LAPPEENRANTA UNIVERSITY OF TECHNOLOGY
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ON OPTION PRICE INFORMATION CONTENT AND EXTRACTION OF IMPLIED PROBABILITY DISTRIBUTIONS

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ABSTRACT

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In this study we used market settlement prices of European call options on stock index futures to extract implied probability distribution function (PDF). The method used produces a PDF of returns of an underlying asset at expiration date from implied volatility smile. With this method, the assumption of lognormal distribution (Black-Scholes model) is tested. The market view of the asset price dynamics can then be used for various purposes (hedging, speculation).

We used the so called smoothing approach for implied PDF extraction presented by Shimko (1993). In our analysis we obtained implied volatility smiles from index futures markets (S&P 500 and DAX indices) and standardized them. The method introduced by Breeden and Litzenberger (1978) was then used on PDF extraction. The results show significant deviations from the assumption of lognormal returns for S&P500 options while DAX options mostly fit the lognormal distribution. A deviant subjective view of PDF can be used to form a strategy as discussed in the last section.

Key words: Derivatives, option, implied volatility smile, implied probability distribution function.

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Tässä työssä käytettiin markkinapohjaisia Eurooppalaisten indeksifutuuri osto-optioiden hintoja implisiittisten todennäköisyyysjakaumien johtamiseen. Käyetty metodi johtaa TN-jakauman kohde-etuuden tuotoille erääntymispäivänä implisiittisestä volatiliteettihymystä. Tällä metodilla testataan Black-Scholes mallin olettamaa lognormaalista tuottojakaumaa. Markkinanäkemystä kohde-etuuden hinnan muodostuksesta voidaan moniin eri tarkoituksiin (Suojaus, spekulaatio jne).

Implisiittinen TN-jakauma johdettiin Shimkon (1993) esittelemällä ns. “smoothing” -metodilla. Analyysissä implisiittiset volatiliteettihymyt saatiin indeksifutuurimarkkinoilta (S&P 500 ja DAX indeksit) jotka standardoitiin. TN-jakaumat laskettiin volatiliteettihymyistä Breedenin ja Litzenbergerin (1978) esittelemällä metodilla. Tuloksien mukaan S&P 500 optioista saadut jakaumat poikkeavat selvästi lognormaalista oletuksesta kun taas DAX optioista johdetut tuottojakaumat olivat yhteneväisemmät. Viimeisessä kappaleessa Implisiittisestä TN-jakaumasta poikkeavaa subjektiivista jakaumaa käytetään eri strategioiden pohjana.

Avainsanat: Johdannaiset, optio, implisiittinen volatiliteettihymy, implisiittinen todennäköisyysjakauma.

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Greed is good. Greed is inevitable. Greed is beautiful since everything in life derives from it and the financial market is its ultimate embodiment. Blessed are those who are privileged to work for such divine concept.

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1. Introduction

An option contract is a financial derivative which price is, as the name implies, *derived* from the value of another asset or instrument. Therefore an option contract does not have a price of its own as it solely depends on the price of the underlying asset. The contract specifies the maturity (time to expiration) of an arrangement as also the strike price (exercise price at the maturity). The profit for an option depends on the price of an underlying asset at the expiration date and has no value if the exercise price is higher than the underlying price (call option, an option to buy) or when the strike price is lower than the underlying price (put option, an option to sell). The option price depends also on the volatility of an underlying asset and on the maturity. The more volatile the underlying price and longer the maturity until expiration, the more probability there is for a specified option contract to expire in-the-money (ITM).

The term *volatility smile* refers to a shape of a curve of implied volatilities presented as function of varying strike prices. While the Black-Scholes option pricing model assumes the volatility to be constant at all strike prices, a differing pattern has been observed from the markets. Notable is that smile patterns tend to vary over time. The smile pattern was not clearly the default volatility curve prior to 1987 market crash, but afterwards literally all markets all around the world started showing smile, skew or smirk patterns (Weinberg 2001). The curves prior to 1987 were usually closer to a constant level as the BS model expects. Many researches have introduced their ideas for this phenomenon and the main argument is that the market quotations of options include some “knowledge” or *information content* of the future volatility of an underlying asset what Black-Scholes model does not take in to account due to its simplifications. In other words, the evolutionary process of volatility is much more complex to model than the simplification assumes. It is a fact that volatility of an asset does vary over time dynamically and the difficult part is to trying to define volatility accurately with mathematical models. Therefore many researchers have turned into studying market implied volatilities and surfaces in order to obtain more accurate means of predicting future asset price changes (Taylor 2005). *Smoothed* smiles and surfaces are nowadays used as pricing tools for other illiquid option series with the same underlying asset and maturity than the observed ones. This approach can be thought as an inverse method when compared to traditional time series based models as ARCH and its variations which use historical price data on predicting future returns and volatility.

Lately, in last 15 years, there has been a wide range of interest in researching *probability density functions* (PDF) extracted from the market premiums. Often the analysis consists of building of a market implied volatility smile and then using it to derive a customized PDF which can be used on pricing option contracts more accurately since the market estimation of volatility during the maturity is the "correct" basis for a pricing model. Implied PDF's tend to have fatter tails on extreme values of strikes and this naturally affects on hedging decisions and speculative strategies. The assumption of lognormal distribution obviously does not take this phenomenon into consideration and therefore the implied distribution function is often seen as a meaningful tool for estimating future returns on an underlying asset (Jackwerth & Rubinstein 1996). Probability distribution functions are discussed more in Section 2.

1.1. History of Derivatives

Although the pricing models for options are relatively new discoveries in the financial world, the idea of being able to do some transaction in future, is very old. The first practical example of an "option contract" is mentioned in the Holy Bible (Genesis 29). In the story Jacob agrees to work for 7 years for an "option" to marry Laban's youngest daughter Rachel. After finishing his obligation, he had to marry Leah instead; the oldest daughter of Laban but Jacob was very fond of Rachel so he agreed on another option of 7 years of maturity for the younger daughter Rachel. So, Jacob did not only introduce the very first option contract but did it actually twice.

A modern example of derivatives usage is often referred as hedging from fluctuations of market prices of commodities in the 1800's. A farmer would buy a forward contract which obligated him to sell his crops at the time of the harvest with a certain predefined price. This enabled him to know for certain how much income he would obtain in time of the harvest. The main fact that differ forward contracts (and futures contracts) from option contracts is the obligation to honor the agreement. The first modern derivatives exchange was formed in Chicago as early as 1848 when the Chicago Board of Trade (CBOT) started trading with forward contracts. Later, in the year 1865, CBOT introduced standardized forward contracts known as futures. The need for such instruments was high since the area of Great lakes was an important market place for farming goods. Nowadays CBOT is the largest derivatives exchange in the world.

During the 1970's, mostly due to the introduction of efficient pricing models, the financial derivatives started to gain popularity among traders, speculators and hedgers. In the last 20 years more complex derivatives instruments have been introduced to meet the demand for customized hedging tools for various risks. These instruments include options on futures, Asian options, barrier options, binary or digital options, lookback options and rainbow options, to name a few. One common tendency that all exotic options share is the fact that they are more complex to price than a plain vanilla equity option and no closed-form solution for pricing usually exists. Therefore numerical tree models are used or the price is derived from vanilla options. There are literally dozens of types of options traded in the exchanges and sold on the OTC (over-the-counter) markets today. In this study, these options will not be discussed any further though. They are merely mentioned to give a good idea of the complexity of modern options markets nowadays.

1.2. Purpose of the Study

The purpose of this study is to examine the information content in the market premiums of options on index futures and to determine if the assumptions made by the Black-Scholes option pricing model (BS model) hold for chosen index futures markets. Especially our aim is to define an implied PDF and to determine how the returns are distributed and if the market view has any deviations from the normal distribution. The analysis is divided into two sections. In the first part, an implied volatility smile is built to determine if the assumption of constant volatilities over maturity holds. The second part consists of building an implied PDF based on the smoothed volatility smile. If the assumptions do not hold and therefore implied volatility smiles exist (e.g. the implied volatility is not constant with all strike prices) we should agree that more advanced measures for volatility estimation is required. And in this case the PDF's come in handy.

1.3. Structure

This study is divided in to four main sections. Section 2 discusses the theory of option pricing, examines two different approaches on option pricing issues and discusses relevant research findings conducted on the area of option pricing theory. Different approaches and aspects for pricing issues are discussed along with the distributions of asset returns and the volatility estimation problem. Also the concepts of risk-neutral pricing and the connection between put and call option prices (put-call parity) is presented in the Subsections along with the theoretical frame behind the implied probability distributions. Section 3 presents the data used in this study and discusses about the methodologies used in the analysis. Section 4 presents the results for our analysis and interprets the observations with illustrative examples of probability trading. The results are also compared to similar researches and differences are discussed and interpreted. Section 5 draws a conclusion of this study and presents the possible subjects of further research not covered in this paper.

2. Option Pricing Theory

The pricing models for option contracts have been available for relatively short amount of time although financial option contracts have been used actively for at least decades earlier. Mainly, two different approaches for pricing options exist; the analytical method of Black-Scholes (1973) and the numerical binomial tree model of Cox, Ross and Rubinstein (1979). Both methods use an assumption of a riskless portfolio of (one long position on an underlying asset and one short option position for the same asset) on determining the value of an option contract (Hull 2003). When we build a riskless portfolio, we can use the risk-free rate of return (e.g. the United States Treasury Bill rates or Euribor rates offered by the European Central Bank) to discount the future value of an option contract to a present. Basically, the logic behind pricing models does not differ from the valuation of any other financial asset.

While Black-Scholes model calculates the closed-form solution for option price of and European call option as a function of strike and share prices, interest rate, volatility and maturity, the binomial tree model uses a numerical approach to approximate the option prices with certain probabilities of future outcomes (thus the so called implied binomial tree model). Note that the Black-Scholes model can only price a European-style option contract; the binomial model turns out to be quite useful on pricing American-style options with a possibility of an early exercise at any moment during its maturity. Also, the binomial approach is quite popular and useful among the problems considering a real option valuation (e.g. investment decisions during multiperiod time frames) (Copeland et al. 2005). One should note that these two pricing methods do not exclude each other out but are used side by side depending on the characteristics of an option contract.

2.1. Binomial Pricing Model

Cox, Ross and Rubinstein introduced their idea of an option pricing model in 1979 in their paper “Option Pricing: a Simplified Approach”. Their approach for pricing option contracts differ a lot from the closed-form model of Fischer Black and Myron Scholes (1973) which uses stochastic differential equations on pricing options and some might argue that it is mathematically demanding to derive. Therefore, in some situations, the binomial tree approach is preferred.

The simplest example of a binomial tree is the one-step model which has only two future outcomes but naturally the amount of steps can be unlimited. Figure 2.1 illustrates such a case with an underlying asset of a common stock.

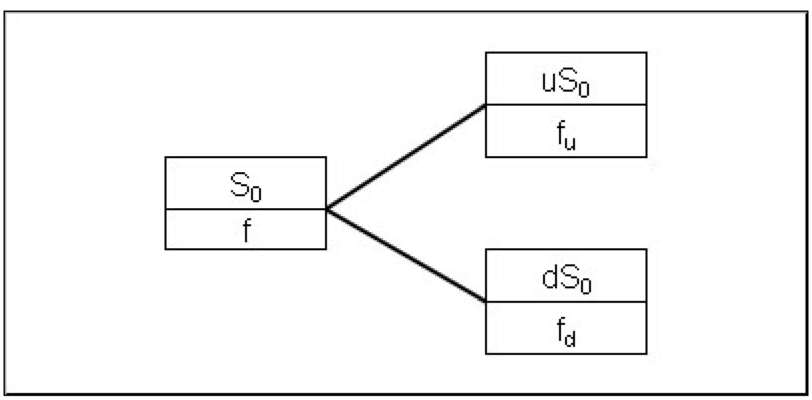


Figure 2.1. One-Step Binomial Tree

In figure 2.1, S_0 is the price of a stock at time 0, f is the price of a call option at time = 0, u and d are the proportional increase and decrease in price. The terms f_u and f_d are payoffs of a call option contract at time = 1. The price of an option contract is calculated by assuming a riskless portfolio; we will short one option and hold a long position of Δ shares (Hull 2003, Copeland et al.

2005). Therefore $uS_0\Delta - f_u$ defines the value of our portfolio after an upward movement and $dS_0\Delta - f_d$ in the case of a downward movement. The two values must be equal stating that

$$uS_0\Delta - f_u = dS_0\Delta - f_d$$

And when we solve the equation according to Δ, we get

$$\Delta = \frac{f_u - f_d}{S_0(u-d)} \qquad (1)$$

Where,

Δ = hedge ratio (multiplier of how many shares should we own [a long position] for one shorted option contract to create a riskless portfolio)

As portfolio is considered riskless, it must obviously yield the risk-free rate. Then the cost of a portfolio at present time must equal the value of a portfolio at time 1 discounted to present time with the risk-free rate.

$$S_0\Delta - f = (uS_0\Delta - f_u)e^{-rT} \qquad (2)$$

When Δ is known (equation 1) and fit into equation 2 we can simplify the equation and get the value of an option contract f at present time.

$$f = e^{-rT}\left[pf_u + (1-p)f_d\right] \qquad (3)$$

Where,

$p = \frac{e^{rT} - d}{u - d}$ = the probability of an upward movement

Rubinstein (1994) later used this binomial tree pricing model on determining an implied PDF from the call prices which is discussed more in Section 2.2.7. As is the case with option pricing, the PDF can be extracted in many ways. The numerical method is more suitable if one wishes to extract the PDF from an American or exotic derivative

2.2. Black-Scholes Pricing Model

The original Black-Scholes option pricing model (sometimes referred as a Black-Scholes-Merton model due to the significant contribution of Robert C. Merton) was introduced by Fischer Black and Myron Scholes in their article "The Pricing of Options and Corporate Liabilities" in 1973. The model soon became a standard in option pricing although it has some limitations due to the assumptions it makes. The following Section is mainly based in "Options, Futures and Other Derivatives" by John C. Hull (2003).

The assumptions of the BS-model include,

a) The volatility of an underlying asset is constant during the maturity of an option
b) The risk-free rate is constant
c) The price of an underlying asset follows a stochastic Geometric Brownian Motion (GBM) with a constant volatility and drift
d) The underlying asset is divisible, e.g. it is possible to buy a fraction of a share

e) Short-selling is allowed

f) Arbitrage opportunities do not exist, e.g. the markets are assumed to be perfect and frictionless and the new information is available to everyone at the same time

g) There are no transaction costs or taxes

h) The underlying stock does not pay any dividends

The equation defines the option call price as a function of the strike price, the share price, volatility, maturity and the risk-free rate of return. The general mathematical form of the equation is presented as (Copeland et al. 2005),

$$C = f\left(S, X, \sigma, T - t, r\right) \quad (4)$$

When the partial derivatives of the option call price are,

$$\frac{\partial C}{\partial S} > 0, \quad \frac{\partial C}{\partial X} > 0, \quad \frac{\partial C}{\partial \sigma} > 0, \quad \frac{\partial C}{\partial (T - t)} > 0, \quad \frac{\partial C}{\partial r} > 0$$

The closed-form solution for the differential equation being,

$$C = SN\left(d_1\right) - Xe^{-r(T-t)}N\left(d_2\right) \quad (5)$$

Where,

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = \frac{\ln\left(\frac{S}{X}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

C = Call option price

S = Share price (price of an underlying asset)

X = Strike price (exercise price)

r = Continuously compounded risk-free rate of return

σ = Volatility of an underlying asset

$T-t$ = The time until expiration (in years)

$N(x)$ = Standard normal cumulative distribution function of x

Basically, what Black-Scholes model does, is that it weighs the components S and X by probabilities $N(d_1)$ and $N(d_2)$. $N(d_1)$ is the inverse hedge ratio (risk-free portfolio can be constructed with 1 long position of a stock and by shorting $\frac{1}{N(d_1)}$ option contracts to the same stock) and $N(d_2)$ denotes the probability of an option contract to be in-the-money at expiration date. Therefore the model calculates the call option price by multiplying the current price of an underlying asset by inverse hedge ratio and by subtracting the discounted strike price multiplied by the probability of being in-the-money at expiration from it. Note that the Black-Scholes model does not need expected return or custom rate of return to discount the future value to present time. Instead, a risk neutral world is assumed and therefore a risk-free rate of return is used. The concept of risk neutral world is discussed in Section 2.2.2.

2.2.1. Put-Call Parity

Put-call parity defines the relation between the prices of a call option and put option (Stoll 1969). Therefore the pricing model for a European put option can be derived from the closed-form solution of the BS differential equation for a European call option. Furthermore, the solution for the differential equation of a put option is not necessary to be derive from the basics when defining the prices for a full option chain with call and put option prices over varying strike prices.

$$P = C - S + Xe^{r(T-t)} \qquad (6)$$

When equation 6 is fit into an equation 5, we get the Black-Scholes pricing model for a European put option,

$$P = -SN(-d_1) + Xe^{-r(T-t)}N(-d_2) \qquad (7)$$

According to equation 6, the price of a put option is a function of the call price, the price of an underlying asset and the discounted strike price (by the risk-free rate). For the relation to hold, the put option of the same strike and maturity needs to have an identical volatility as the call option. This is very important if one is constructing an option chain for both, the call prices and the put prices. If pricing differences (arbitrage opportunities) would exist, they would be exploited in the efficient markets until they would vanish. Ahoniemi (2007) studied Nikkei 225 index implied volatilities for both call and put options and came to a conclusion that the put-call parity does not hold necessarily and differences do exist. Her paper consists of time-series analysis (back-ward looking method) with prediction performance estimation with illustrative examples for options trading.

The put-call parity is an important tool when pricing illiquid options since a liquid call options price can be used to determine the implied volatility not only for the call option but also for a put option of same maturity and strike price.

Often quotations near the one to be calculated are used on determining the "fair price". Obviously, we should not price many option series based on a few trading events since the accuracy of the volatility curve (and the probability distribution function) is affected and the strategies based on inaccurate estimations do not fundamentally differ from guessing.

2.2.2. Risk Neutral Valuation

The risk neutral valuation is a fundamental concept in derivatives and bonds pricing. In a risk neutral world investors are neutral and indifferent to risks between various investments. In other words, the investors need no compensation for the risk taken and therefore risk-free rate can be used on discounting the future values of derivatives (Hull 2003). This argument is based on the work of Cox and Ross (1976). They compared the two approaches in option pricing, the method presented by Samuelson (1965) and the one by Black and Scholes (1973). Samuelson derived the option price with an expected rate of return and used a custom rate on discounting. Black and Scholes did not make such assumptions and thus their model did not require known expected rate of return for an underlying asset nor did it require a custom rate for discounting[1]. Cox and Ross noted that the two methods provided the same price for an option contract and argued that the investors do not require any extra compensation for the added risk (the expected rate of return and the discount rate then cancel each other out).

As we know, the present value of any financial asset is equal to future earnings and the future value (the payoff) discounted to a present time. If the option pricing model assumed investors not to be risk neutral in their behavior, it would complicate the pricing process significantly since a model which takes into account all investor's risk preferences would be extremely

[1] See chapters 2.2.3 and 2.2.4 for more thorough explanation.

complicated, if not impossible to form. Therefore it is justified to assume the discount rate to be a risk-free rate.

Although clearly a simplification, an assumption of a risk neutral valuation enables us to price options without knowing all risk preferences that the market participants possess. As we know, the economical theories are not based in 100 percent observable phenomenon as is the case in classical physics and thus models that will explain utterly and completely the behavior in the financial markets are not possible to form. We can not therefore model the market activity by predicting the actions of individuals, and we even should not but we can make fairly good estimations on the market average behavior based on the actions of many.

2.2.3. Geometric Brownian Motion

The basics for the *Geometric Brownian Motion* are in physics. It was first observed by a botanist Robert Brown in 1827. He studied minute particles in fluids and their continuous random movements and collisions to each other. He noted that the movement, although seemed random, had some pattern in it. Much later, in 1905 (the paper was published again in English in 1956) the phenomenon was studied again by Albert Einstein, who managed to form an equation for the Brownian motion by studying heated molecules. Around the same time, a French mathematician Louis Bachelier (1900) presented an idea that the prices on stock markets could follow a Brownian motion. In his PhD thesis, he derived the Wiener process and managed to price an option contract based on an assumption that the price process of an underlying asset is stochastic (Forfar 2002).

There are many types of models on predicting the future development of asset prices. These models (or processes) often assume the evolution of the stock price to follow a stochastic tendency, making the changes in prices random events. Therefore, the future prices is assumed to follow a random walk hypothesis which states that the price of the stock yesterday can not be used to predict the price tomorrow. This simplest model of a stochastic process is called the *Markov process* which states that only the price today matters when we try to predict the price of tomorrow. The Markov model is widely accepted as it is consistent with the random walk and weak form efficient market hypotheses (EMH, first presented by Bachelier [1900] and later by Fama [1965]). Other types of stochastic processes also exist (mean reversion with or without jumps for example). Additionally, stochastic processes can be continuous or discrete, but due to the infinite maturity of stocks, the price evolution process should be considered to be a continuous model. These models will not be discussed further in this paper.

As noted above, it is fair to assume that the future stock prices are uncertain and although predicting future is impossible, we can find patterns which will correlate with real life observations reasonably well. To be able to price option contracts with the BS model, some type of assumption on the underlying asset prices evolution has to be made. BS model states that the price of an underlying instrument follows a continuous stochastic model, geometric Brownian motion (sometimes referred as generalized Wiener process which is the mathematical form of GBM). GBM adds a new variable of volatility in to the Markov process. Remember how the Markov process assumes the price evolution process to depend only on the price of the stock at present time (S_t). To be more precise, the continuous stochastic variable will be able to get any random value without limits according to a change in time. Thus, with Geometric Brownian Motion, jump diffusions in the process are possible. These jumps are often observed when new, surprising information arrives to market. The Geometric Brownian Motion process of S_t can be presented mathematically as (Al-Harthy 2007),

$$dS_t = \mu S_t dt + \sigma S_t dz \tag{8}$$

Where,

μ = constant drift (expected rate of return)

σ = constant volatility (standard deviation) of S_t

dz = $\varepsilon\sqrt{dt}$, ε = Wiener process

dt = change of time

dS_t = change in price of a stock

The first term ($\mu S_t dt$) defines the expectation term and the second term ($\sigma S_t dz$) the variation term. The first term therefore presents the expected drift rate of μ for S_t while second term adds noise and uncertainty to the stock price evolution process. And when we note that the basic Wiener process follows a Markov process with a constant, predefined drift of 0 and a volatility of 1.0 (the process follows a normal distribution, $\phi(0,1)$), we can define the dz. Thus, when the drift of the price evolution process equals to zero, the expected value of S_t in the future will also be zero. Also, if the basic Wiener process (ε) has a volatility of 1.0, then the continuous stochastic model will have a volatility of σ. Basically, the Geometric Brownian Motion process adds more volatility to the model when the time horizon increases due to the fact that the probability of value changes will also increase.

2.2.4. Itô's lemma

Itô's lemma can be used to derive the actual Black-Scholes model from the Wiener process[2]. The general form is presented mathematically as (Taylor 2005),

$$dx = a(x,t)dt + b(x,t)dz \qquad (9)$$

Where a and b are the functions of x and t ; dz denotes the Wiener process discussed earlier. Therefore the drift of the process is a and the volatility b (and therefore the variance is b^2). Itô's lemma describes G as the function of x and t, therefore,

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz \qquad (10)$$

Where,

dG = Change of a function G

$\left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right)$ = Drift rate of the price process

$\frac{\partial G}{\partial x} b$ = Volatility ($\left(\frac{\partial G}{\partial x} \right)^2 b^2$ =variance)

When Itô's lemma is fit into an equation 8 we get,

[2] The following derivation should not be seen as a detailed description of mathematics behind the BS model. If interested in the complete solution please see, Black and Scholes (1973) and Itô (1951).

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz \quad (11)$$

Note that when the function $G = \ln S$ (logarithmic or continuously compounded change in stock price) then,

$\frac{\partial G}{\partial S} = \frac{1}{S}$ = the 1st derivative of function $G = \ln S$

$\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$ = the 2nd derivative of function $G = \ln S$

$\frac{\partial G}{\partial t} = 0$

Then the equation 11 simplifies into,

$$dG = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dz \quad (12)$$

Then the constant drift is $\left(\mu - \frac{\sigma^2}{2} \right) T$ and the constant volatility is $\sigma \sqrt{T}$ ($\sigma^2 T$ = variance) where T is the moment of time in the future. In other words, the function $G = \ln S$ is normally distributed with the mean $\left(\mu - \frac{\sigma^2}{2} \right) T$ and volatility (standard deviation) of $\sigma \sqrt{T}$ during the period of $T_0 - T$, therefore,

$$\ln S_t - \ln S_0 \approx \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right] \qquad (13)$$

Equation 13 presents mathematically the lognormal distribution of the changes in a price of a stock. The equation enables us to calculate the lognormal probability distribution for the price change of a stock if expected return, volatility, the time period and the current price of a stock is known. Since returns are normally distributed, we know at the 95% confidence level that the returns will be within 1.96 standard deviations from the mean.

