partial_\xi \theta = \partial_\xi^2 \theta + Q_d \quad (\text{A3})$$

where, $Q_d = \frac{Q k_m}{\hat{u}^2 (T_i - T_0)}$. This equation is simplified by introducing the solution of Eq. (A3) as $\theta(\xi, \zeta) = e^{(2\xi - \zeta)/4} K(\xi, \zeta)$ [25]. Consequently, the simplified version of energy equation is presented below:

$$\partial_\zeta K = \partial_\xi^2 K + Q_d e^{(\zeta - 2\xi)/4} \quad (\text{A4})$$

with initial and boundary conditions of:

$$\begin{aligned} K(\xi, 0) &= 0 \\ K(0, \zeta) &= 1 \\ \partial_\xi K(\infty, \zeta) + K(\infty, \zeta)/2 &= 0 \end{aligned} \quad (\text{A5})$$

The above equation is solved by applying Laplace transform technique. Getting Laplace transform over ζ and considering initial condition of problem, κ which is the Laplace transition of K is

obtained as:

$$\kappa = C_1 e^{\xi\sqrt{s}} + C_2 e^{-\xi\sqrt{s}} + Q_d \frac{e^{-\xi/2}}{(s - 1/4)^2} \quad (\text{A6})$$

where, C_1 and C_2 are constant to be determined by applying boundary conditions. Considering boundary conditions, κ is rewritten as:

$$\kappa = \frac{e^{-\xi\sqrt{s}}}{s - 1/4} - Q_d \frac{e^{-\xi\sqrt{s}}}{(s - 1/4)^2} + Q_d \frac{e^{-\xi/2}}{(s - 1/4)^2} \quad (\text{A7})$$

Finally, K is obtained by getting inverse Laplace transform of κ as (see Ref. [26] for advanced inverse Laplace transforms):

$$K(\xi, \zeta) = \frac{1}{2} \left(e^{(\zeta-2\xi)/4} \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\zeta}} - \frac{\sqrt{\zeta}}{2}\right) + e^{(\zeta+2\xi)/4} \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\zeta}} + \frac{\sqrt{\zeta}}{2}\right) \right) - \frac{Q_d}{2} e^{\zeta/4} \left((\zeta - \xi) e^{-\xi/2} \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\zeta}} - \frac{\sqrt{\zeta}}{2}\right) + (\zeta + \xi) e^{\xi/2} \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\zeta}} + \frac{\sqrt{\zeta}}{2}\right) \right) + Q_d \zeta e^{(\zeta-2\xi)/4} \quad (\text{A8})$$

where, erfc is complementary error function, which is defined as $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$. Eventually, the temperature distribution is in the form of:

$$T(x, t) = T_0 + \frac{T_i - T_0}{2} \left(\operatorname{erfc}\left(\sqrt{\frac{\sigma}{k_m}} \frac{x}{2\sqrt{t}} - \frac{\sqrt{\sigma k_m}}{u} \frac{\sqrt{t}}{2}\right) + e^{\hat{u}x/k_m} \operatorname{erfc}\left(\sqrt{\frac{\sigma}{k_m}} \frac{x}{2\sqrt{t}} + \frac{\sqrt{\sigma k_m}}{u} \frac{\sqrt{t}}{2}\right) \right) - \frac{Q_d(T_i - T_0)}{2} \left(\left(\frac{\hat{u}^2 t}{\sigma k_m} - \frac{\hat{u}x}{k_m} \right) \operatorname{erfc}\left(\sqrt{\frac{\sigma}{k_m}} \frac{x}{2\sqrt{t}} - \frac{\sqrt{\sigma k_m}}{u} \frac{\sqrt{t}}{2}\right) + \left(\frac{\hat{u}^2 t}{\sigma k_m} + \frac{\hat{u}x}{k_m} \right) e^{\hat{u}x/k_m} \operatorname{erfc}\left(\sqrt{\frac{\sigma}{k_m}} \frac{x}{2\sqrt{t}} + \frac{\sqrt{\sigma k_m}}{u} \frac{\sqrt{t}}{2}\right) \right) + Q_d \frac{\hat{u}^2(T_i - T_0)}{\hat{\sigma} k_m} t \quad (\text{A9})$$

Acknowledgment

The authors would like to acknowledge the financial support for this work by the SIM-Platform at LUT, and also Academy of Finland under Grant No. 311138.

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