The Itô's lemma can be used to derive the actual Black-Scholes differential equation by assuming the price of a call option to be a function of the underlying stock price and time, $df = (S, t)$. Then by fitting the variables into the equation 11 we get,

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S dz \qquad (14)$$

Black and Scholes formed a closed-form solution for their differential equation (an actual usable analytic formula) by proving that a riskless portfolio is possible in the perfect markets by selling short one derivative and buying long one share of a stock. The assumption of a riskless portfolio was proved by the fact that the Wiener processes of a long share and a short derivative eliminate each other out (as does expected rate of return, μ). Thus, it was argued by Black and Scholes that the rate of return is a nonstochastic variable (Copeland et al. 2005). When equations 14 and 8 (the price evolution process of a derivative and a stock, respectively) are combined we get the Black-Scholes differential equation.

$$rf = \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \qquad (15)$$

Where, f , is the price of a European call option, r is the risk-free rate, S is the price of an underlying stock, σ is volatility and t is maturity. When a maximization problem $f = MAX(0, S - X) \geq 0$ is solved we get the equation 5 which is the solution for the Black-Scholes differential equation. Note that the equation 15 does not include any variables which depend on investor's personal preferences of risk; therefore the differential equation and its solution solely rely on an assumption of a risk neutral world discussed earlier.

2.2.5. Volatility

The term volatility in Black-Scholes model refers to a standard deviation from the mean returns of an underlying asset or in more general terms, price variability over some period of time (Taylor 2005). Therefore it describes how fluctuating the returns of an underlying asset are and moreover, can be used to measure the risk involved in investing on such an asset. The fact that nowadays practitioners (traders) use volatility to compare the option prices instead of using their quoted dollar amounted market prices, tells how important variable volatility is.

All variables in the Black-Scholes model are observable (strike and current prices, risk-free rate of return, maturity) except the volatility. As noted earlier, the original Black-Scholes model made an assumption, what is today thought as being an oversimplification, that the volatility was constant during the time of maturity. Many researchers have tested this hypothesis (as we will do the same) and have noted a shape of a smile right after the 1987 market crash (Egelkraut et al. 2007, Pirkner et al. 1999). Thus, the name *volatility smile* which refers to a graphical presentation of implied volatilities as a function of option strike prices (that is, $\sigma(X)$). Since then researches have argued that the shape of this curve has changed to a skew or even to a smirk in some cases (Taylor 2005).

The volatility is the trickiest variable to estimate and greatly impacts in the price of an option calculated by the BS pricing model since when the volatility increases, the probability of an option contract to be ITM at expiration also increases. An extensive amount of work has been done in this area of research, especially on developing models on estimating stochastic volatility process from historical data (ARCH model and its multiple variations, stochastic volatility model by Heston etc.). Basically, two different approaches for volatility estimation exist, backward-looking and forward-looking methods. Backward-looking method refers to techniques which use historical volatility data on estimating future evolution process of standard deviation of an underlying asset. These methods include the basic historical volatility estimation (or realized volatility) which uses logarithmic changes (returns) in stock prices on estimating the volatility for a chosen period of time in past. ARCH (Autoregressive Conditional Heteroscedasticity) model uses a more advanced technique since it assumes the volatility process to be time-varying and conditional on historical observations. Realized volatility is used to benchmark the models against it.

The forward-looking method uses an inverse approach on estimating the volatility during the maturity of an option. Implied volatility extracted from the observed market premiums of an option contract can be then seen as the market expectation of the average volatility during the maturity. The BS model assumes a constant volatility during the maturity and the market-observed option premiums tend to differ slightly from the premiums calculated by the Black-Scholes model. This indicates that the market premiums have some information content in addition to a BS premium (Weinberg 2001, Vähämaa 2004). Basically, no closed-form solution for volatility extraction from observed option premium exists and iteration is needed on calculation of volatilities imposed by the BS model. The forward-looking method is discussed more in Section 3. Many researchers suggest that the ARCH models give a fair estimate only on short-time volatility while implied volatilities from observed option premiums give a better estimation and more long-term predictability (Chang and Tabak 2002, Vähämaa 2004). Chang and Tabak argue that this is due to the fact that the market implicit variables adjust more rapidly to new information or situations unlike the historical models which have a lag in their estimations since they need to have historical data for their forecasts. Taylor (2005) also agrees that the implied volatility is by far more superior method than the historical ones. Others might criticize that the estimations extracted from the market premiums are only views of the future volatility at certain moment of time and can not be used on estimating the time-varying process of volatility which ARCH-models try to achieve.

The future volatility process is an important concept to internalize for practitioners since it affects directly the hedge ratios of a portfolio (delta hedging). As Egelkraut et al. puts it,

> *"Understanding (of) future volatility patterns is important to market participants for a variety of reasons including the need to determine effective hedge ratios, and for assessing the relative costs and risks of hedging in different periods. Increased volatility can lead to more frequent margin calls, putting a greater portion of wealth at risk by shortening the time that investors have to respond with new funds.*

Information about future volatility also provides insight into whether holding a position or portfolio is consistent with risk preferences. Moreover, understanding longer-term behavior of volatility and the predictability of its magnitude and change are critical for effective commodity marketing and derivative pricing".

With delta-hedging it is essential to keep the ratio of underlying asset and the derivatives possessed optimal and to be able to read the changes in implied volatilities from the market gives an advantage over maintaining the correct amount of risk preferred by the hedger.

For market participants, it is important to know how the volatility changes over time. The term structure of volatility defines the volatilities over traded maturities. Technically, the term structure of an option series does not differ much from the term structure of interest rates. Since the volatility smile presents the implied volatilities for traded strike prices at a certain moment of time; by combining the option term structure and volatility smiles, we can graph the volatility change over multiple maturities and strike prices. This 3D-plot is referred as a *volatility surface* and is used by practitioners on illustrating how the volatility changes over time. Implied volatility surfaces derived from plain European vanilla options are also used on pricing more exotic option contracts on the same underlying asset. In this study, a volatility surfaces are constructed in order to interpret the changes in volatility and to study if it differs from the theoretical value.

2.2.5.1. Historical estimation

Mathematically historical standard deviation of an underlying asset can be calculated with logarithmic changes in historical (realized) prices. Logarithmic change in prices can be defined as (Taylor 2005),

$$r_t = \ln(P_t) - \ln(P_{t-1}) = \ln\left(\frac{P_t}{P_{t-1}}\right) \qquad (16)$$

Where,

$t = 1,2,3,...,n$ observations

r_t = Continuously compounded rate of return (logarithmic change in stock price)

P_t = Price of a stock at time t

P_{t-1} = Price of a stock at time $t-1$

After calculating the logarithmic returns, the standard deviation is defined as,

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(r_t - \bar{r})^2} \qquad (17)$$

Where,

σ = Standard deviation

$\bar{r}$ = Mean of r_t

As noted earlier, usually in statistical analysis, the realized volatility is the benchmark against the prediction model.

2.2.5.2. Advanced Historical Volatility Models

In addition to basic historical volatility estimation model, more advanced methods of historical estimation also exist. The ARCH (Autoregressive

Conditional Heteroscedasticity) model is based on an assumption that the evolution process of variance is not constant but time-varying. The future variance is assumed to be partly conditional (or autoregressive) on observed historical data (Engle 1982, Paronen 2003, Taylor 2005). The key notation in ARCH model is that the evolution process of variance is assumed to have continuous peaks and drops and that these tendencies can be estimated for future purposes.

Another advanced stochastic volatility model was introduced at 1993 by Heston. Unlike the ARCH model which assumes the variance to vary over variance, the Heston's stochastic model assumes the variance to vary over the square root of variance. The closed-form solution can be derived in the similar way than the original BS model with the exception that the Heston model assumes a different drift for the geometric Brownian motion (equation 8). In Heston model, the volatility variable σ is replaced by the square root of variance ($\sqrt{\sigma^2} = \sqrt{v_t}$) (Taylor 2005). These models can be also adjusted to take large jumps in price changes into consideration (Poisson jumps). Further analysis of these models will go beyond the boundaries of this paper. It is justified to present these in this context since the implied distributions are often benchmarked against the continuous stochastic volatility models and/or realized volatility.

2.2.6. Option Greeks

The Greeks are used on determining the risk exposure by indicating the change in option price when a certain variable in the function changes (MacDonald 2006). There is a ceteris paribus assumption behind every Greek. That is, when a key variable changes, the option price changes the defined amount all other variables staying constant. The connection between the Option Greeks and the BS model can be defined easily mathematically;

the Greeks are derivatives of Black-Scholes model with respect to an individual input (the partial derivatives discussed earlier in Section 2.2). There are total of five different Greeks; Delta, Gamma, Vega, Theta and Rho[3].

Delta (Δ) is the easiest one to internalize as it is already been discussed in the Section 2.2. Delta is often described as the hedge ratio (note that the $N(d_1)$ in BS model is the inverse hedge ratio) as it defines the number of shares long against one short option contract (riskless portfolio). $\frac{\partial C}{\partial S} > 0$ describes the mathematical relation to call option price. Delta must be positive for a call option; if the stock price increases, the option price must also increase since the possibility to buy at a certain price becomes more valuable for the purchaser. Naturally, there is an inverse relation for a put option; Delta has to be negative for put options since if the stock price decreases, the price of a put option has to decrease also. Delta can be then defined also as a sensitivity of a change in option price when the underlying price changes.

Gamma (Γ) is the mathematical derivative of Delta or the second derivative of call price with respect to a stock price (the change of Delta when the stock price changes, $\frac{\partial^2 C}{\partial S^2}$). Gamma is always positive, since when the stock price increases, the price of a option rises. Deep in-the-money options will be most likely exercised, therefore Delta is close to 1. I.e. the assumed riskless portfolio consists of almost an equal amount of shares long and options short. As Gamma denotes the change in Delta when the stock price change, Gamma in this case is very close to zero (Delta can not change very rapidly since it is close to 1 already). If an option contract is deep out-of-the-money, Delta is close to zero and the assumed portfolio does not have many shares

[3] The complete mathematical definitions of the Greeks can be found from the appendix A.

in it. For the same reason, Gamma is also zero. Note that Delta also changes according to the maturity of an option since the option with a longer maturity has a greater probability to end up being in-the-money at expiration date. Gamma is very useful on determining how often a Delta neutral portfolio should be adjusted. Therefore Gamma of zero is preferred over it being close to one. If Gamma is close to one, it means that the Delta neutral portfolio has to be adjusted frequently and with high number of individual assets which will increase the costs of hedging.

Vega measures the sensitivity of a call price to volatility; the increase in volatility of an underlying stock will increase the price of an option ($\frac{\partial C}{\partial \sigma}$). This happens since the greater volatility increases the probability of an option contract to be in-the-money at expiration date. Volatility is therefore assumed to be time-varying, not constant as the Black-Scholes model assumes. Vega measures the sensitivity and it should be observed carefully.

Theta (Θ) is the sensitivity of a call price to the maturity (or the change of time, $\frac{\partial C}{\partial (T-t)}$). Theta is quoted in days, usually per one day, so it can be interpreted as a price change of a call option in one day. Rho (P) is the partial derivate with respect to a risk-free rate. That is, it measures the sensitivity of a call price to a change in risk-free rate ($\frac{\partial C}{\partial r_f}$). Although a practitioner of a successful derivative portfolio should be familiar with all the Greeks, Delta, Gamma and Vega can be considered the most important partial derivatives to know and internalize.

2.2.7. Implied Probability Distribution Function

Implied probability distribution functions are used in order to better understand the nature of asset price dynamics. Since the market consists of numerous different traders, the price process of an underlying is difficult to model mathematically. Implied PDF can be used to see the current, although rapidly changing, market view of the risk and probability involved. This view is argued to be superior since it includes all the risks that investors include in the prices in the markets. Shimko (1993) developed a practical method for extracting the implied PDF from the observed option premiums from the work of Breeden and Litzenberger (1978). They argued that the risk-neutral PDF of an underlying asset S_T, $g(X)$, can be calculated from the second derivative of the call option price with respect to strike price if the price has a continuous probability distribution. We come to this solution by forming butterfly spread option portfolios of two sold call options with the exercise price $X = S_t$ and two bought call options, one with an exercise price of $X - \delta$ and one with $X + \delta$. When quoted for all strike prices in the option chain (with a very small change between the two observations) we obtain the risk-neutral distribution function[4] for the returns of an underlying asset at expiration date. The derivation of the equation is presented mathematically as (Hull 2003),

$$C = e^{-rT} \int_{S_T = X}^{\infty} (S_T - X) g(S_T) dS_T \tag{18}$$

Where, C is the call price, S_T is the price of and underlying asset at time T, X is the strike price, r is the constant risk-free rate of return and $g(X)$ is the risk-neutral distribution function of S_T. Differentiating once with respect to X,

[4] In addition to the original article by Breeden and Litzenberger, see Pirkner et al. (1999) for an illustrative example of a butterfly spread.

$$\frac{\partial C}{\partial X} = -e^{-rT}\int_{S_T = X}^{\infty} g(S_T)dS_T$$

Differentiating again with respect to X we get,

$$\frac{\partial^2 C}{\partial X^2} = e^{-rT} g(X)$$

When solved for $g(X)$, we get the probability density function used in our analysis.

$$g(X) = e^{rT}\frac{\partial^2 C}{\partial X^2} = e^{rT}\frac{C_1 + C_3 - 2C_2}{\delta^2} \qquad (19)$$

Where, C_1, C_2 and C_3 are call price with the same maturity, T, and strike prices $X-\delta$, X and $X+\delta$, respectively. Note that delta (δ, the constant change in strike price) is assumed to be very small and it can affect the accuracy of the distribution negatively if the absolute change in strike price is too high. In other words, the closer the observations (or interpolated prices) are to each other, the better estimation for probability distribution can be obtained. Also, the Breeden-Litzenberger relaxes the assumptions of the evolutionary process of an underlying price by only assuming perfect markets (short sales allowed, no transaction costs, no taxes and infinite borrowing at risk-free rate of return) (Miranda & Burgess 1998, Bahra 1997).

2.2.8. Black-76 model

Fischer Black introduced his model of pricing options on futures (the so called Black-76 model) in 1976. The model is a variation of the original Black-Scholes model. Since the data used in this study consists of option premium observations for index futures, the following Black-76 model will be used for the iteration of volatilities from observed option premiums. The model is presented mathematically as,

$$C = e^{-r(T-t)}\left[FN(d_1) - XN(d_2)\right] \tag{20}$$

Where,

$$d_1 = \frac{\ln\left(\frac{F}{X}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{F}{X}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

C = Call price

F = Futures price

X = Strike price

σ = Volatility of an underlying futures price

If familiar with the Black-Scholes model, the Black-76 is very straightforward to internalize and to use, although, a few characteristics of the model should be kept in mind. Firstly, the underlying asset is futures contract issued on some other asset, therefore making futures option a so-called "derivative on derivative" contract. Also, the evolution process of the futures price is assumed to follow the same lognormal property than the stock prices in the original Black-Scholes model. Secondly, a futures contract as an underlying asset differs from the stock due to its finite nature. The option on futures contract requires a futures contract with a longer maturity than the maturity of the option. Obviously, since it is difficult to price an option contract which does not have an underlying asset to derive the price from. With a stock, this is not taken into an account since the corporation (and the publicly-traded stock) is assumed to have an infinite lifetime. A stock then is defined to be a perpetual financial instrument. With these few alterations, the Black-76 model does not differ much from the original BS model and therefore we will keep on referring to the original model instead of the Black-76.

3. Data and Methodology

The next section discusses the data gathering and filtering process and introduces the methods for smoothing the rough volatility smile from observed option prices. Later on, the smoothed smile is used for PDF estimation.

3.1. Data

The data used consists of two different European call option chains with daily settlement prices (various strike prices for each maturity). Both series are index options on futures contracts. This type of “derivative on derivative” contract was chosen due to the liquidity of the markets since high liquidity ensures a better estimation for volatility smile and surface. Due to the liquidity issues, options are not usually issued straightly for a stock index (as S&P 500, FTSE 100, Nikkei 225 or DAX) but on the futures contracts instead. Futures markets are highly liquid and therefore they give a good basis for option contracts to be priced correctly and fairly. The underlying assets of selected option series are,

1. One S&P 500 stock index futures contract (CME S&P 500 options)
2. One DAX stock index futures contract (EUREX DAX options)

Standard & Poor 500 stock index consists of 500 large capital companies mainly from the United States and is gathered from the two largest stock exchanges in the United States, the New York Stock Exchange (NYSE) and NASDAQ. Many mutual funds are benchmarked against the S&P 500 index return which is often seen as the main indicator of the economy in the USA as a whole. DAX (Deutscher Aktien Index) consists of 30 major blue chip companies traded in the Frankfurt Stock Exchange. S&P 500 index option settlement prices were provided by Chicago Mercantile Exchange (CME), a major financial derivatives exchange based in Chicago and DAX index option settlement prices by EUREX[5], the European derivatives exchange based in Zurich, Switzerland. The both data sets were obtained on 22nd of February 2008. Maturities of options range from 1 month to 11 months (From March 2008 to December 2008, 7 series for S&P 500 options and 6 series for DAX options). The CME S&P 500 series is floor-traded (pit-traded) while the EUREX DAX series is electronically traded which gives slightly thinner trading for CME options.

The option series traded in EUREX have 30 to 70 actively traded strike prices for a given maturity while the CME pit-traded options have amount of observations as low as 10 to 20. Although, the observation sets might seem limited, similar research has been conducted with data sets of only 3 observed strike prices on same maturity (Chang and Tabak, 2002). Therefore, the CME option series are more influenced on interpolation methods as the rough smile is smoothed. Although both data sets were consistent after filtering process, some extreme values can be noticed from both series. Usually this happens when there is a lot of speculation on extreme strike prices with a long maturity series. These problems with outlier observations and corresponding extrapolation overshoots can be seen from the volatility surface with the most extreme strike-to-index ratios. Also, observations with longer maturity (December 2008 option series with 301 days until expiration) tend to suffer from thin-trading more than options with shorter maturity. Naturally, the shortest maturity is the most actively traded in both exchanges. Although these extreme values for implied volatility were clearly outlier observations, they were included in the analysis due to the active trading on them.

[5] For more information on contract specifications, see www.cme.com and www.eurexchange.com. For information about S&P 500 stock index and DAX stock index, see www.standardandpoors.com and www.deutsche-boerse.com, respectively.

The CME option contracts are also traded in GLOBEX platform which is an electronic system like EUREX. In this paper, we try to observe if there are any notable differences in prices (i.e. information content) between pit-trading and electronic trading so theoretically less liquid CME options were chosen as basis for comparison.

3.2. Methodology

Firstly, the observation sets were screened for illiquidity. That is, the observations that had an option volume and open interest[6] of zero were immediately removed. If the daily volume is zero and/or there is no open interest then the price obtained is old or estimated in some other ways and does not give a good estimate for volatility. If there is no trading or even open interest on a specified strike, there can not either be any “information content” over the Black-Scholes price in the premium. At this phase, the obtained call option prices could also be used to determine the corresponding put prices and volatilities with the put-call parity. In this study, this is not the case and we will only consider call prices.

Secondly, a smoothing of the volatility curve via interpolation/extrapolation is needed for filling the gaps between the observations. As volatility surfaces are used on determining the expensiveness of an exchange-traded and quoted option contracts, it is essential that the market data is adjusted according to the best estimation of volatility between the observed points. The relative strike-to-index ratio is more informative and comparable for us in the same way than the implied volatility is a better measure for comparing option contract expensiveness rather than absolute quoted prices from the market.

[6] By definition, option volume is the total amount of contracts traded during the day and open interest is the total amount of contracts issued (and held by the market participants) on the specified strike and maturity at the end of trading day.

Sometimes the implied volatility is built with respect to hedge ratio (Delta) which is merely the same thing than the X/S ratio.

This smoothing method is consistent with the one introduced by Shimko (1993). He used the same type of estimation for the volatility smile. He later on used this estimated smile curve to recover a risk neutral PDF for a given moment in time. In other words, he assumed that the option premiums observed from the market would have some type of information content not included in the theoretical premiums suggested by the Black-Scholes model. His analysis was divided into four parts (Shimko 1993, Chang and Tabak 2002, Buttimer 2008),

1. Implicit volatilities are iterated from observed market premiums with the Black-76 model[7]
2. Rough volatility smiles are smoothed with a quadratic least squares method (extrapolation if required)
3. The prices of the options are calculated for standardized points in the curve with the Black-76 model
4. With the Breeden-Litzenberger (1978) method (equation 19), the risk neutral density function is formed

The approach to standardize the strikes to a ratio has its advantages and disadvantages. While volatility can be computed for all strike prices from the smoothed curve, the method does not take into a consideration the time difference of option series with exact strike prices in the market during the maturity (Tompkins 2001). It should be noted that this approach does not define its inferior nature compared to time dependent models, but the complexity of the concept, implied volatility, and its time-varying nature. Naturally, if it was possible to observe constant and standardized strike prices

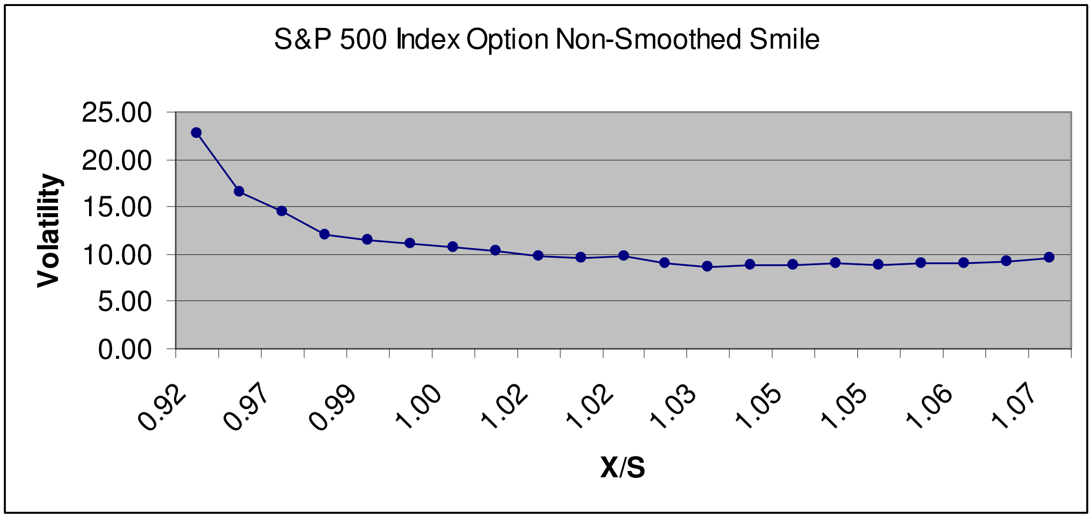
[7] Chang and Tabak used Garman-Kohlhagen model since they analyzed currency options but in this study, Black-76 model is used for options on index futures.

for several periods during a consistent time period, there would not be any problems for estimating the needed values. Our approach differs from the one presented by Shimko as we did not assume the volatility to be constant outside the observed range. This is done due to a practicality since our data is probably not as comprehensive as the one used by Shimko. In practice, these continuous and standardized quotations hardly ever exist in any market since the price of an underlying asset fluctuates and option contracts with new strike are issued to meet the demand of hedgers, speculators and other market participants (Jackwerth & Rubinstein 1996).

3.2.1. Estimation of Volatility

As there is no reverse, closed-form solution for solving volatility from the Black-76 model (nor for the BS-model), an iteration of volatility sets from prices is required. Basically, it is required to insert a wide range of volatilities to narrow down the prices so it will match with the observed one. This can be extremely time-consuming for high-frequency data so, for practical reasons, with data sets of high number of observations an option pricing program of choice is essential for computation of the volatility smile.

Graph 3.1 presents the "rough" smile based purely on market quotations for S&P 500 index option series expiring at March 2008 (maturity of 28 days) on 22nd of February 2008. X/S refers to a ratio between the prices of a strike and an underlying asset (in this case futures price for the stock index).



Graph 3.1. S&P 500 Index Option Volatility smile before smoothing. March 2008 option with 28 days of maturity.

Note the limited amount of strikes available for this series, X/S being between 0.92 and 1.07 only. Also, the in-the-money (ITM) call options (X/S ratio < 1.0) tend to have more volatility than the out-of-the-money (OTM) call options (X/S ratio > 1.0)[8]. This can be explained by the market estimations of the future trend for the underlying asset (e.g. the practitioners tend to trade heavily on ITM options and hedgers might expect or at least fear the index to fall in near future). The shape of the index option series is actually a skew, rather than a smile. This observation has been noted from many index options and options on index futures after the stock market crash in 1987 (Jackwerth & Rubinstein 1996, Weinberg 2001). Such extreme events as the 1987 crash suggest that the pricing models have understated the importance of huge drops in the markets. Statistically speaking, the probability of a crash is nearly impossible but they still have to be considered, as they will happen in real life, and this is why we can observe a skew pattern in implied volatility smiles.

[8] Note that this is vice versa for a put option. When X/S ratio is below one the put option is OTM (a right to sell cheaper than the market price) and when the ratio above one, the put option is ITM (a right to sell with a price higher than the market price).

3.2.2. Smoothing Process and Extraction of PDF's

When the volatility matrix data is built from observed prices, we need to smooth the curve which goes through the observed points and find the volatilities according to the standardized strike prices and which will be on the smoothed curve. Linear interpolation is too simple method for this purpose and it is often assumed that the volatility smile curves follow some function with a polynomial order of two or higher. In literature, two separate approaches for smoothing have gained popularity; the smile method (Shimko 1993, Chang & Tabak 2002) and the two mixed lognormals method (Miranda & Burgess 1998)[9]. The two lognormals mixture method basically assumes, as the name implies, that the fat-tailed and peaked implied probability distribution is a mixture of weighted averages of two independent Gaussian lognormal distributions. The assumption is that the price evolution process can be estimated significantly more precisely this way. This method will not be discussed thoroughly in this paper[10]. As mentioned earlier, our analysis will follow the smoothing method of Shimko.

Graph 3.2 shows the basic concept of a polynomial interpolation (quadratic). The line with dots presents for example the volatilities with given strike prices. The dashed line describes the smoothed polynomial trend line based on the observations.

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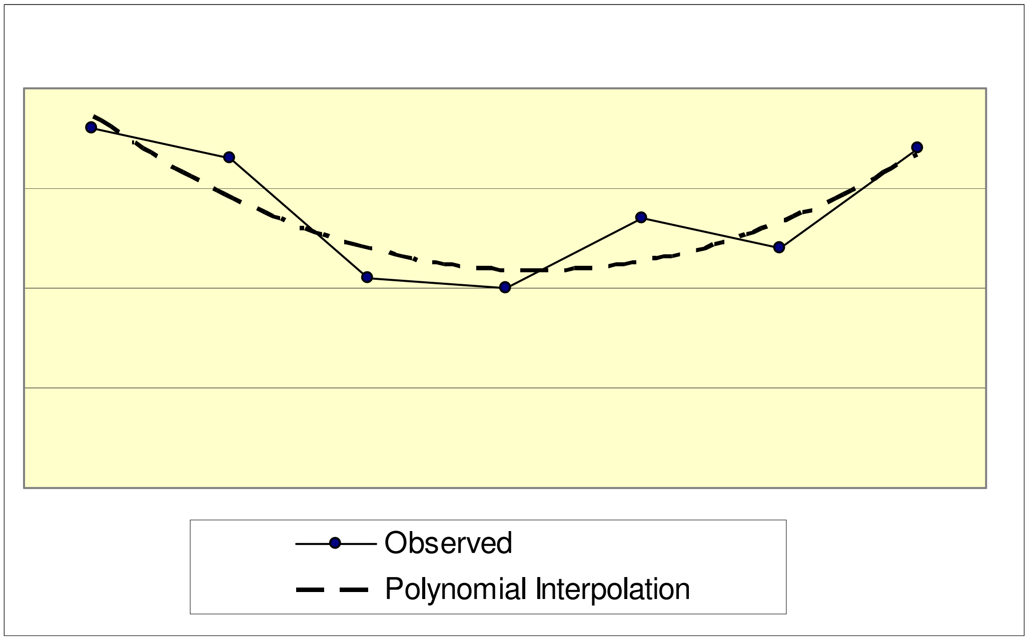
[9] The risk-neutral PDF can also be obtained directly as described by Rubinstein 1994, Jackwerth & Rubinstein 1996 and Pirkner et al. 1999. This approach is referred as a mixture binomial trees method and it is a numerical approach for PDF estimation.

[10] For more information on the two mixed lognormals method, see Ritchey 1990 or Pirkner et al. 1999.



Graph 3.2. The Concept of Basic Polynomial Interpolation

Two different interpolation methods come into consideration for the smoothing process,

1. Quadratic polynomial
2. Cubic spline

Quadratic (power of two) interpolation assumes the first derivatives to be continuous and uses linear interpolation for extrapolation (estimation of values outside the observed data range) while cubic (power of three) spline method uses the first and second derivatives and should create a smoother curve. Mathematically, the quadratic interpolation process assumes the first derivatives (the slopes) of the function to be the same in the meeting knots (the observed points of X's and corresponding Y's) of the functions. Cubic interpolation assumes also the second derivatives (the acceleration) to be continuous in the meeting knots of the functions. Least squares fit method is a quadratic interpolation method which minimizes the sums of squares of the residuals of observed points in the curve (Kelly 1967)[11].

In our case, our data sets need an assumption of a nonlinear curve and thus, a method of nonlinear quadratic least squares fitting is used. The cubic extrapolation method tends to overestimate the curve significantly, when the estimated extrapolated values do not lie close to the observed ones. More specifically, the curve from the data set seems to be sensitive to higher order interpolation. Thus, cubic spline method is in this case useless as it creates outlier values to our volatility surface (especially with the S&P 500 index option data which has limited observations). This phenomenon is not so important with the DAX index option series since they have significantly more observations with wider strike-to-index ratios, and often the observed ranger is wider than the one needed for standardization. Therefore extrapolation is not always needed. Due to the reason of fair comparison between the two surfaces, we will use the same nonlinear least squares method for both data series.

Note that even though interpolation methods discussed might seem too much of a simplification considering the complex nature of a volatility surface, many researches have came into a conclusion that a quadratic interpolation (and nonlinear least squares method) gives a fairly good estimation for the curve (Chang and Tabak, 2002) although Weinberg (2001) could not determine clearly which method was more superior to another. Bahra (1997) came to same conclusion; differences between smoothing methods are small. Smoothed volatility smile ensures that the call price function $c(X)$ derived from it is monotonic, continuous and can be differentiated twice, therefore meeting the prerequisites of the Breeden-Litzenberger PDF extraction method (Bahra 1997).

[11] See Bliss and Panigirtzoglou (2000) for more information about the cubic spline method.

The observed volatilities are not synchronized with the strike prices of series with different maturities, which can lead to a very disordered and moreover, useless, volatility surface. That is, instead of using exact strike prices, we interpolate volatilities for new strike prices based on percentage ratio of strike-to-index. In other words, we standardize the strike prices and calculate the new values with a percentage of an observed price of an underlying asset. In our study, the moneyness change is one percentage point denoting that 40 % ITM and OTM strike spread and 80 standardized observations. The observed ratios for S&P 500 index options and to-be-standardized ratios are presented in the table 3.1 along with the amount of observations,

Table 3.1. Observed Strike-to-Index Ratios for All Series, S&P 500 Index

	MAR	APR	MAY	JUN	JUL	SEP	DEC	Standardized
Low	0.92	0.89	0.98	0.85	1.00	0.81	0.85	0.60
High	1.07	1.19	1.17	1.26	1.13	1.41	1.46	1.40
Observations	21	34	10	22	11	16	16	80

Table 3.1 shows that the observed strike prices differ quite a lot during the maturities. The series with least maturity until expiration tends to have more observations and the series with longer maturities have less (naturally since trading is more active close to the exercise date and there is more demand for options on varying strike prices). The moneyness of option series is also wider with the longer maturity options. Table 3.2 presents the same data for DAX index option series,

Table 3.2. Observed Strike-to-Index Ratios for All Series, DAX Index

	MAR	APR	MAY	JUN	SEP	DEC	Standardized
Low	0.66	0.91	0.93	0.90	0.92	0.23	0.60
High	1.62	1.18	1.19	1.60	1.44	1.73	1.40
Observations	71	36	20	61	52	63	80

The DAX index option series have significantly more price observations as is the case with the wideness of strike price moneyness. Therefore there is no need for large scale extrapolation, unlike with the S&P 500 observations.

Mathematically, series of X's and Y's (the observed values of strike prices and volatilities, respectively) are used on determining the new value of Y (volatility) which matches the new given value of X (the percentage strike-to-index ratio price). The new value of Y is interpolated to lay on the polynomial curve fit by the least squares method (minimizing the sum of squares of deviation from the mean).

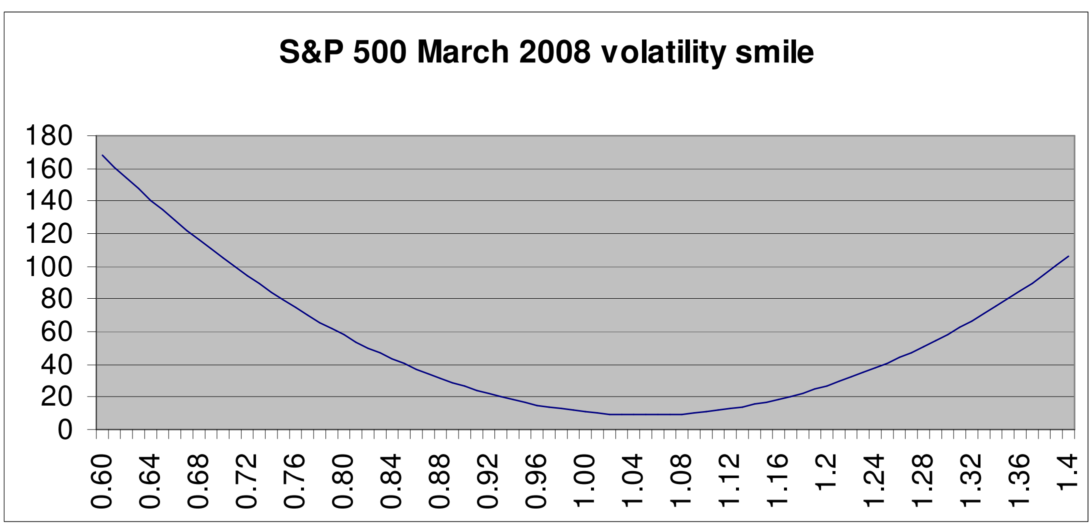
The smoothed smiles were used for calculating the PDF's with an equation 19. Note that since three call price observations are needed for calculation, the probabilities for the most extreme values were not possible to be derived. According to the equation, we will form a butterfly spread of two bought call options (with strike prices of $X-\delta$ and $X+\delta$) and two sold call options with strike price of X. In this case the spread is small, one percent in ratio of X/S, which is in index points around 13 and 69 for S&P 500 and DAX, respectively. After we process the equation for all strike prices we obtain the PDF $g(X)$. The results for analysis are discussed in the next section.

4. Results

The results for analysis are discussed in this section. The main hypothesis was to determine if BS model does a fair estimate in pricing the options on index futures markets in the United States and in Europe. The indices chosen present two different types of markets, a pit-traded and an electronic market, and the objective was to compare the pricing tendencies on different trading platforms. In other words, to determine if one's implied volatilities differs more from the premiums suggested by the Black-Scholes model and if it correlates with the liquidity of the market. Implied volatility smiles are used to determine implied probability distribution functions which are interpreted and based for trading strategies in the last part of this section.

4.1. Volatility Smiles and Surfaces

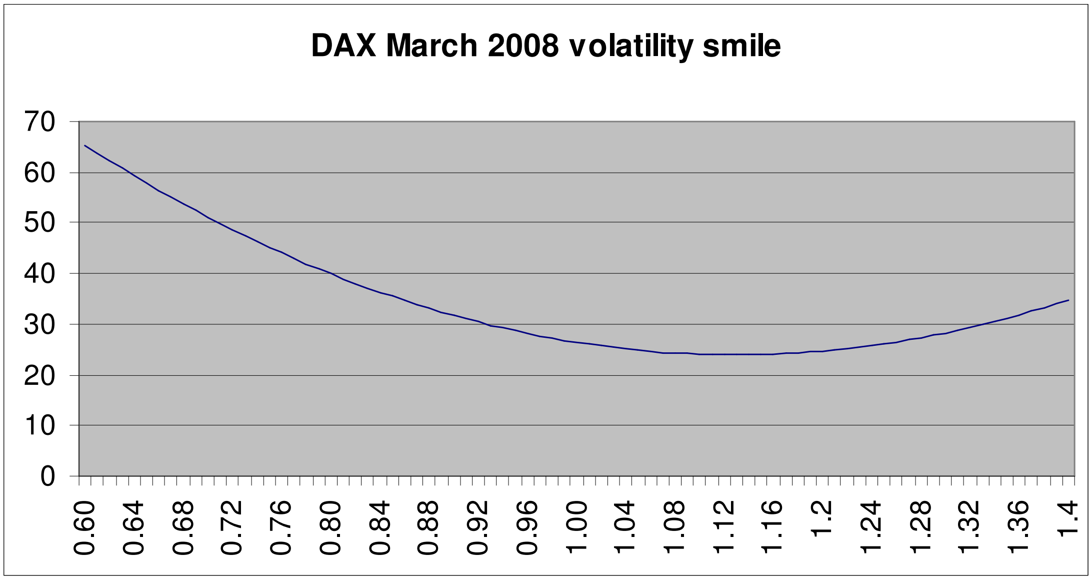
The graph 4.1 presents the smoothed volatility smile for March 2008 S&P 500 index option (expiration in 28 days).



Graph 4.1. S&P 500 smoothed volatility smile, March 2008

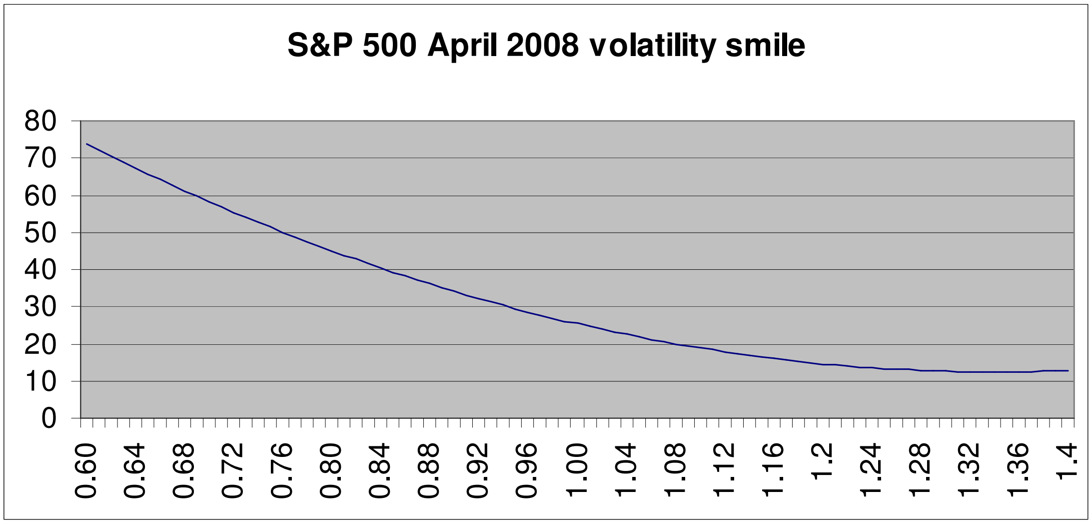
X-axis denotes the strike-to-index ratio and Y-axis the implied volatility. The implied volatility smile curve follows a quadratic order and is estimated with a least squared method described earlier. When compared to original, “rough” and non-standardized volatility smile, one should note the heavily extrapolated tails (graph 3.1). The original data set had observations ranging within the ratio of 0.92 to 1.07 and after the smoothing and extrapolation; the moneyness ranges 40 % in-the-money and out-of-the-money. Naturally, the reliability of estimation suffers when the curve is extrapolated for far outside of the range of the original data set. But in the other hand we can not either assume the volatility to be constant outside the observed range as Shimko (1993) does in his research. As we know from the original observations, the volatility seems to have some kind of increasing nature towards ITM. Therefore the assumption of constant volatility outside observed range can not be used. The curve has a smile shape (the function is convex) and there is a noticeable difference between the ITM and OTM volatilities for a given moneyness[12]. The most extreme ITM volatility is well above the corresponding OTM volatility (over 160 % versus 100 %) Graph 4.2 shows the volatility smile for DAX index option series with the same maturity.

[12] Note that we are discussing about call options here.



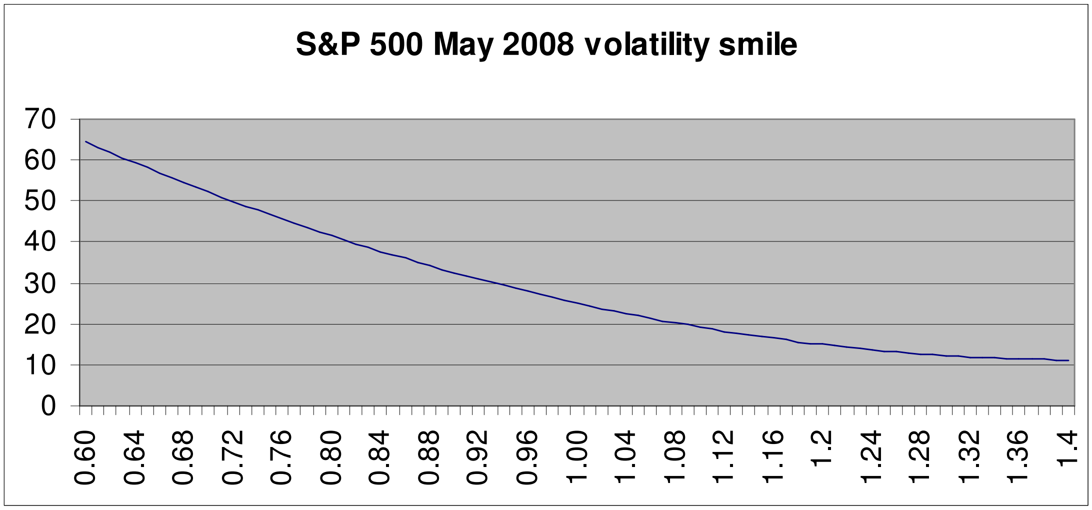
Graph 4.2. DAX smoothed volatility smile, March 2008

Due to the electronic trading (and possibly more active trading also), the estimated volatility for DAX index futures options is significantly lower on tails and higher at-the-money (ATM) than is the case with the S&P 500 smile. There is a significant ATM volatility difference between the two series, S&P 500 ATM volatility being less than 11 % while DAX series have ATM volatility of over 26%. This indicates clearly in this point that the DAX data has a lower kurtosis than the S&P 500 series and it might follow the standard normal distribution better than the S&P 500 data. That is, the S&P 500 data then seems to be leptokurtic (high kurtosis denoting a higher peak and fatter tails than the normal distribution).

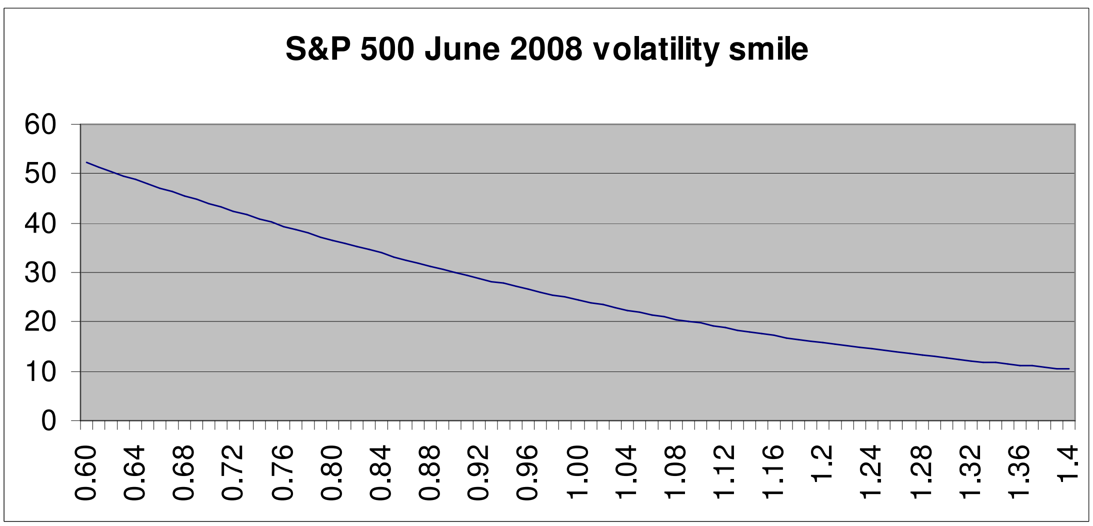


Graph 4.3. S&P 500 smoothed volatility smile, April 2008

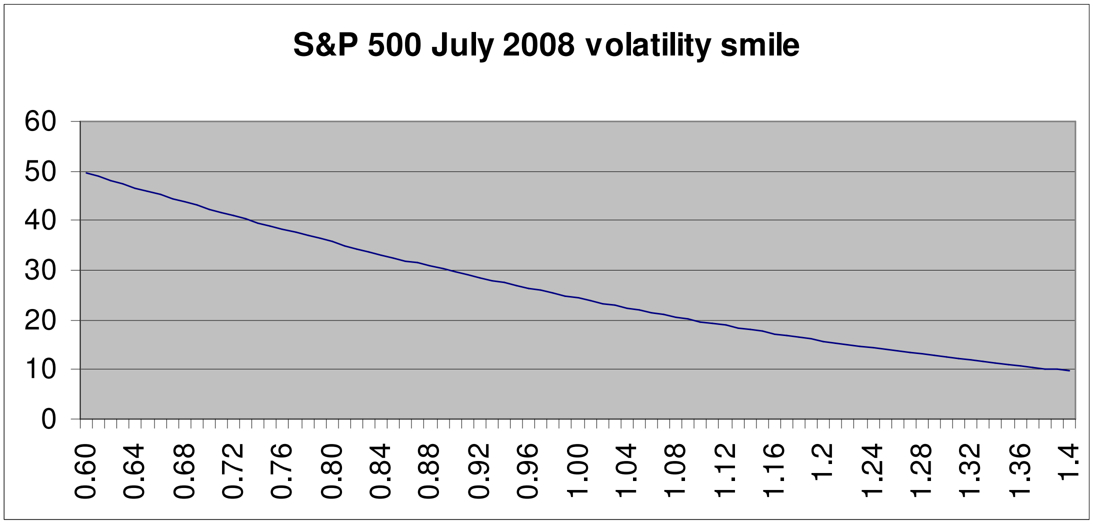
The volatility curves for S&P 500 option series with longer maturities differ from the March 2008 clear smile-shaped curve (graphs 4.3 to 4.8). The shape of the curve is more like a skew than a smile. This behavior of options on index futures is also noted by Vähämaa (2004). The volatility is high on deep ITM strike prices and the trend is declining when the strike increases. April 2008 volatility smile forms a slight curve shape on extreme OTM strikes but this can not be observed from option chains with longer maturities which have clearly skewed shapes and steadily decreasing volatilities over increasing strike.



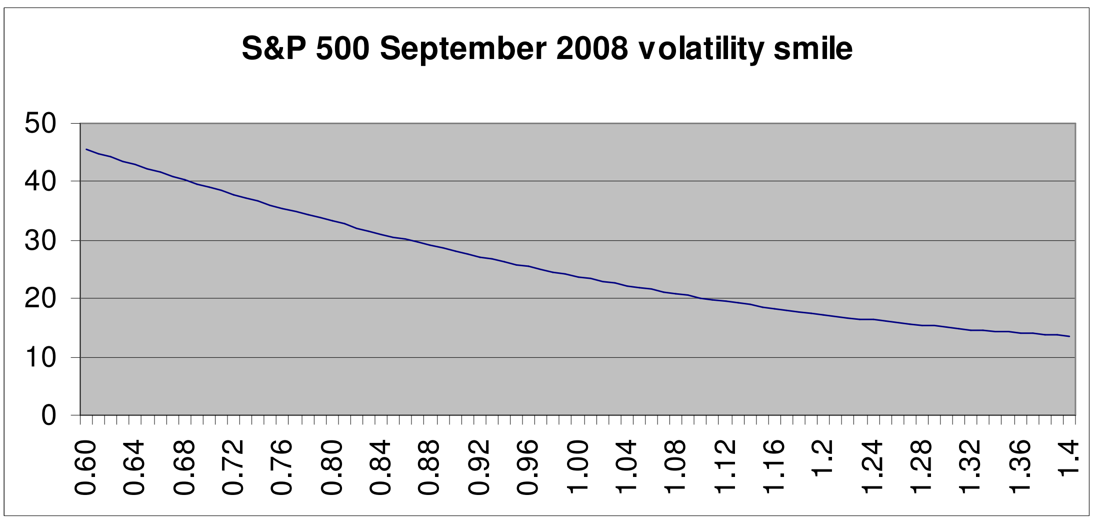
Graph 4.4. S&P 500 smoothed volatility smile, May 2008



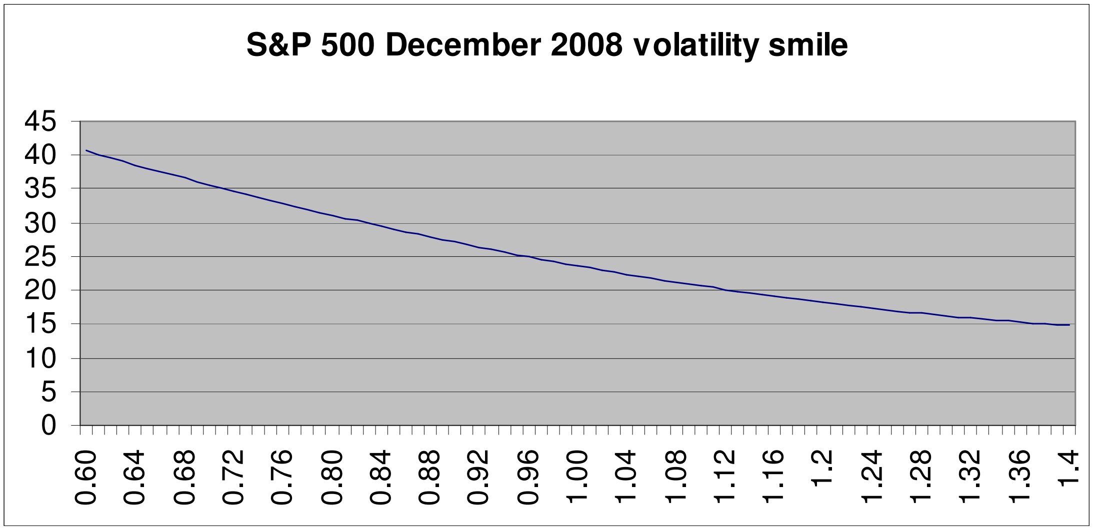
Graph 4.5. S&P 500 smoothed volatility smile, June 2008



Graph 4.6. S&P 500 smoothed volatility smile, July 2008

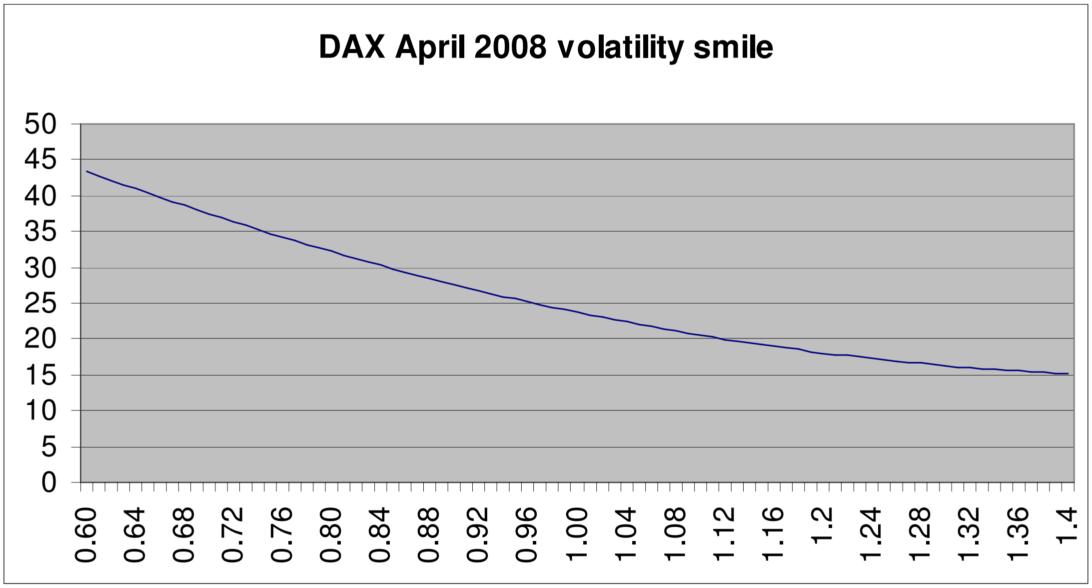


Graph 4.7. S&P 500 smoothed volatility smile, September 2008

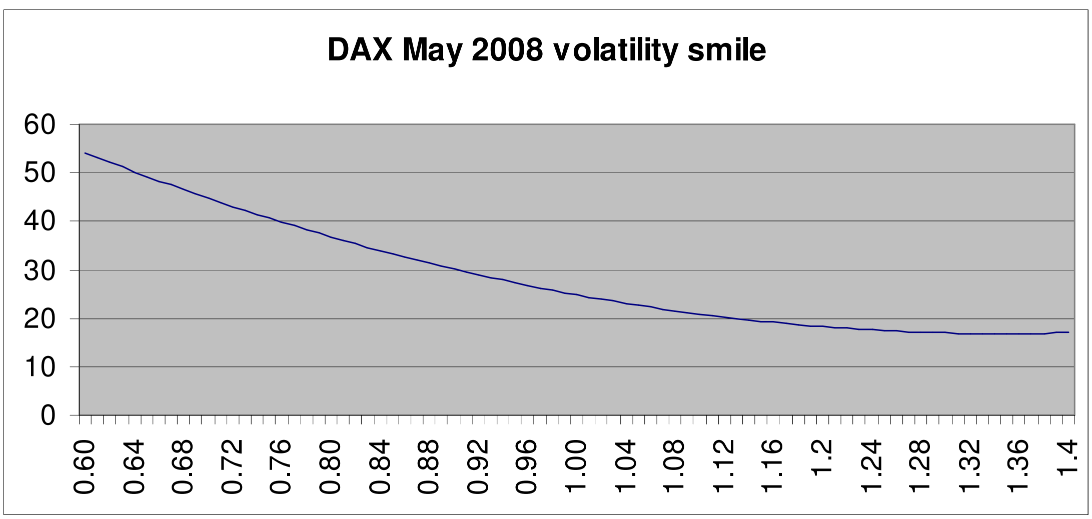


Graph 4.8. S&P 500 smoothed volatility smile, December 2008

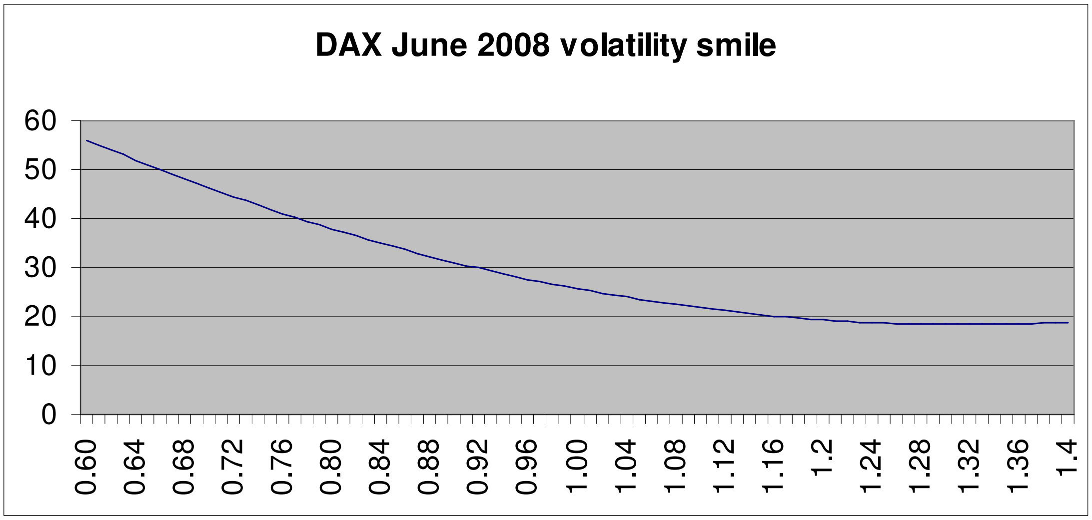
We can determine from smoothed volatility smiles that the volatility level seems to be an increasing function of time (graphs 4.4 to 4.8). When the time passes (the maturity shortens) the volatility level increases steadily. Volatility change is higher on deep ITM strike prices than with the deep OTM strikes which are quite conservative during most of the lifespan of an option. Some decreasing tendency can also be noticed from OTM volatilities during the maturity but in general, they are within the range of 10 to 15 %. This study concentrates on different option chains in a discrete moment of time so this notation should be confirmed with a time series of observed quotations, preferably with a research consisting of daily observations from the whole lifespan of the option chain. That is, to be able to transfer the observed pattern into a successful trading or hedging strategy. Graphs 4.9 to 4.13 present the implied volatilities for DAX option chains from April to December.



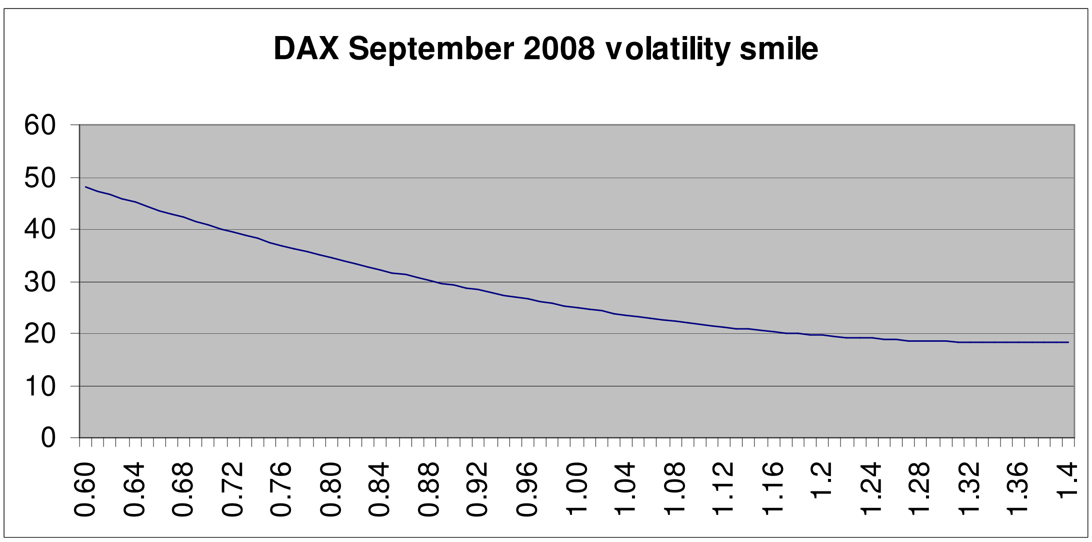
Graph 4.9. DAX smoothed volatility smile, April 2008



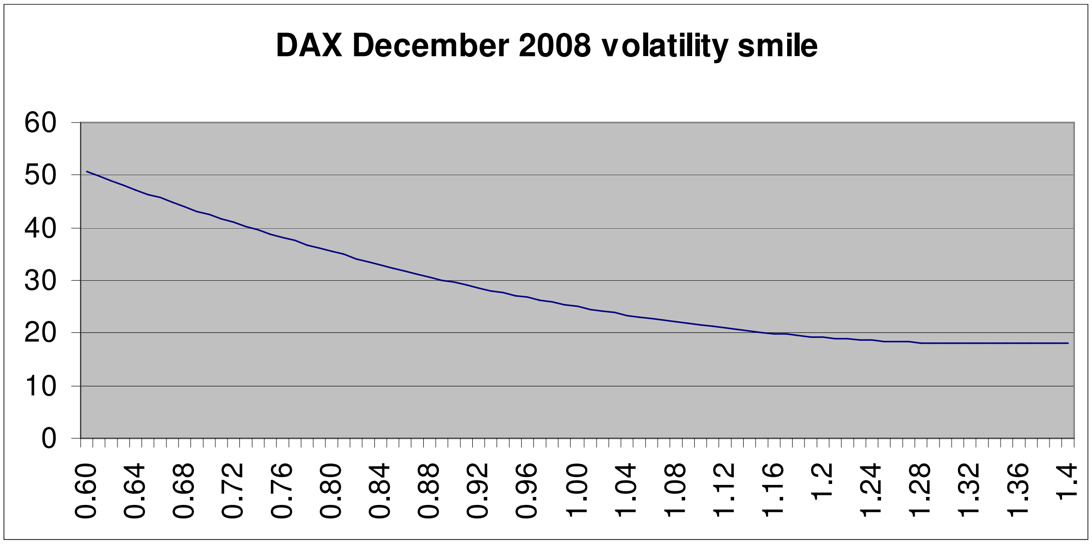
Graph 4.10. DAX smoothed volatility smile, May 2008



Graph 4.11. DAX smoothed volatility smile, June 2008

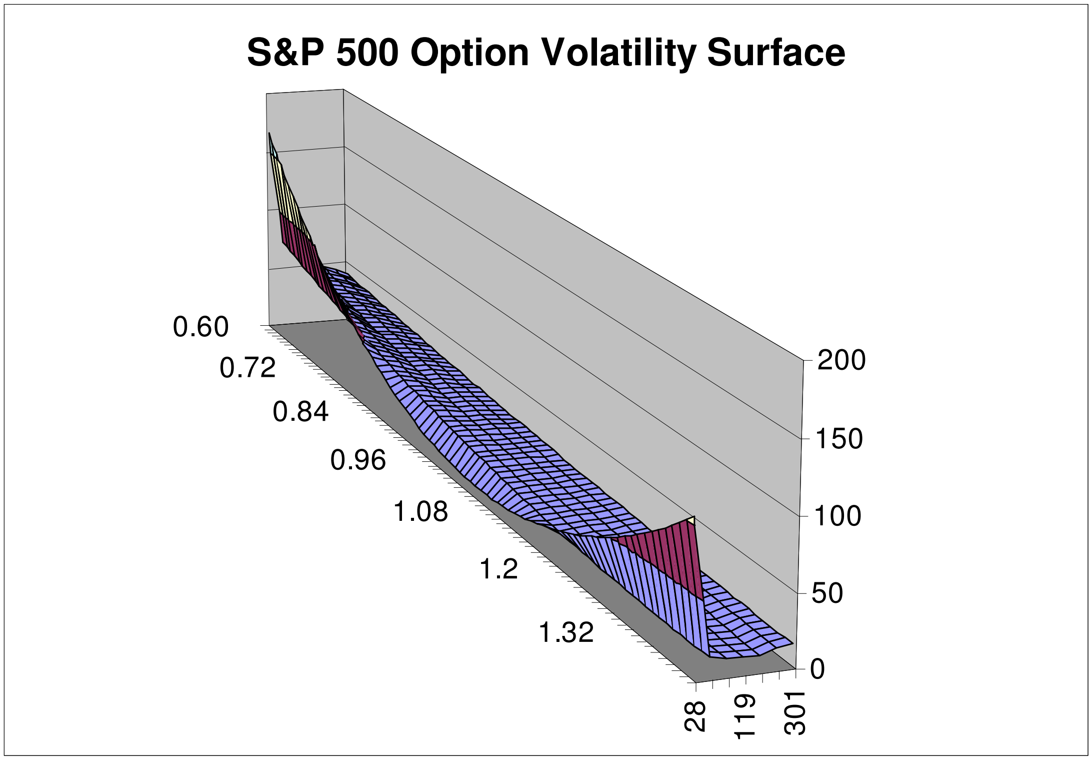


Graph 4.12. DAX smoothed volatility smile, September 2008



Graph 4.13. DAX smoothed volatility smile, December 2008

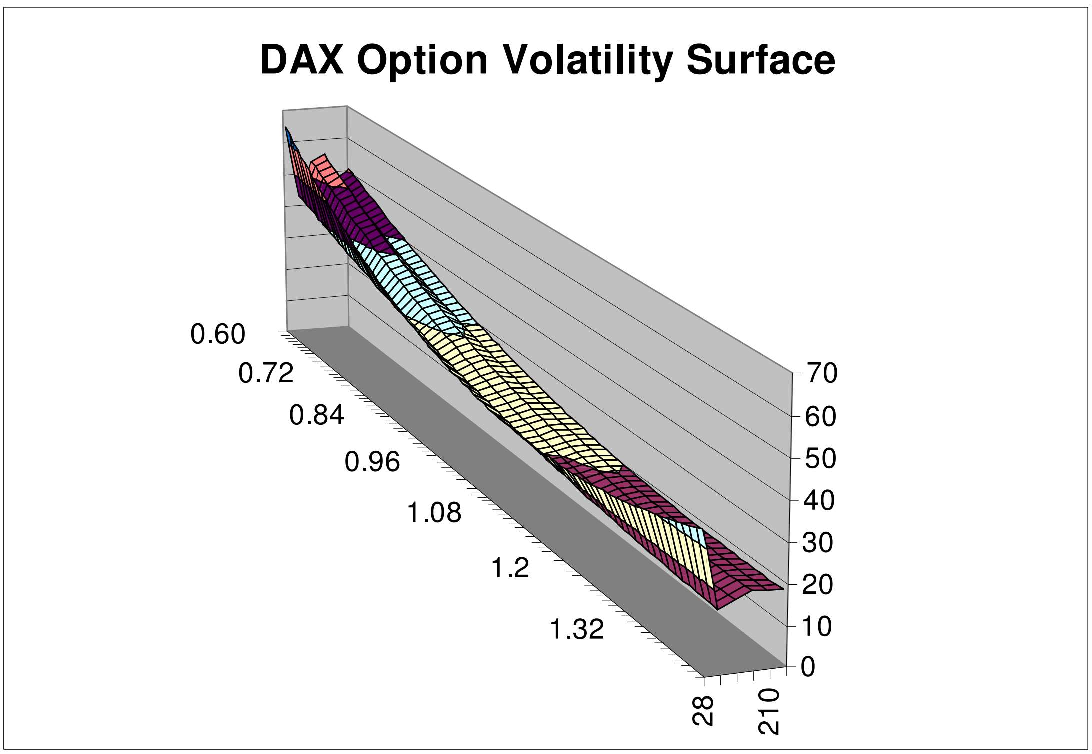
The pattern is similar to S&P 500 option series; the option chain with the shortest maturity shows a convex volatility curve while the longer maturities form a skew. The increment in the volatility level over time could not be observed so clearly in DAX series thus indicating more harmonious volatility change between option series than with the S&P 500 series. As noted above, the DAX option series tend to be less leptokurtic and therefore they should fit the normal distribution better than the S&P 500 data. The market-implied volatility surfaces are plotted in graphs 4.13 and 4.14.



Graph 4.13. S&P 500 option volatility surface on 22nd of February 2008.

In graph 4.13, X-axis denotes the strike-to-index ratio, Y-axis the maturity (from March to December, 28 days to 301 days) and Z-axis the implied volatility. The volatility surface presents the market estimation of the implied volatility for traded maturities at one discrete moment in time and does not describe the future evolution process of volatility. For such analysis, a historical daily data set for a long period of time would be required, preferably observations from the whole period of maturity. Also, the predictive power of an outcome of such research should be significantly higher. This type of analysis can be both time and resource-consuming since daily volatility smiles are needed for one series.

It can be easily seen from the graph how the option series with shortest maturity tends to be the most heavily traded one. The volatility tends to vary more over strikes with the option series expiring in 28 days than with the options with longer maturities. This happens due to the extrapolation problem described earlier. There are significant variations in implied volatilities. This is understandable when we take into consideration two facts that will explain the differences in volatility; the uncertainty of an underlying price and the process of stochastic volatility over time. That is, the probability for an option contract to be in-the-money at the expiration date is greater for options with a longer maturity than with the ones with a shorter maturity. Similar changes in volatility between the maturities can be detected also for the DAX option chains (graph 4.14),



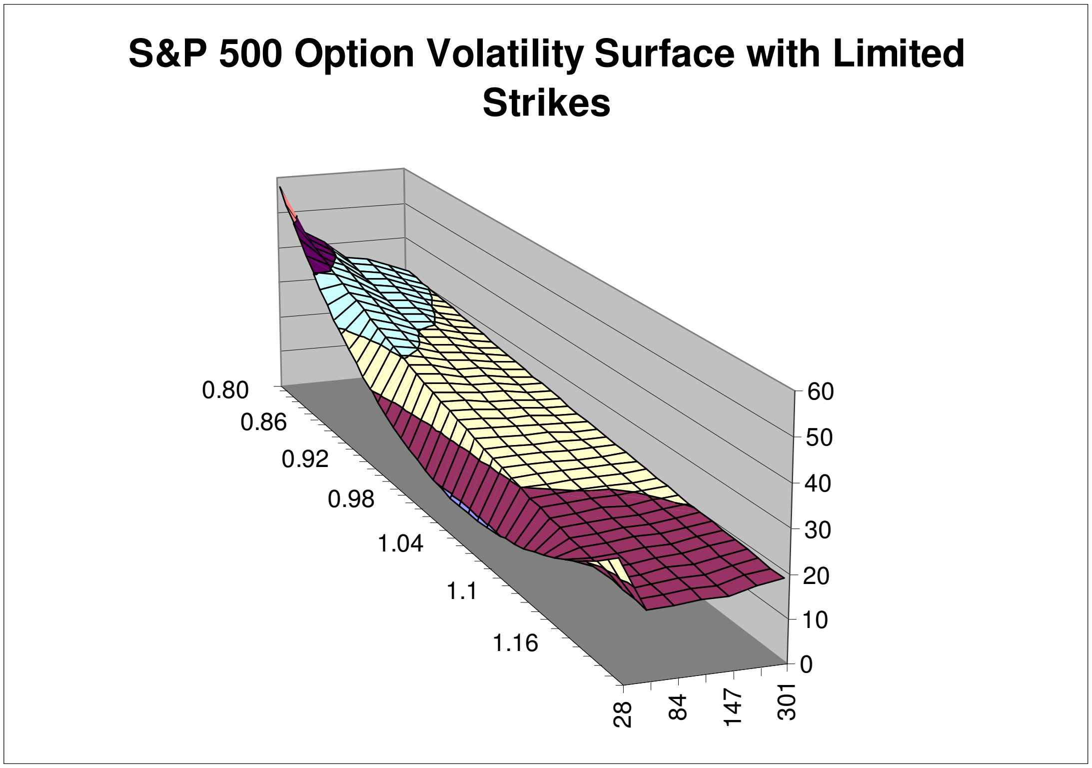
Graph 4.14. DAX option volatility surface on 22nd of February 2008.

The volatility variation tends to be more conservative with options of longer maturities, but shows significant increase when approximating the expiration date. Thus, our surfaces indicate two facts that a practitioner should keep in mind. Firstly, Vega, the sensitivity of a call price to an increase in implied volatility, is more important variable if one's portfolio consists of options with shorter maturities. Furthermore, Vega should be carefully monitored since the price changes rapidly when Vega changes (Shimko 1993).

Secondly, S&P 500 options have a higher Vega than DAX options since the surface seems to possess a lot of variation in volatility. This is indicating that one should carefully examine the market (the index) since different underlying assets have individual volatility surfaces which in addition, are always in a continuous and dynamic transformation (stochastic volatility). Traders could then make bets against the market and try to profit from the changes in implied volatility level.

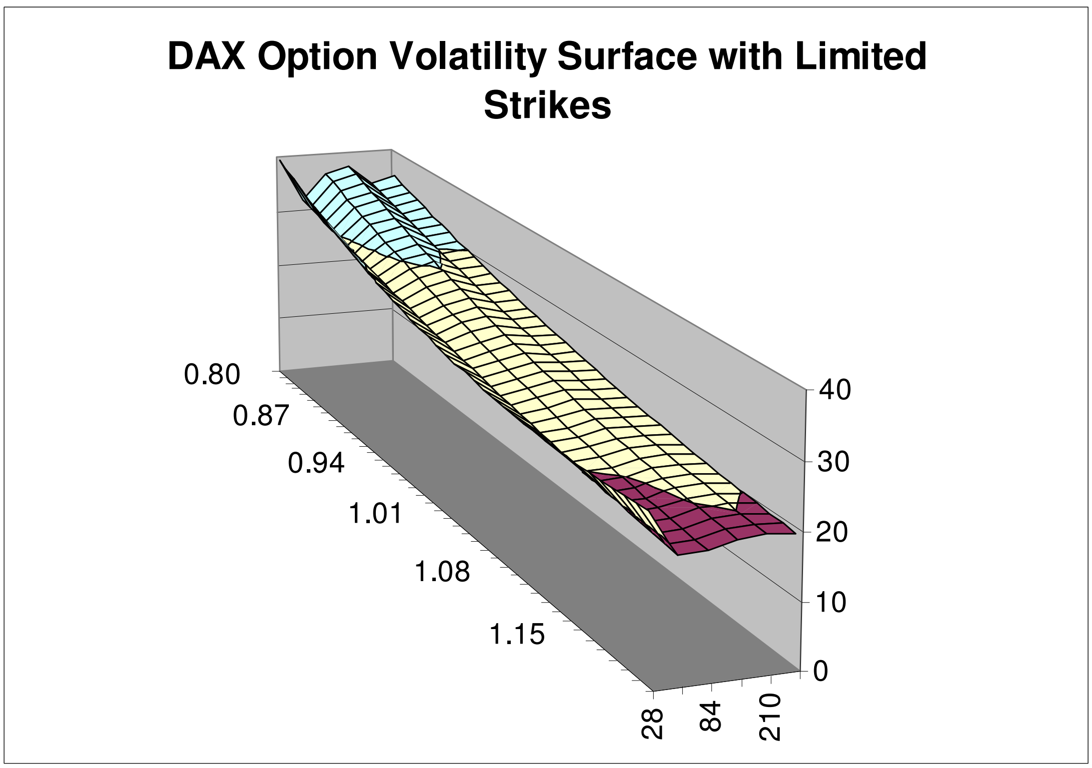
Generally, these *volatility trading strategies* are based on the shape of the current implied volatility curve or surface and the trader's estimation of the future outcome. The trader takes a position that generates profit if his perception of the future change in volatility curve is correct. For a Delta hedger, it is important to understand the relation of implied volatility and Delta and not to forget Gamma which measures the change in price when Delta changes. Delta defines the sensitivity of change in the value of an option when the underlying price changes, for example $1. If implied volatility increases, the delta will approximate to 0.5 (for call option, -0.5 for put option). This means that the ITM option has more probability to be OTM at expiration and vice versa. A portfolio consisting of one short option contract and 0.5 long underlying assets will be then neutral to a price change of an underlying. Similarly, when implied volatility decreases the ITM option Delta will approximate to 1.0 and OTM option to zero (-1.0 and zero for a put option). The probability of an option to stay ITM or OTM at expiration increases. When the volatility changes, hedger will adjust his position accordingly and the aim should be to have the correct hedge ratio estimated for the portfolio and a Gamma of zero. In this case the portfolio is truly neutral and the Delta is not going to change rapidly which decreases the need to constant adjustment to a portfolio. Implied volatility smiles and surfaces come handy in estimating the current volatility level and what the volatility level might be in future. In practice, if the portfolio is formed with short ITM call options and long underlying assets and the current implied volatility surface is similar to ours in graph 4.13, we could assume that the implied volatility level has an inverse relation to maturity. The term structure of volatility would be then skewed towards the longer maturities. The increment in implied volatility means that the Delta of our option contract will decrease. Therefore we need less long positions in an underlying asset for one short option contract in order to have a portfolio Delta close to a zero and indifferent to a price change of an underlying. Incorrect volatility estimation would lead to an incorrect Delta ratio and to a non-optimal portfolio. Therefore it is crucial for a hedger to know the implied volatility and its relation to a hedge ratio. Vähämaa (2004) studied the inverse relation between the stock price and volatility and the effect of it to hedge ratios. He came to a conclusion that since the volatility is not constant as assumed by BS model the hedge ratios used will be exaggerated. The trading strategies based on implied variables are discussed more thoroughly in the next Section.

For illustrative purposes, the graph 4.15 presents the S&P 500 volatility surface with a strike-to-index ratio closer to ATM (20% ITM and OTM) and which will describe more clearly the difference between options with different volatilities and maturities and without the extreme values obtained by extrapolation.



Graph 4.15. S&P 500 option volatility surface with limited strike prices.

We can see from the graph the same things that were noted earlier. The series with shortest maturity has a clear smile pattern while the series with longest maturities form a skew pattern with an increasing volatility towards the expiration date. In both cases (graphs 4.15 and 4.16), the implied volatility is higher for ITM options. When examined closer to ATM, the S&P 500 surface has a lower volatility than the longer maturities. The DAX surface differs in this matter, the ATM volatility is slightly higher for March than for April option chain. Also, the DAX March curve shows only a slight smile shape and the overall surface tends to be significantly more harmonious than the surface for S&P 500 index futures. One explanation for this could be the trading system used and the settlement pricing process of option series as discussed earlier. Pit-traded options could be priced more individually, based on the supply and demand, unlike in electronic trading places like EUREX where most of the option price quotations could be calculated from the nearby strikes and not entirely individually by the supply and demand in the market place. The fact that observed DAX strike quotations are extremely standardized when compared to S&P 500 quotations is strongly supporting this perception. Moreover, this could lead to a reduced possibility of option-implied information in the traded quotations as the importance of "market knowledge" is reduced via automated trading.

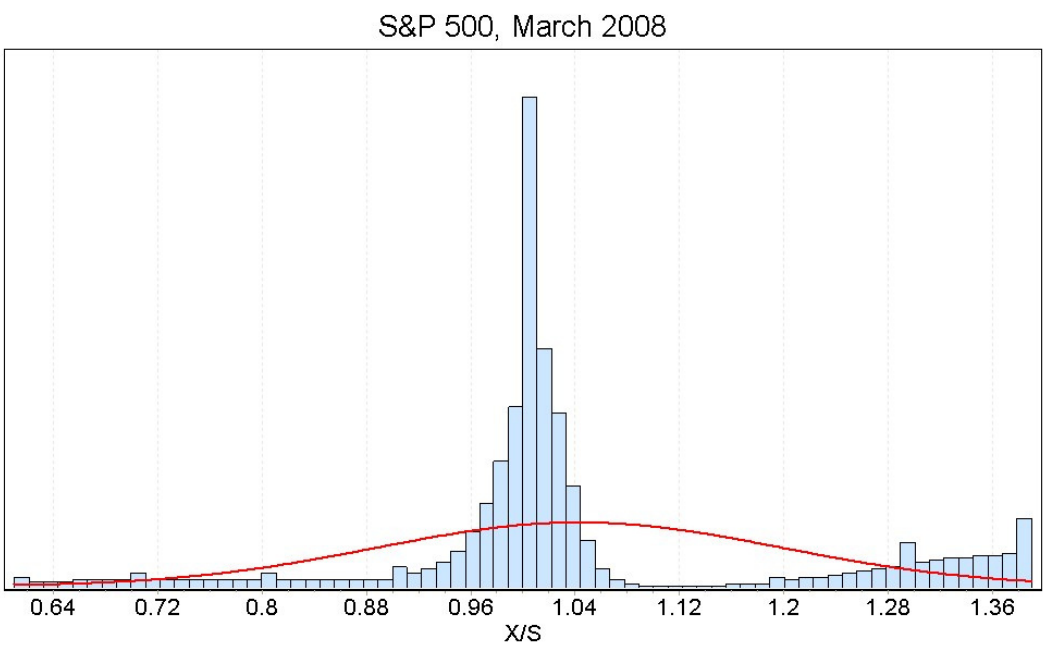


Graph 4.16. DAX option volatility surface with limited strike prices.

Moreover, the existence of non-flat volatility surfaces for both option series clearly demonstrates that the assumption of the BS model does not hold and needs to be relaxed. These findings are consistent with the findings of other researches and the literature of inaccurate pricing by BS model is extensive. Vähämaa (2004) argues that these inconsistencies are caused by market imperfections such as bid-ask spreads, illiquidity and non-continuous trading. These reasons could affect our smiles too, especially the illiquidity and non-continuous trading. In next section we define the implied probability distribution functions based on the implied volatility smiles presented in this section.

4.2. Risk-Neutral Probability Distributions

The risk-neutral probability density functions were constructed with the Breeden-Litzenberger method described earlier in Section 2.2.7. The following graphs compare the obtained distributions to a standard Gaussian normal distribution. The main interest is to determine if there is any difference from a normal distribution in the distributions based in market quotations. Graph 4.17 describes the obtained results for S&P 500 March 2008 option chain.

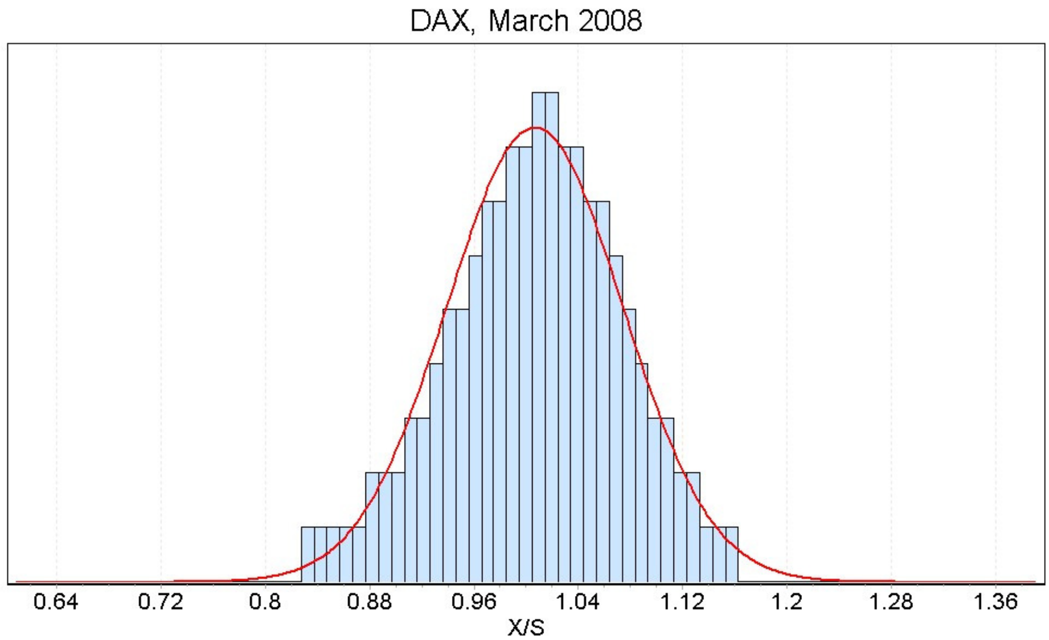


Graph 4.17. S&P 500 Probability Distribution Function, March 2008

The histogram shows the probabilities of different strike prices based on the data and compares it to the standard normal distribution which is denoted by the curve. The histogram is highly leptokurtic (extremely high peak and fatter tails than the normal distribution). It is significantly more probable to have situations in which the strike-to-index ratio is more than 1.30 while the difference in deep ITM moneyness is not as significant. Also, the high peak also suggests that the normal distribution highly underestimates the probabilities very close to ATM moneyness while overestimating the probabilities of X/S ratio being in the ranges of 0.75 to 0.95 and 1.05 to 1.25 at the expiration.

Chang and Tabak (2002) have noticed a deviant pattern from the normal distribution in currency options markets but their proprietary distribution shows a clear pattern of bimodality. Ritchey (1986) has also observed similar tendencies. In our study the pattern is not as obvious as is the case with the results of Chang and Tabak as we have not been able to identify the bimodal behavior so clearly. Also their constructed distribution has a second peak close to the ATM moneyness (X/S close to 1.20) while our S&P 500 data shows higher probabilities for extreme deep OTM moneyness only (X/S over 1.30). These bimodal distributions exist when there are two clearly separate and individual possible future outcomes for an underlying price. A good real life example could be a presidential election where only 2 candidates exist. Although bimodal distributions also have been recorded from the financial markets, they tend to be rare cases. Not many markets can be expected to have two clearly separate outcomes. Naturally, one has to keep on mind our extrapolation method and its limitations while interpreting our results.

The PDF for DAX March 2008 option series shows a different pattern (graph 4.18). The implied distribution follows the normal distribution extremely well. There can not be observed any type of high peaks nor fatter tails.



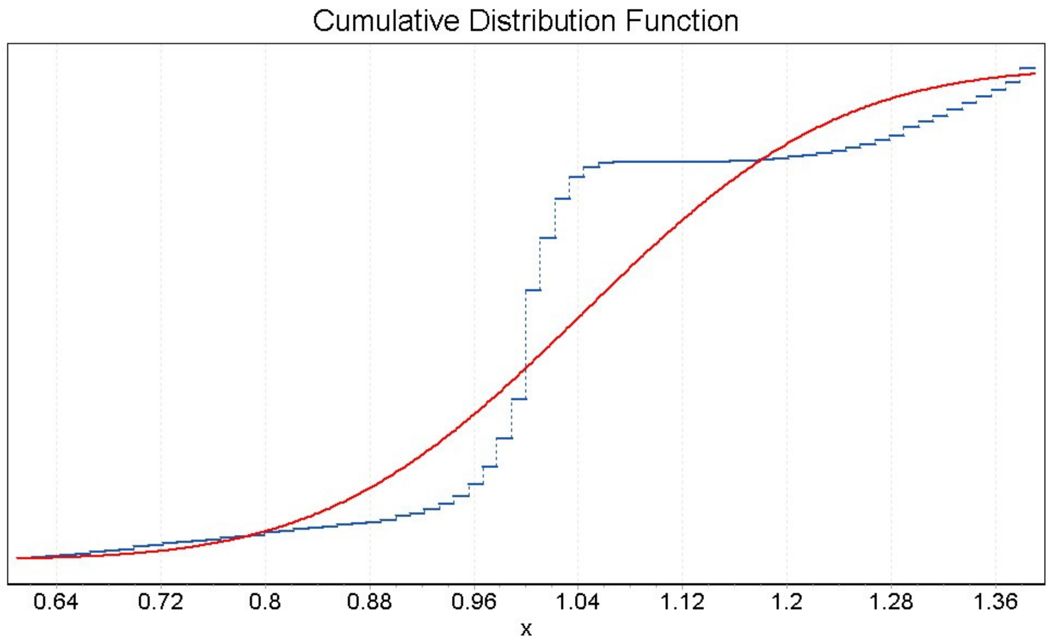
Graph 4.18. DAX Probability Distribution Function, March 2008

The original data set is therefore highly correlated with the normal distribution. And when taken into consideration the fact that the original observations for DAX March 2008 options range from 4,500 index points to 11,000 (or strike-to-index ratio of 0.66 to 1.61) we still could not find any pattern which would differ from the graph 4.18. The probabilities beyond 1.40 ratios tend to be extremely low, as the normal distribution suggests. The extrapolation factor does not concern the DAX option series then. Table 4.1 defines skewness and kurtosis by numbers for all option chains,

Table 4.1. PDF Descriptive Statistics

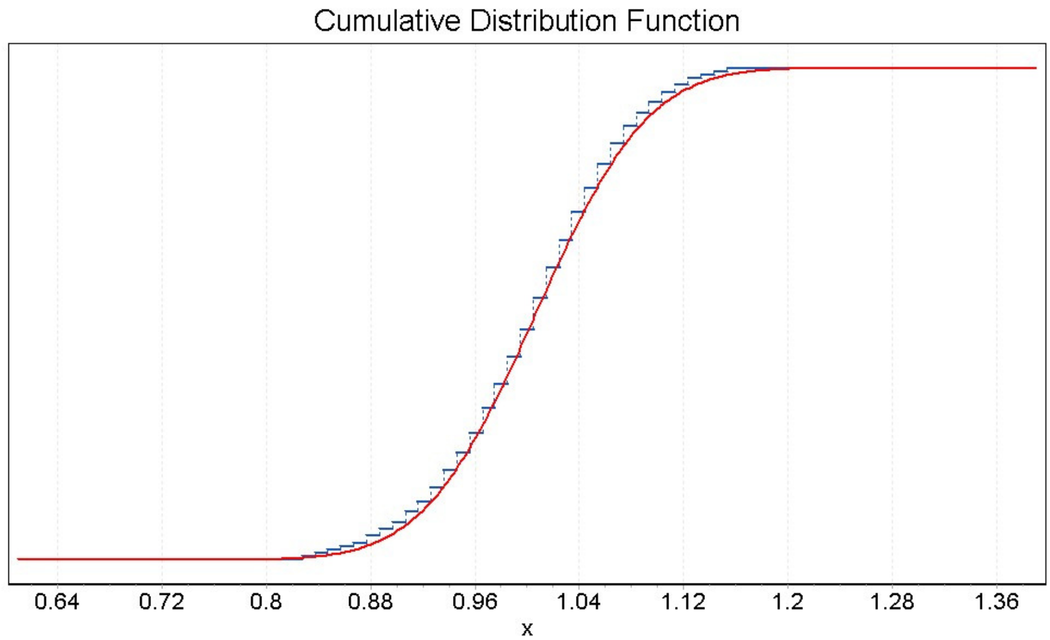
S&P 500	March	April	May	June	July	September	December
Observations	79	79	79	79	79	79	79
Skewness	3.103	1.230	0.989	0.711	0.605	0.378	0.158
Std. error of skewness	0.271	0.271	0.271	0.271	0.271	0.271	0.271
Kurtosis	9.187	0.072	-0.049	-0.098	-1.112	-1.341	-1.453
Std. error of Kurtosis	0.535	0.535	0.535	0.535	0.535	0.535	0.535
DAX	**March**	**April**	**May**	**June**	**September**	**December**	
Observations	79	79	79	79	79	79	
Skewness	1.385	0.991	0.816	0.631	0.344	0.268	
Std. error of skewness	0.271	0.271	0.271	0.271	0.271	0.271	
Kurtosis	0.474	-0.538	-0.837	-1.103	-1.377	-1.393	
Std. error of Kurtosis	0.535	0.535	0.535	0.535	0.535	0.535	

Since the normal distribution is symmetric in shape, it has a skewness and kurtosis of zero. All distributions are skewed to the right since they have a positive skewness furthermore, suggesting that the OTM moneyness probabilities are higher than ITM probabilities. In other words, there is more probability mass on the right side of the mean of the distribution. Kurtosis is extremely high on S&P 500 March distribution but decreases steadily over maturity. The high kurtosis can be also noted from the histogram in graph 4.17. The kurtosis is significantly lower for DAX distributions and the decreasing tendency is similar to S&P 500 series. This indicates that the S&P 500 March option chain does have a high peak while the DAX chain is only statistically speaking slightly peaked.



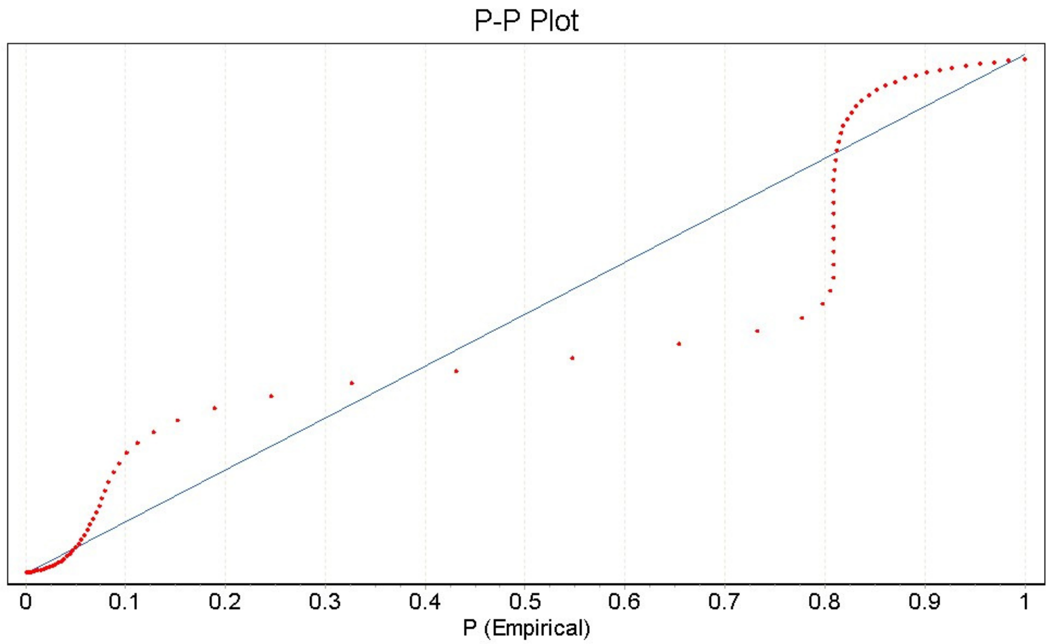
Graph 4.19. S&P 500 Cumulative Distribution Function, March 2008

Graph 4.19 shows the cumulative distribution function for the S&P 500 March series. It shows the exact same information than the probability distribution function but describes how the probabilities should accumulate when summed together if the distribution is symmetric in shape. As we are talking about probabilities, the sum of cumulative probability is exactly 1.0. The deviation from the normal distribution is more easily observed from CDF graphs. E.g. note the overestimation of the normal distribution between the ranges 0.75 to 0.95 and 1.05 to 1.25 discussed earlier (the ranges differ some due to the accumulation). Graph 4.20 confirms the earlier notation of tight positive correlation between the DAX distribution and the normal distribution. Only slight deviation can be observed.



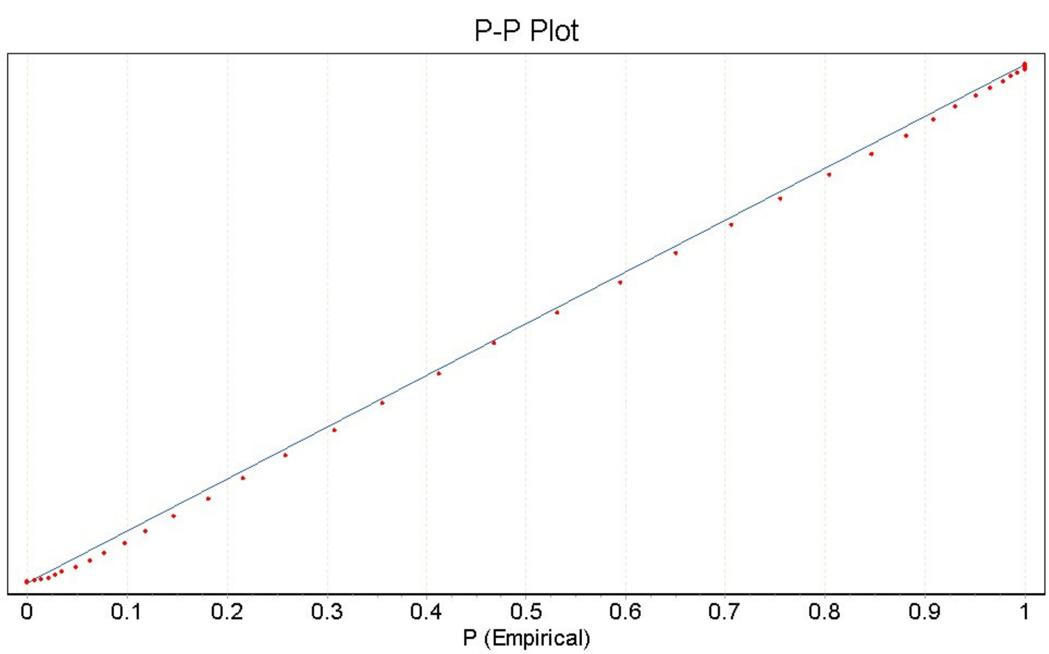
Graph 4.20. DAX Cumulative Distribution Function, March 2008

In addition to PDF and CDF graphs, also P-P (Probability-Probability) plots are presented to further explain the degree of fit between the option-implied distributions and the normal distribution. P-P plot differs from CDF graph in the way that it plots the distributions' cumulative proportions.



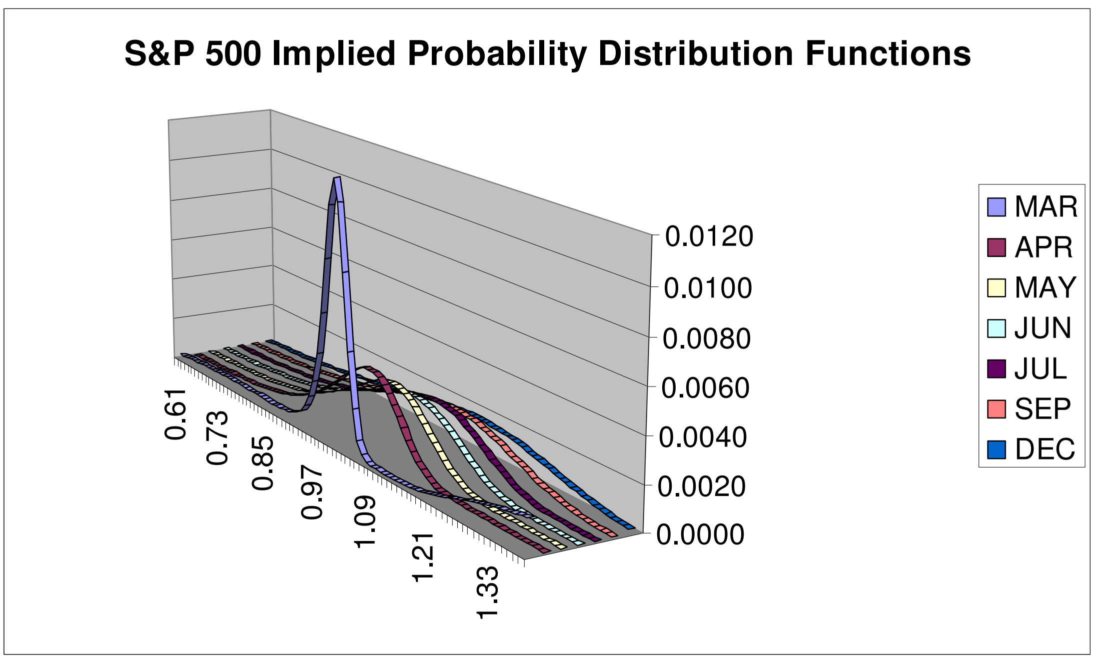
Graph 4.21. S&P 500 P-P Plot, March 2008

Our customized distribution of S&P 500 shows significant deviation from the normal distribution also in P-P plot (graph 4.21) and DAX distribution fits the normal distribution quite nicely (graph 4.22).

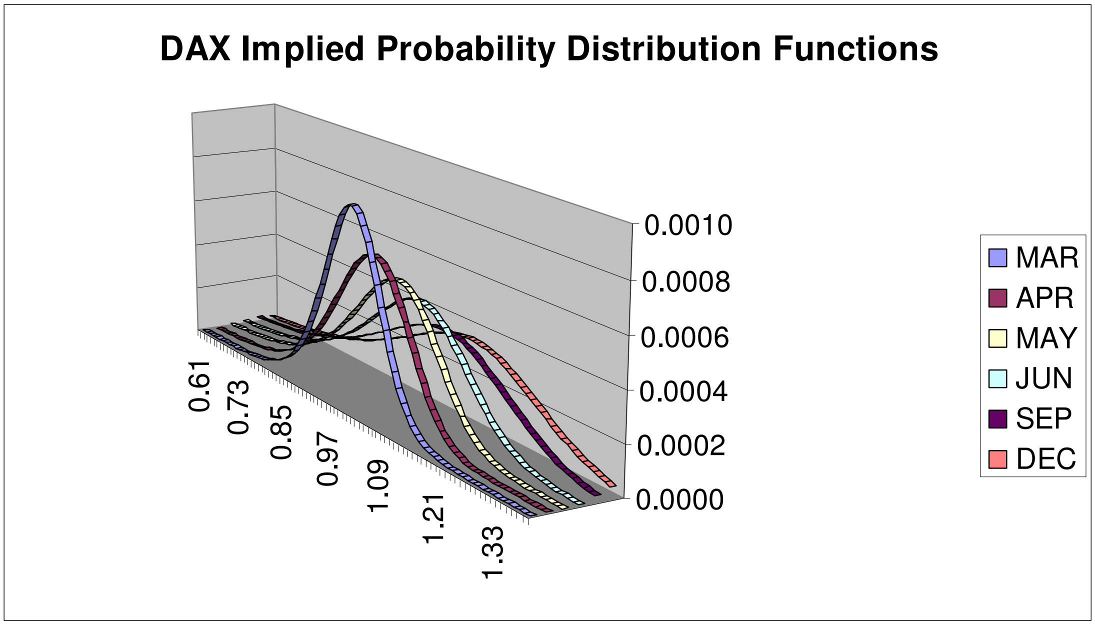


Graph 4.22. S&P 500 P-P Plot, March 2008

The graphs for rest of the maturities can be found from the appendix D (PDF, CDF and P-P plots). The results on rest of the DAX distributions were highly similar to one discussed in this section and no significant differences from the normal distribution could be detected. The same pattern of decreasing deviation from the normal distribution could also be observed from S&P 500 distributions. Graphs 4.23 and 4.24 present the distributions for all maturities in the S&P 500 and DAX series. The graphs clearly illustrate what is the time-varying nature of implied probability distribution functions and how heavier trading affects the probabilities.



Graph 4.23. S&P 500 Implied Probability Distributions



Graph 4.24. DAX Implied Probability Distributions

Since one might wonder why market implied probability distributions are important for practitioners, the next section will explain the strategies based on implied distributions further. It was discussed earlier how forward-looking methods are considered superior predictors for future changes in the underlying asset price when compared to historical models. Implied volatility can be used as an indicator of current market trends, e.g. the cheapness or expensiveness of an option contract can not be observed from the underlying price only but from volatility instead. Also, the price of an option contract today is useless when compared to the price in past because the time factor is a very important variable which has to be taken into a consideration. The riskiness of a given option is then used as a standardized pricing tool for similar option contracts.

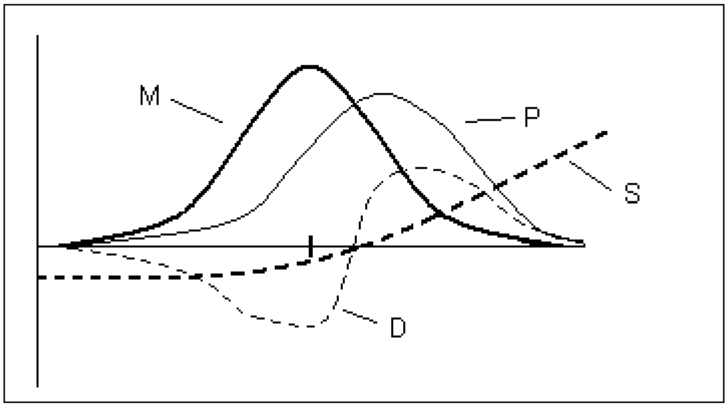
4.3. Probability Trading Strategies

Traders can use smiles, surfaces and implied distributions for quickly estimation if their option contract is expensive or not and to adjust their taken positions accordingly. The concepts of implied volatility smile and probability distribution are closely tied together. The strong convex smile function indicates that the PDF will have fat tails and higher peak. If practitioner's view of the future probability of prices and volatility does not follow the market view, he can design a strategy to bet against it. Of course, this requires that the implied volatility surface (and smile) is based on a sufficient amount of observations (no illiquid option series) so a good estimation for the "correct market probability distribution" can be described. In practice, this becomes very difficult since even the most liquid option markets are not quite liquid enough. It would be more than absurd to make bets against the market if the strategy is not based on a precise smile or distribution. Basically it would not differ from tossing a coin when making investment decisions. In other words, the importance of precise distribution estimation is crucial when practitioner *"buys undervalued and sells overvalued probability"* (Shimko 1994).

The optimal trading strategy can be calculated from the difference between the proprietary and market probability distributions (equation 21) (Shimko 1994). The risk aversion term in the equation is used on reducing the position taken for more risk-averse practitioners.

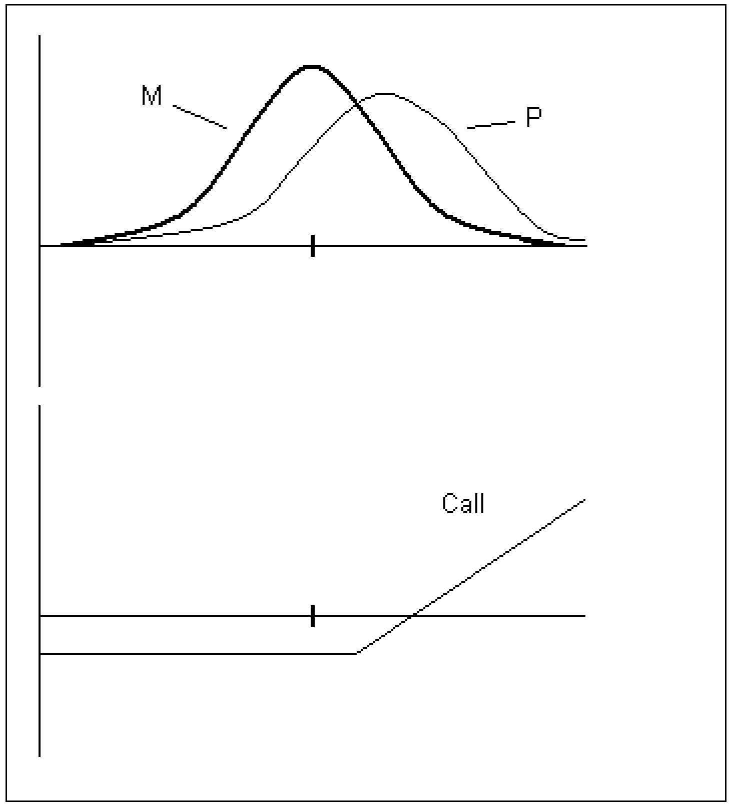
$$Payoff = \frac{\Pr oprietary\ probability - Market\ probability}{Market\ probability} * \frac{1}{Risk\ Aversion} \quad (21)$$

The difference can be fitted in the same graph with the both views, and then an optimal strategy is easier to define with the equation 21 and to present graphically (graph 4.25).



Graph 4.25. Optimal strategy based on a proprietary view

In graph 4.25, X-axis denotes the underlying price and Y-axis the probability. P curve is the proprietary view of the distribution. The trader assumes that the market implied distribution (M, the market view) is heavily underestimating the probability of an underlying asset to be significantly higher at expiration date. The thin dashed line (D) denotes the difference between the views and the bolded dashed line (S) the optimal strategy. The following graph describes the same situation with a simple long call option strategy (graph 4.26).

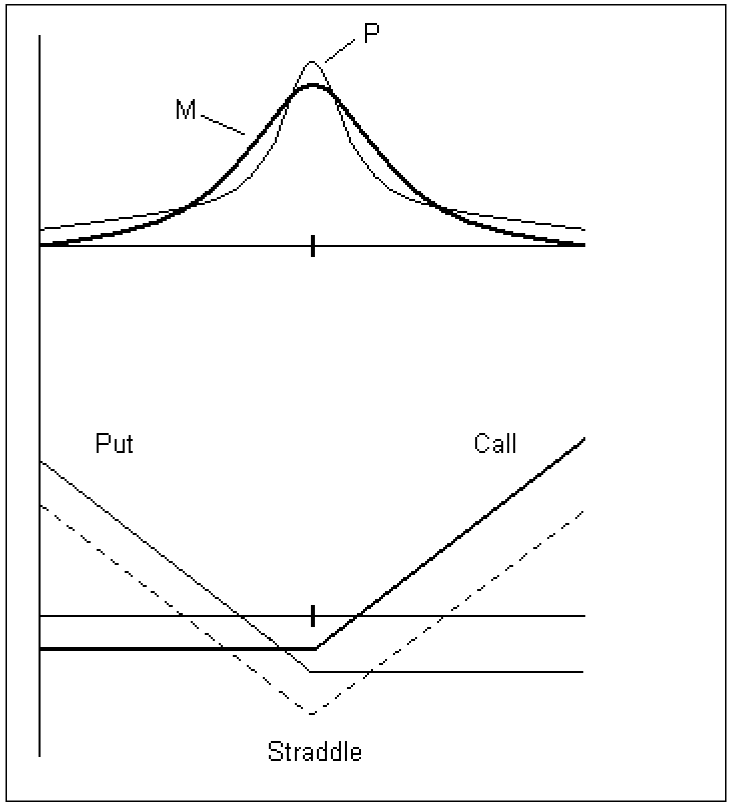


Graph 4.26. Simple long call option strategy with a solid proprietary view

The upper graph shows how the trader believes strongly in a rise in the price of an underlying but also in some rise of volatility; the proprietary view has significantly more probability mass on X/S ratio over 1.0. If volatility was considered to rise significantly and there were no means of determining in which way the price changes, the proprietary distribution would show heavy tails and the strategy should be adjusted to match this. Trader enters into a long call option position with a strike price adjusted to his needs. Note that the optimal strategy here consists of a simple long call position for illustrative purposes; a long forward position could be also considered. The lower graph shows the payoff graph for long call. In reality, to be able to match the calculated optimal strategy the best, a combination of instruments is required.

Another example of the usage of implied distributions and volatility smiles for practitioners is the straddle strategy. A long straddle is built from an equal amount of bought call and put options with the same strike and maturity and short straddle by selling an equal amount of same call and put options (MacDonald 2006, Ahoniemi 2007). With a long straddle strategy, the practitioner makes profit with both, the upward and downward movements in price and he expects the volatility to increase. The call option generates profit when the underlying price increases and in the same manner the put option will profit if the underlying price decreases. Or in other words, the strategy will generate profit when the volatility level increases.

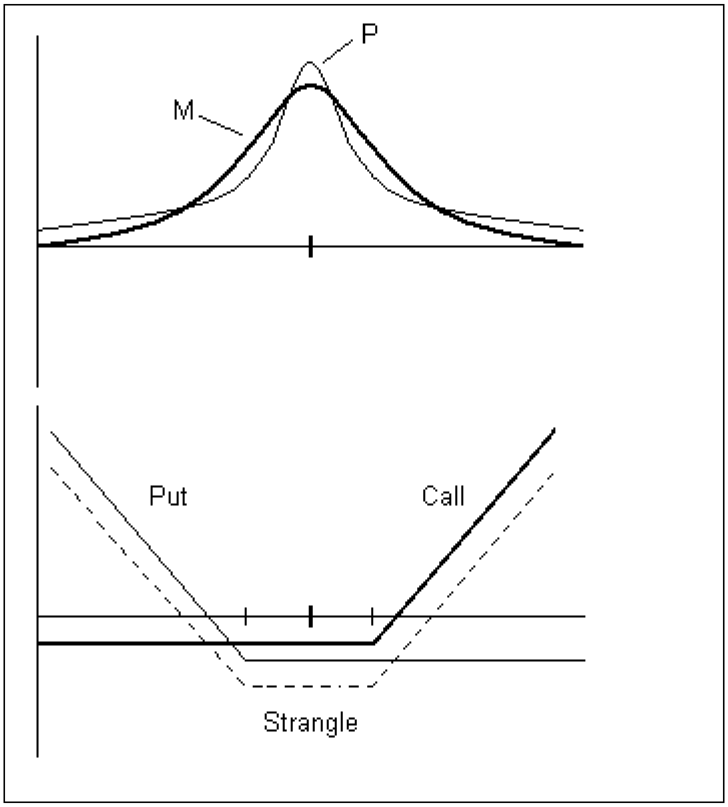
The payout graph is the opposite for the short straddle strategy and the profit consists of received premiums from the shorted contracts. Therefore the short straddle is used when the practitioner estimates the future volatility to be less than the market expectations. The payout graph for the long straddle strategy is a V-shaped curve where the loss consists of paid premiums if the underlying price does not differ greatly from the current level at expiration date and the contracts are not exercised. The long straddle example is presented in graph 4.27.



Graph 4.27. Long straddle strategy with a specified proprietary view

The strategy is based on a practitioner's view of the distribution of the underlying asset price at the expiration date which has a leptokurtic shape (fatter tails *in both sides* of the distribution and a high peak). The market view underestimates the probability of extreme movements in an underlying price but also the prices being within a close range to ATM at expiration date.

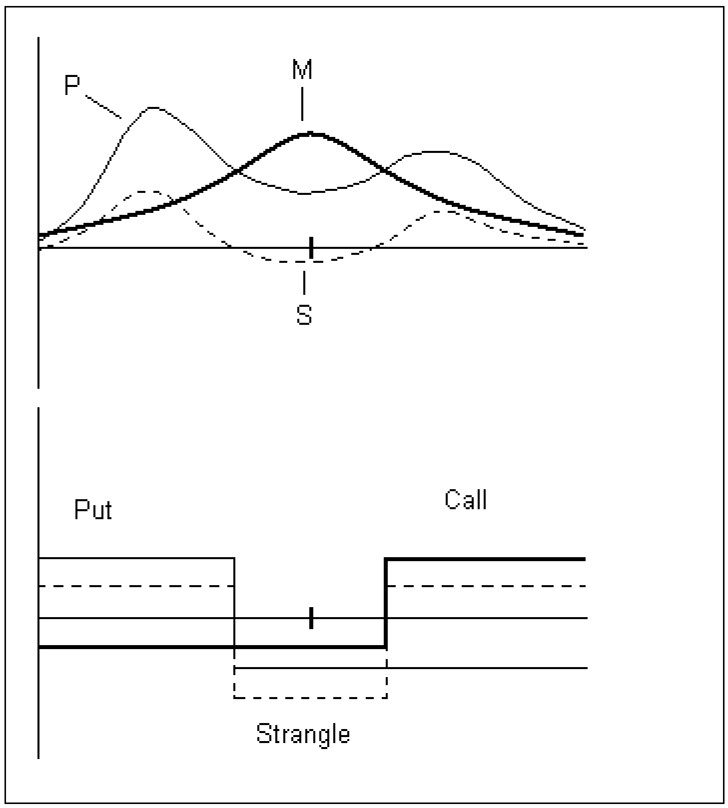
The lower graph describes the payout for a long straddle strategy (the dashed line) and shows the profit when payout graphs for call (the bolded line) and put (the thin line) options with the same strike and maturity are combined. He would then bet against the market view that the volatility is going to be higher in future (and therefore the price should deviate significantly from the current level) although he does not know if the price will rise or fall significantly. In other words, the long straddle strategy assumes the probabilities of extreme movements of an underlying asset price to be significantly higher and to profit from this. An example of such a situation could be an extreme uncertainty of the policies of an upcoming government in an unstable political atmosphere. A strangle strategy (a straddle with differing strike prices) should also be considered for proprietary distributions with heavy tails (graph 4.28).



Graph 4.28. Long strangle strategy with a specified proprietary view

In both examples, the trader here is buying an undervalued probability in the tails of the distribution. In this case the trader also assumes the probability of possible lower volatility also to differ from the market view so he is taking a risk of an underlying price to be in within a narrow range around the current price at expiration date. In addition, it has to be noted that trader's proprietary view could also be more aggressive towards extreme movements and less aggressive to situations with a moderate or low deviation from the current underlying price thus either minimizing or eliminating this dilemma. The proprietary distribution curve would flat out but maintain the fat tails. Either way, the example discussed can be seen as a good presentation of a real life trading situation based on implied probability distributions since no proprietary view will be as simple as the graph 4.26 suggests.

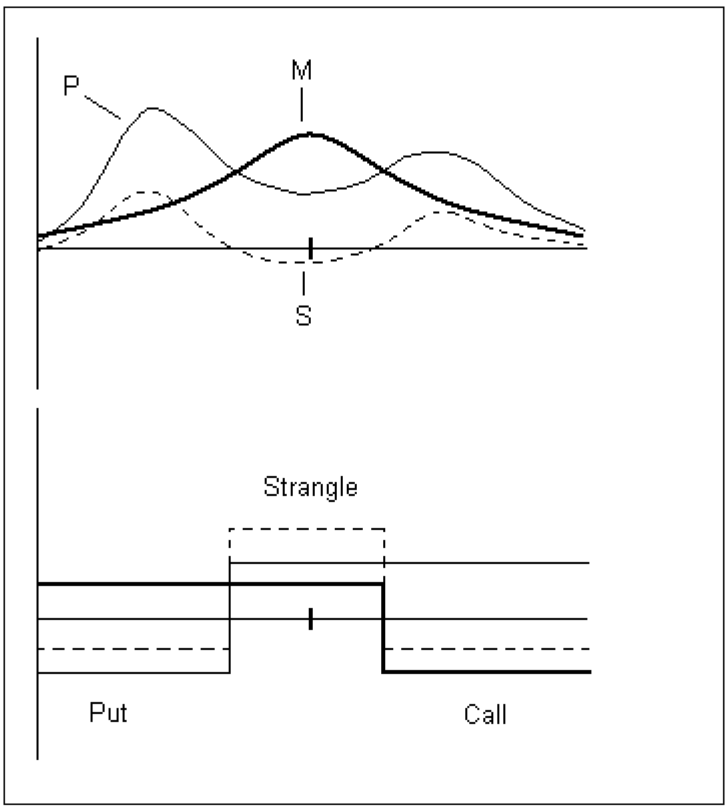
Shimko (1994) defines more advanced strategies depending on the view of the future probability distribution. He argues that to be able to build an optimal strategy, one has to use combined derivatives strategies. The next example presents a case of bimodal distribution and an optimal strategy by usage of binary options[13].



Graph 4.29. Long binary strangle strategy with a bimodal proprietary view

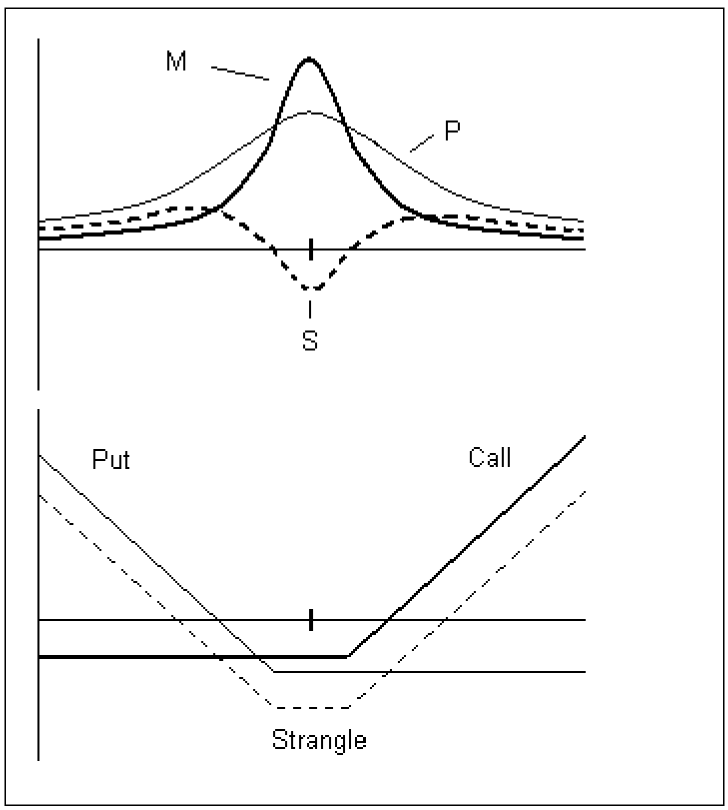
[13] A binary option (sometimes referred to a digital option) pays a predefined fixed amount when the underlying price rises or falls in to a certain range. The cash-or-nothing option pays a specified amount of cash if the option is in-the-money at the expiration date. The asset-or-nothing option pays the value of an underlying asset (Hull 2003, Wilmott et al. 1995).

The upper graph shows the bimodal proprietary distribution with an optimal strategy curve. Lower graph present the profit for long strangle strategy (the dashed line) with cash-or-nothing binary options. The strategy consists of equal amounts of binary call and put options on the same underlying but on different strike prices. Trader's view is suggesting high volatility and extreme movements in price and the strategy will generate a predefined amount of profit in such a situation. The option contracts could also be shorted and the profit would consist of gained premiums (graph 4.29).



Graph 4.30. Short binary strangle strategy with a bimodal proprietary view

The presented strategies demonstrate how useful probability distributions can be for a professional trader with a thorough knowledge of the market trends. If we had a clear view of probability based on earlier experience or fundamental analysis, we could base our strategy on the estimated probability distribution functions for S&P 500 and DAX indices (graphs 4.23 and 4.24). For example the S&P 500 March 2008 option chain has a very distinctive leptokurtic probability distribution function (graph 4.17). Let's assume that we know for some reason (based on a fundamental analysis for example) that the probability of a high peak is heavily overestimated and the implied probability distribution should be flatter. The market participants expect the future price to be very close to the current price or to deviate significantly in 28 days. Our subjective view is that the recent uncertainty in the US market indicates that the volatility during the month will be higher than expected thus enabling deviation from the current price to be more probable. Our proprietary and market-implied views would look like the graph 4.31.



Graph 4.31. S&P 500 long strangle strategy

Since we have a specific subjective view of the future outcome, we make a bet against the market for example, with a long strangle strategy. The optimal strategy would follow more or less the bolded and dashed curve in the upper graph (denoted by the S). The strategy would generate profit when there is deviation from the current underlying price at expiration. At the same time, the loss is limited to paid premiums of the two contracts. Also, it would be possible to build a straddle or either one with binary options instead of normal European option contracts. If the proprietary view defined specifically the direction of a price change, also forward and single option strategies would be useful. The next section draws a conclusion for this research.

5. Conclusions

The most of the research in the area of option pricing and volatility estimation has concentrated on development of advanced models for modeling the evolutionary process of future volatility of returns of an underlying asset. The research interest aroused from the study of Engle (1982) in which he introduced his ARCH (Autoregressive Conditional Heteroscedasticity) model. Later on, at 1993, Heston introduced his Stochastic Volatility model. Both models study the historical time series and assume the volatility to follow stochastic process in future. The probability distributions based on these should give a fair estimation for prediction purposes. In this paper we took an inverse approach to this dilemma and studied the so-called forward-looking method of volatility estimation. In our analysis we used market premiums for obtaining a probability distribution function specified from the given data set at the given moment in time. Our goal was to determine if the market prediction of the process of an underlying price change has deviations from the simplified assumption of normality in distribution.

Two data sets used in analysis consisted of option prices for S&P 500 and DAX stock index futures. The observations for option premiums were obtained on 22nd of February 2008 and the data sets consist of option series with several maturities for both cases. The data sets were chosen to represent two different trading systems from two different market areas, pit-traded exchange in the USA and electronically traded exchange in Europe.

Our research consisted of four different parts. First, the data was screened for illiquid observations and afterwards the volatilities were iterated from liquid observed option premiums. Secondly, the strike prices were standardized and the volatilities with respect to new strike prices were interpolated with an assumption that the volatility curve follows a polynomial order of 2. The interpolation method used was quadratic least squares. Finally, the risk-neutral probability distributions for the observed moment were calculated via the second derivatives of call option prices with the Breeden-Litzenberger method (1978).

We found out in our research that the implied volatility differs from the assumptions made by the BS model between the two markets. The same type of observation has been noted in numerous studies and practically in all markets worldwide from as early as mid-1970's. The smoothed volatility curves showed significant deviations from the constant level and the smile and skew patterns could be detected. This notation is consistent with the earlier research conducted in this area (Weinberg 2001). Moreover, there seems to be clearer smile pattern in the curve when liquidity increases while volatility decreases on extreme strike prices. Therefore we can safely conclude that market-implied option premiums contain some information over the BS model, especially with the shortest maturities when trading is at the most active. Shimko (1993) argues that hedgers will benefit from the information that observed premiums contain. The hedgers would consistently be using overestimated deltas for adjusting their portfolios if deltas suggested by BS model were used. Vähämaa (2004) comes to the same conclusion; the BS deltas are too high in comparison to ones derived from option prices.

And in addition it has to be remembered that the BS model does not calculate the "correct price" for an option contract, the market and its numerous participants do. In other words, the participants in the market (i.e. the supply and demand) will determine the fair price and BS model is used on estimating this price. The BS model should not be seen then as a complete answer to option pricing issues but a tool that gives remarkably good estimations for the option market premiums. The aim is to try to understand and model the market behavior as perfectly as possible and in this process; the implied distributions come handy since they give the market view at a discrete moment of time. One could argue that because of this, the market-implied methods will always be superior to historical models, no matter how advanced they get. As always in the economical theory, exact measurable results are not possible to obtain, only educated guesses that will explain the phenomenon statistically speaking well enough. This kept in mind, the implied PDF approach can be seen as a good alternative.

Our risk-neutral probability distributions showed a noticeable difference between the two option chains. While the S&P 500 data differed greatly from the assumption of normality in the distribution, the DAX data followed it precisely, especially with the longer maturities. The S&P 500 options had a tendency of being skewed to the right for all maturities. In the case of DAX options, the results were not so self-explanatory although some skewness could be detected. As the risk neutral PDF describes the market estimated probability of an underlying price change for the given maturity, we can clearly see that with the S&P 500 March call option has more probability mass on the deep OTM strikes. This notation is convergent with findings of Miranda & Burgess (1998). This kind of pattern could indicate that arbitrage possibilities exist. That is, if the time-varying change in volatilities was taken into consideration and the results still showed a deviation from the normal distribution. The idea behind this is that the used pricing models would constantly *underestimate* the probabilities of events very close to the mean but also the extreme events occurring during the maturity and *overestimate* the probability of observations between these two probability zones. Weinberg (2001) suggests that this is due to the fact that the market participants are willing to pay more to hedge from the high fluctuations in the prices. Also, what was not clearly determined is the importance of trading costs to observed distributions (Jackwerth & Rubinstein 1996).

Further research on this matter could try to obtain a daily data for liquid option chain and to build daily distributions and therefore to determine patterns in the time-varying data. This could be done for a longer period of time depending on the historical quotations available. Other type of research interest would be a construction of a risk neutral time-varying distribution for one option series only during its maturity, from the moment of issuance to expiration. Different markets (exchange rates, interest rates etc.) could be also tested in the same manner that has been already done for the volatility curves. The characteristics of the market could be interpreted and one might base trading to these findings. It is important to mention here that the reliability or "predictive power" based on option premiums of index futures contracts might not fully represent the future dynamics of index returns. It has to be kept in mind that the implied volatility calculated from option premiums presents the price dynamics of *futures contracts on the index,* instead of the index itself. It is difficult to determine how well the futures contracts price process differs from the one of the index, if any.

What comes to the volatility curve estimation, a study for different smoothing methods and their impact on distributions could be an interesting point of view. Especially, if the method of two mixed lognormals was compared to methods of quadratic least squares and cubic spline. These varying types of distributions could be then compared against other methods of the price process estimation (e.g. ARCH and stochastic volatility models) and to try to determine which method produces the most precise predictions in a short-term and long-term basis when compared against the realized volatility[14].

[14] For more information about prediction efficiency, see Weinberg, 2001.

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Appendices

Appendix A

- **Quadratic interpolation example**

1. Given the (x_0, y_0), (x_1, y_1), … , (x_n, y_n), solve the required value of y if x is given. The polynomial equations between the observed dots are,

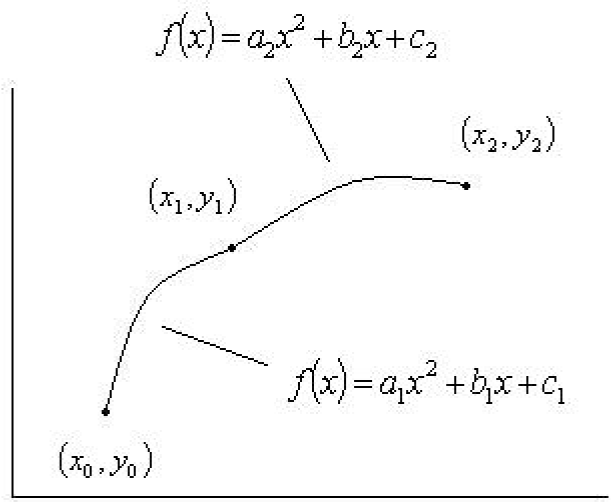
$$f(x) = a_1x^2 + b_1x + c_1, \qquad x_0 \le x \le x_1$$

$$f(x) = a_2x^2 + b_2x + c_2, \qquad x_1 \le x \le x_2$$

.
.
.

$$f(x) = a_nx^2 + b_nx + c_n, \qquad x_{n-1} \le x \le x_n,$$

if the curve is continuous. In this case only three points are observed, (x_0, y_0), (x_1, y_1), (x_2, y_2). Graphical presentation being as follows,



Find a_i, b_i, c_i.

2. Since the polynomial equation goes through the points observed,

$$f(x_0) = a_1 x_0^2 + b_1 x_0 + c_1 = y_0$$
$$f(x_1) = a_1 x_1^2 + b_1 x_1 + c_1 = y_1$$

$$f(x_2) = a_2 x_1^2 + b_2 x_1 + c_2 = y_2$$
$$f(x_3) = a_2 x_2^2 + b_2 x_2 + c_2 = y_3$$

The first derivatives of the splines are,

$$f'(x) = 2a_1 x + b_1$$
$$f'(x) = 2a_2 x + b_2$$

At point $x = x_1$ the rates of change in the $f(x_i)$ (the first derivatives) are equal in the same point. That is,

$2a_1 x_1 + b_1 = 2a_2 x_1 + b_2$ which simplifies to,

$$2a_1 x_1 + b_1 - 2a_2 x_1 - b_2 = 0$$

Polynomial equations $f(x) = a_i x_i^2 + b_i x_i + c_i$ go trough points (0, 0), (5, 5) and (8, 8) so we can set up the equations and form the matrix solution,

$$a_1(0)^2 + b_1(0) + c_1 = 0 \qquad (1)$$
$$a_1(5)^2 + b_1(5) + c_1 = 5 \qquad (2)$$

$$a_2(5)^2 + b_2(5) + c_2 = 5 \qquad (3)$$
$$a_2(8)^2 + b_2(8) + c_2 = 8 \qquad (4)$$

Continuous derivatives at point (5, 5) for both spline equations,

$$2a_1(5) + b_1 - 2a_2(5) - b_2 = 0 \qquad (5)$$

The first spline is assumed to be linear,

$$a_1 = 0 \qquad (6)$$

With above 6 equations we can form the matrix solution which can be solved with an adequate mathematical program,

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 25 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25 & 5 & 1 \\ 0 & 0 & 0 & 64 & 8 & 1 \\ 10 & 1 & 0 & -10 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 5 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

Appendix B

- **The Least Squares Method (Quadratic Interpolation)**

The data set is given by $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ with the restriction of $n \geq 3$. It is assumed that the curve follows a polynomial order of two, i.e.

$$y = a + bx + cx^2 = f(x)$$

Mathematically the best fitting curve $f(x)$ has the least square error by,

$$\Pi = \sum_{i=1}^{n} [y_i - f(x_i)]^2 = \sum_{i=1}^{n} \left[y_i - \left(a + bx_i + cx_i^{\,2}\right)\right]^2 = Min$$

x_i's are known while a, b and c are unknown. To define the least square error a, b and c, the first derivatives according to a, b and c must equal to zero,

$$\begin{cases} \dfrac{\partial \Pi}{\partial a} = 2\displaystyle\sum_{i=1}^{n} \left[y_i - \left(a + bx_i + cx_i^{\,2}\right)\right] = 0 \\ \dfrac{\partial \Pi}{\partial b} = 2\displaystyle\sum_{i=1}^{n} x_i \left[y_i - \left(a + bx_i + cx_i^{\,2}\right)\right] = 0 \\ \dfrac{\partial \Pi}{\partial c} = 2\displaystyle\sum_{i=1}^{n} x_i^2 \left[y_i - \left(a + bx_i + cx_i^{\,2}\right)\right] = 0 \end{cases}$$

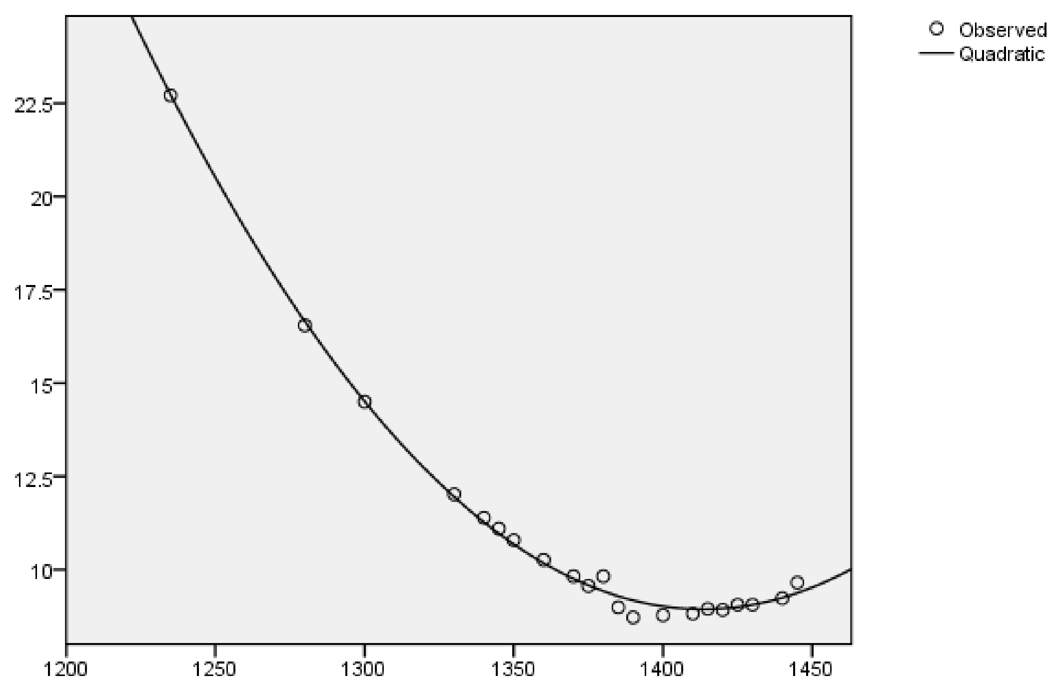
Which can be expanded to,

$$\begin{cases} \sum_{i=1}^{n} y_i = a\sum_{i=1}^{n} 1 + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} x_i^2 y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^4 \end{cases}$$

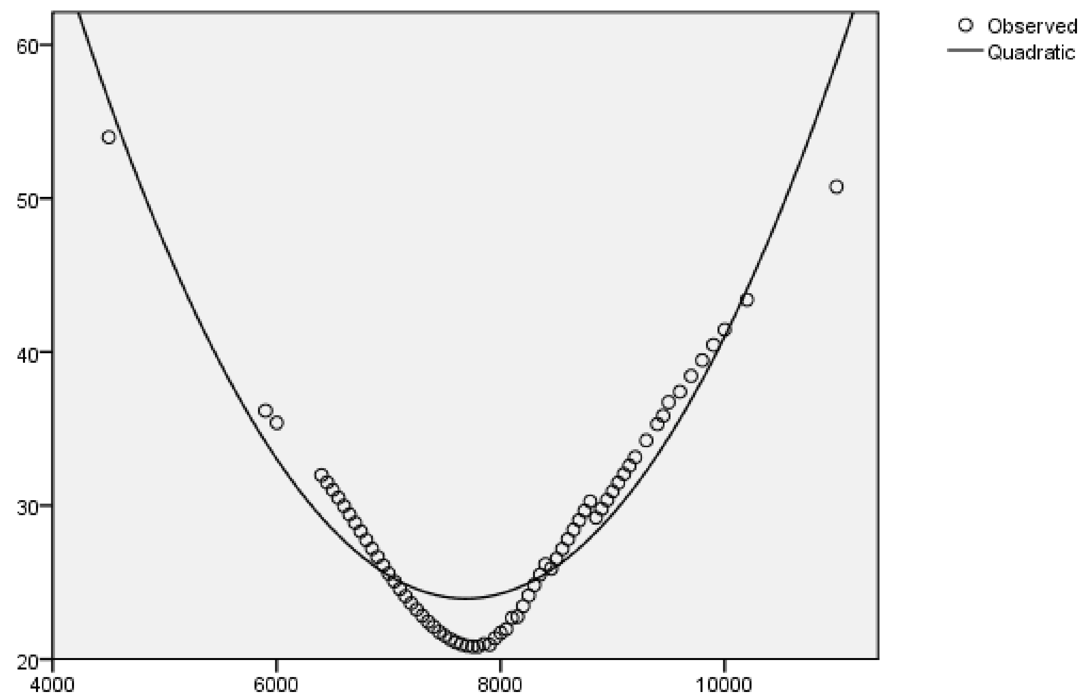
By solving the above equations we can obtain a, b and c.

Appendix C

- S&P 500 index option volatility smile, March 2008. Original non-extrapolated smile with observations. Underlying price is 1347 index points.

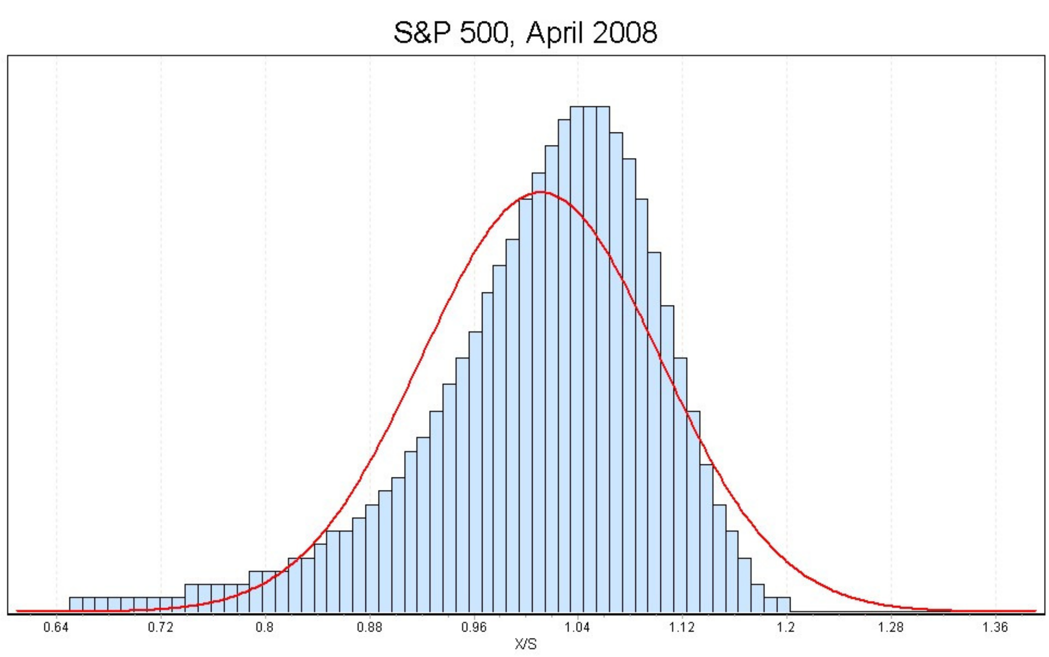


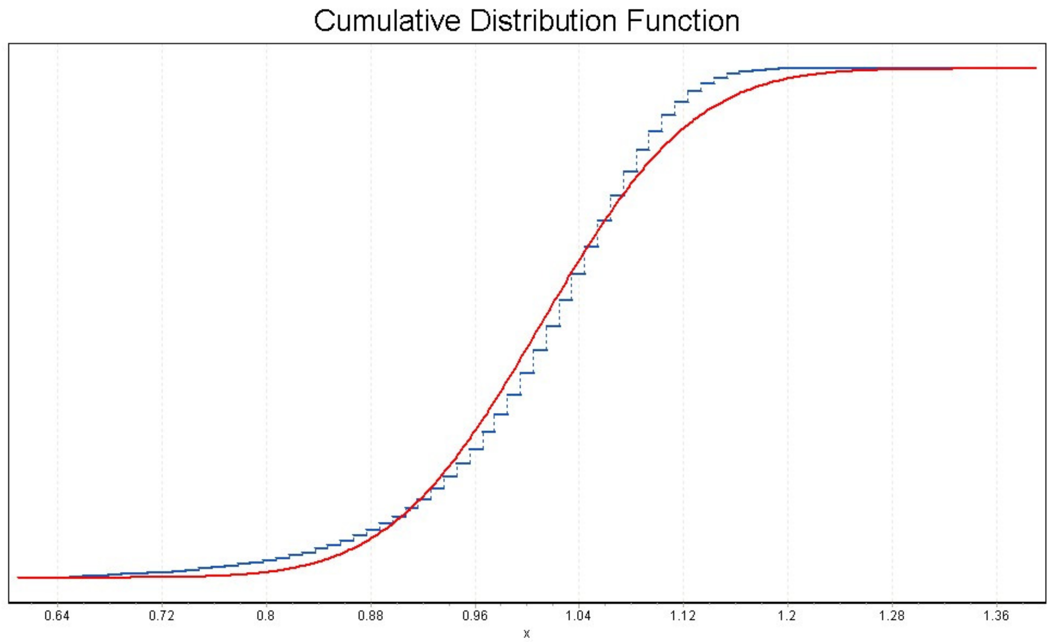
- DAX index option volatility smile, March 2008. Original non-extrapolated smile with observations. Underlying price is 6809 index points.



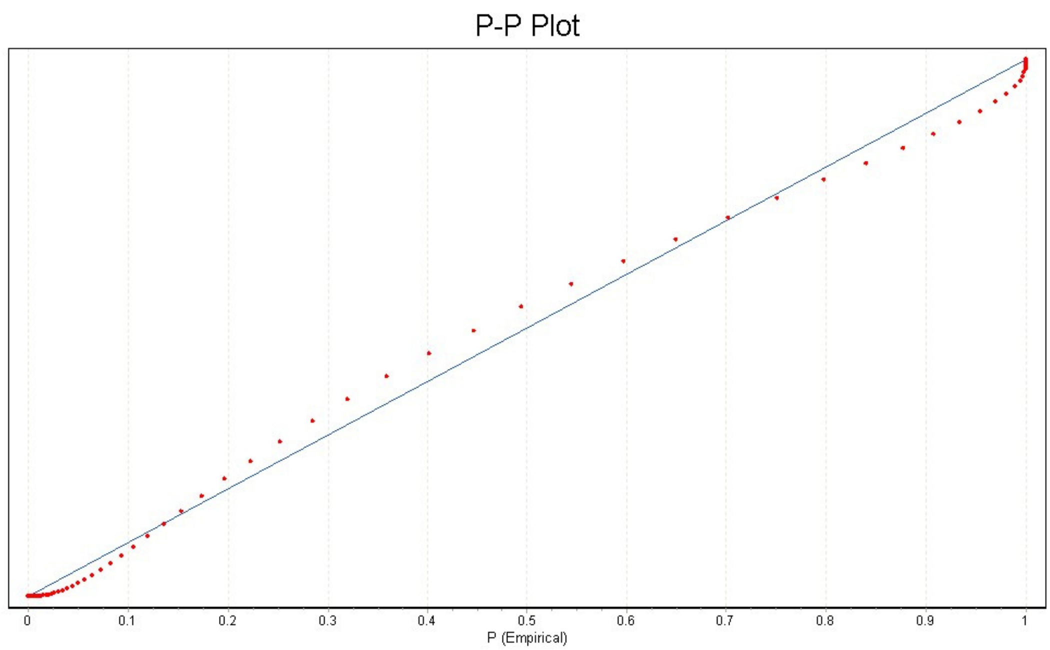
Appendix D

- **S&P 500 Implied Probability Density Functions**



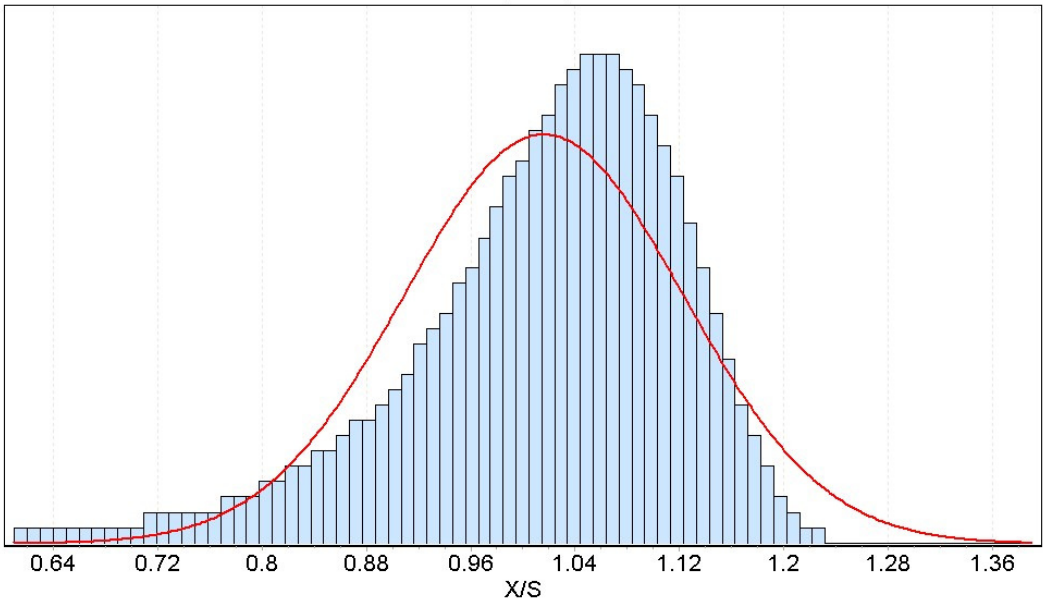


Cumulative Distribution Function
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
x

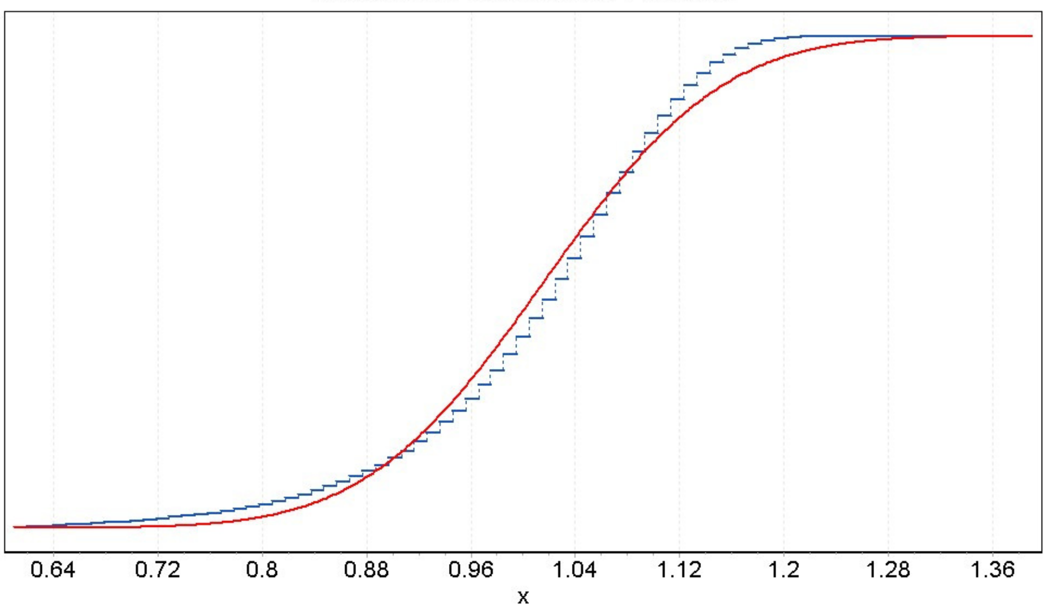


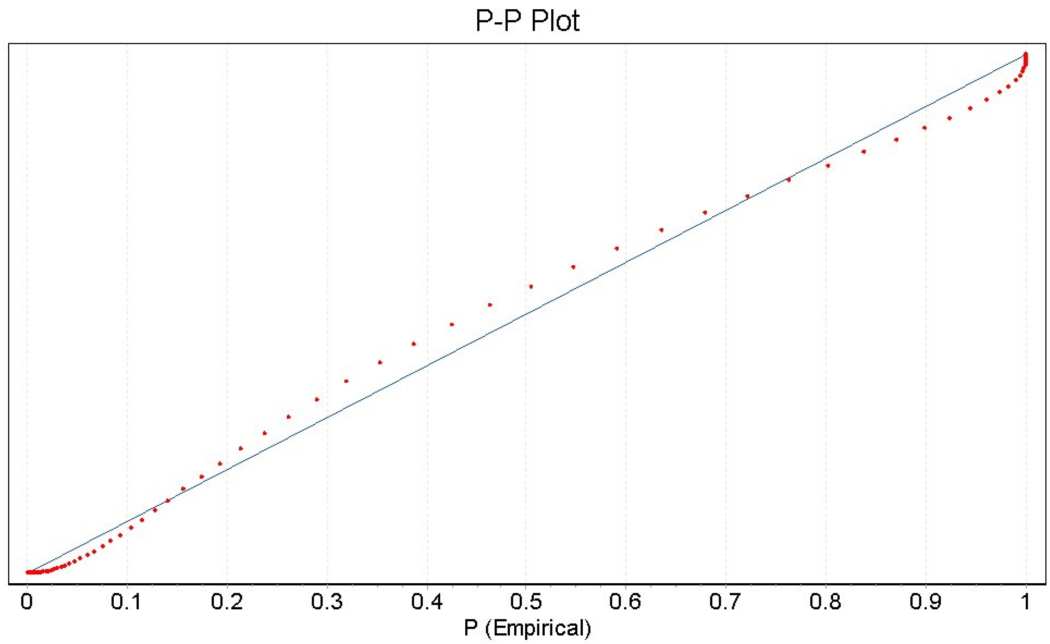
P-P Plot
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
P (Empirical)

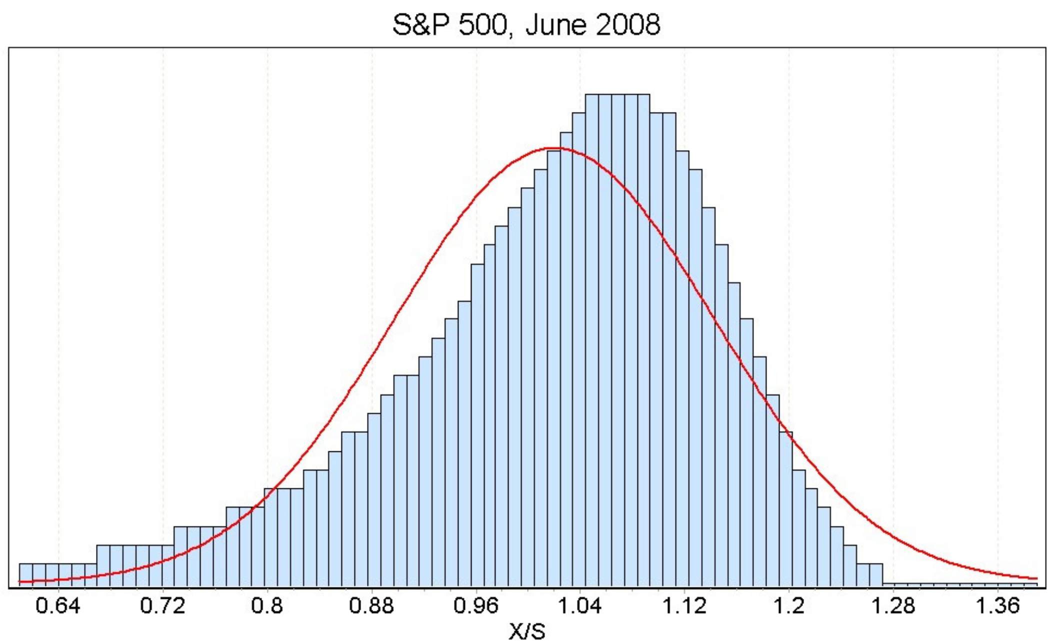
S&P 500, May 2008

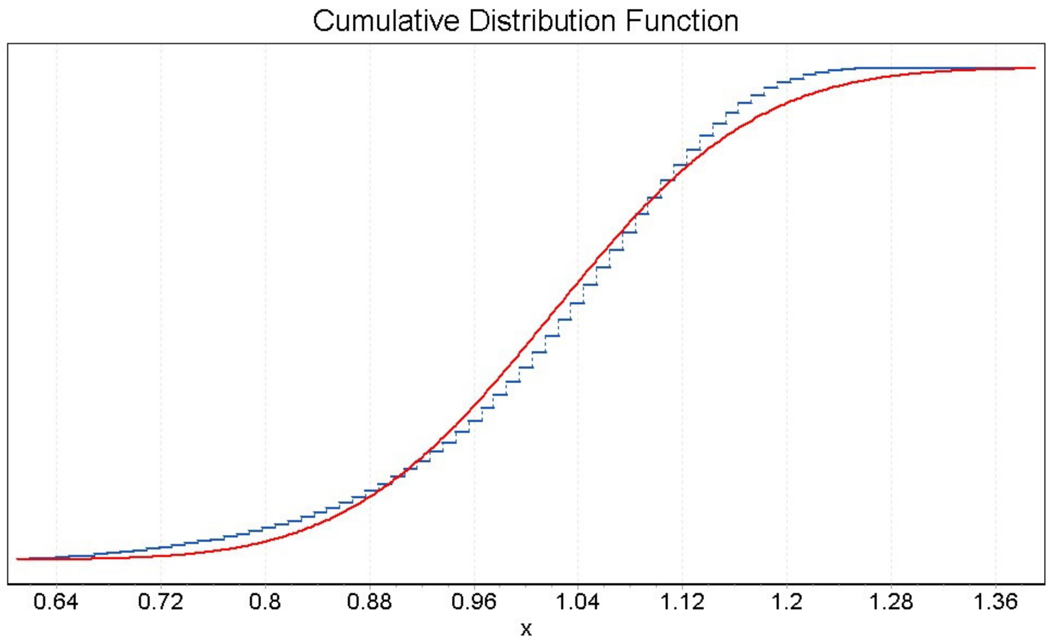


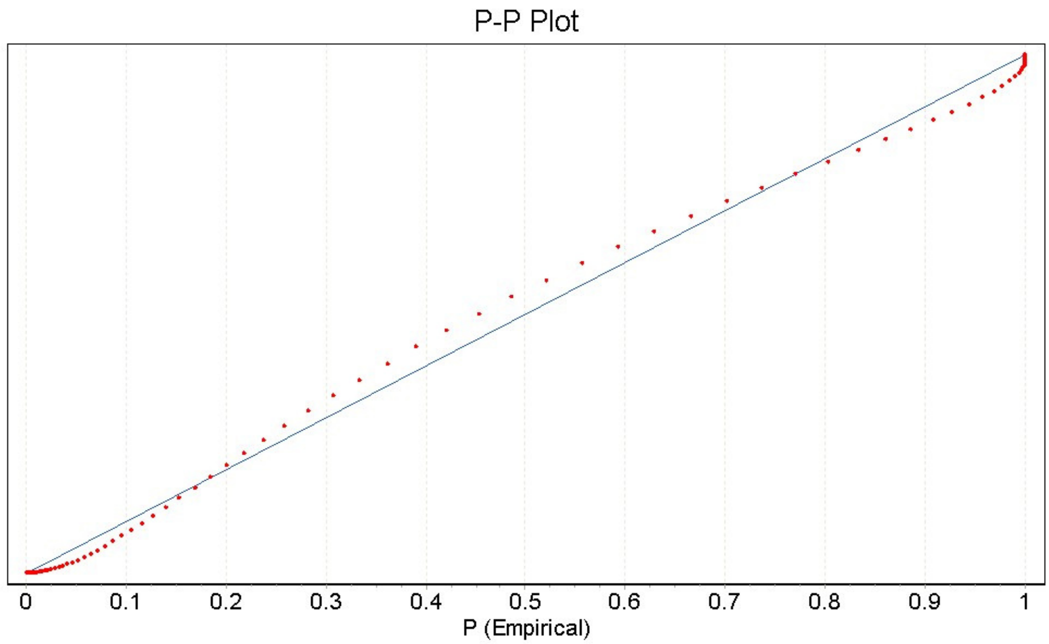
Cumulative Distribution Function

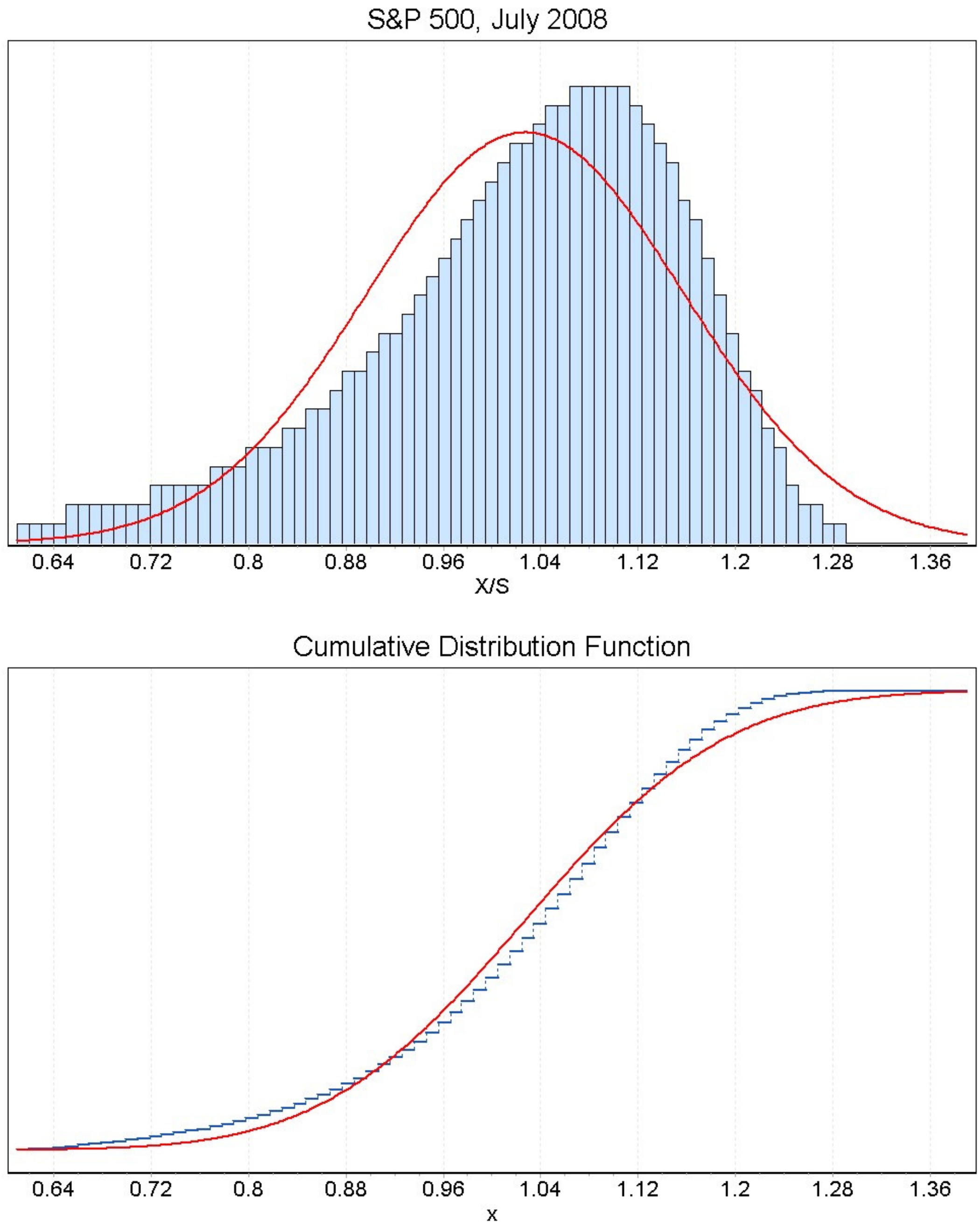


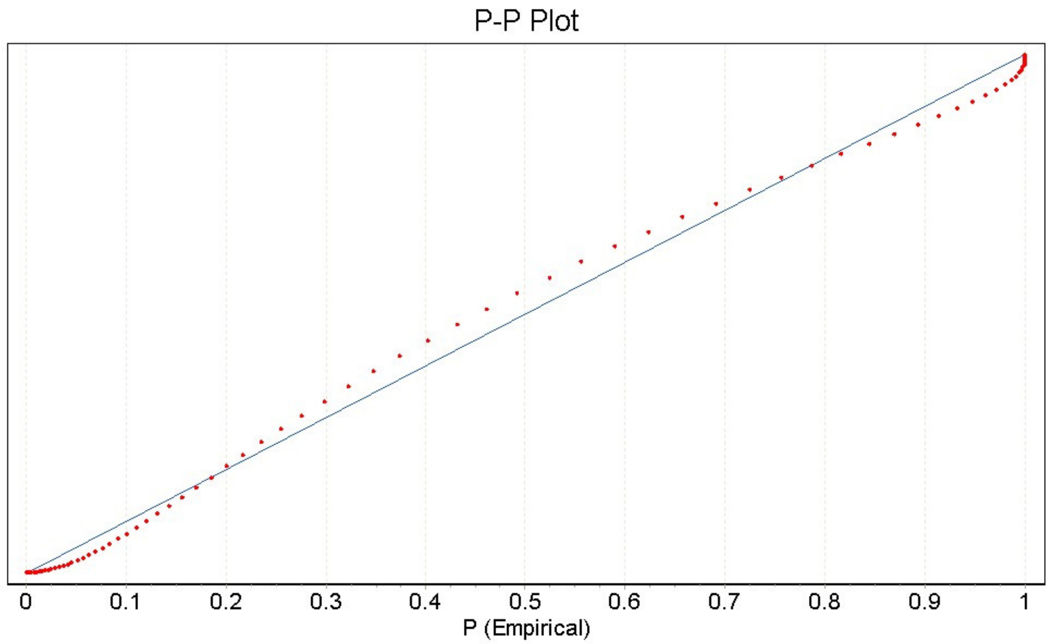

P-P Plot
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
P (Empirical)

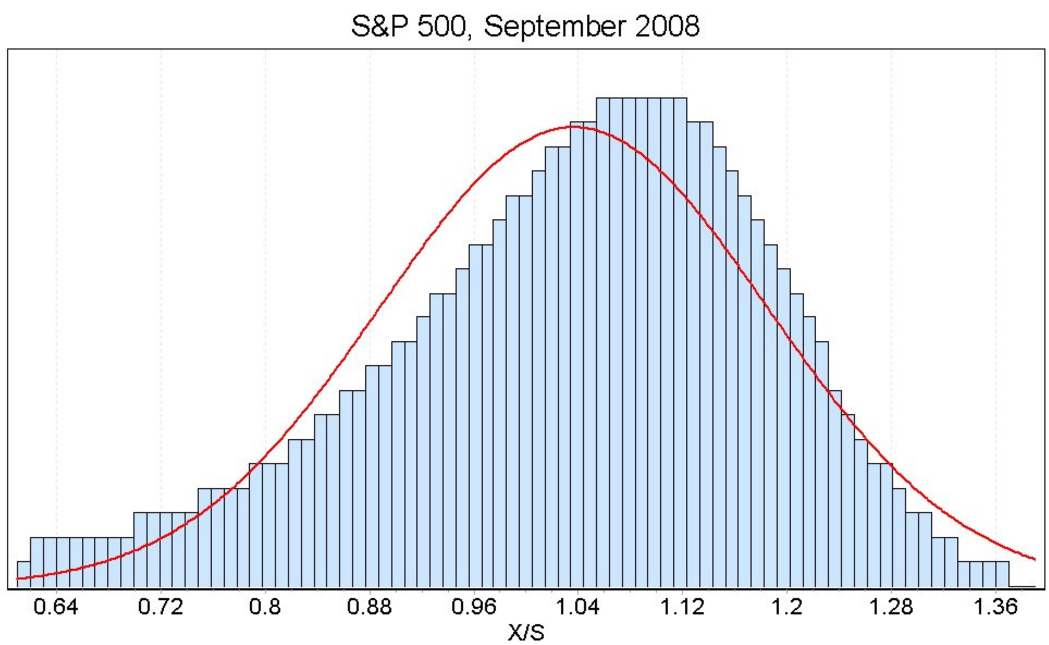

S&P 500, June 2008
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
X/S

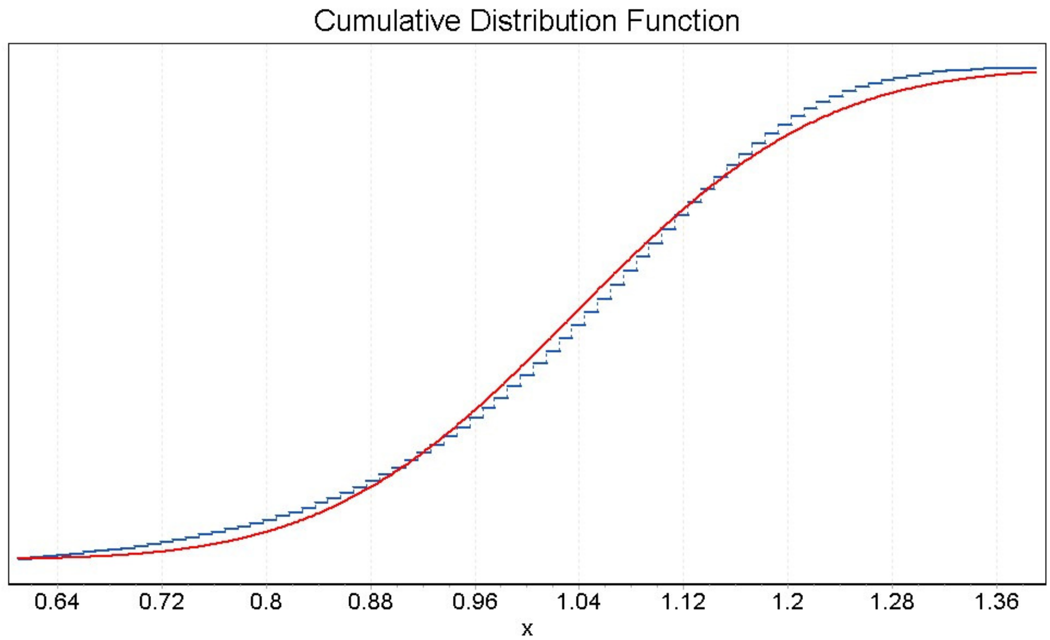

Cumulative Distribution Function
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
x


P-P Plot
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
P (Empirical)

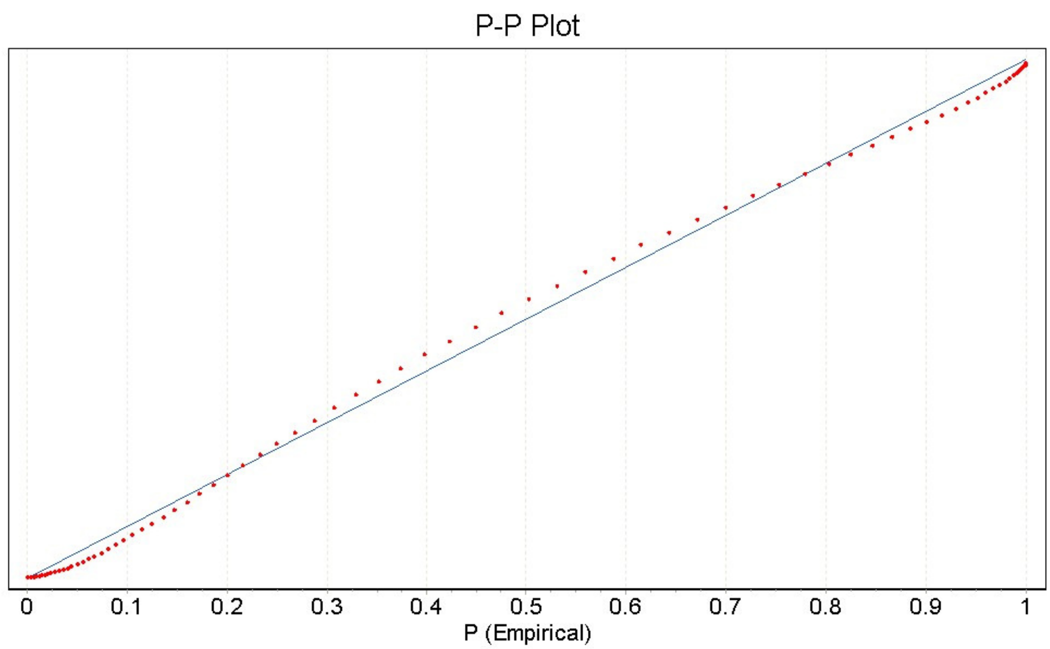

S&P 500, July 2008
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
X/S
Cumulative Distribution Function
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
x


P-P Plot
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
P (Empirical)

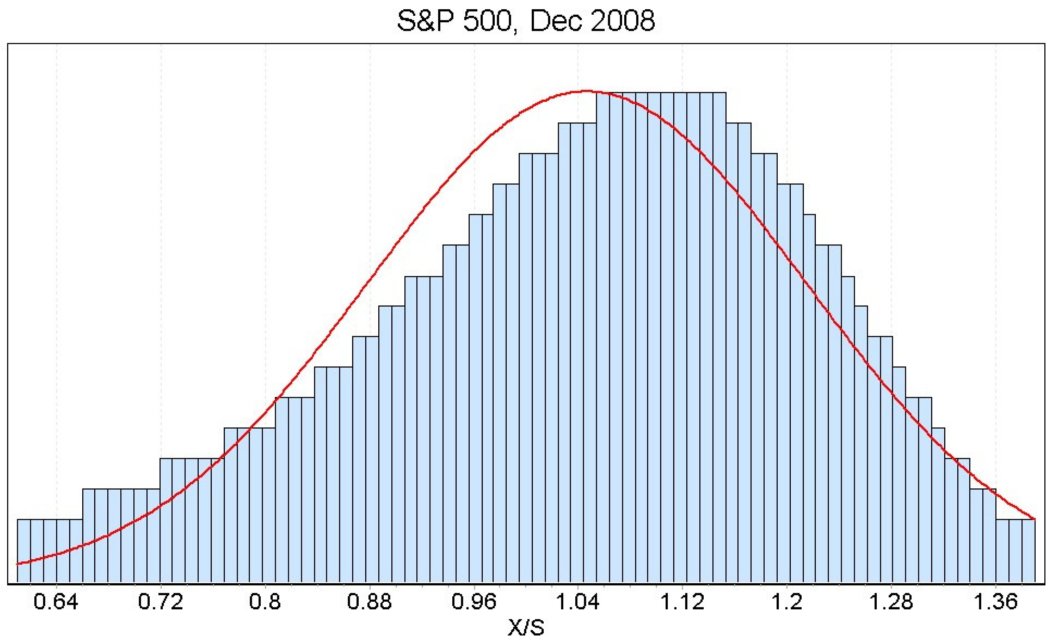

S&P 500, September 2008
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
X/S

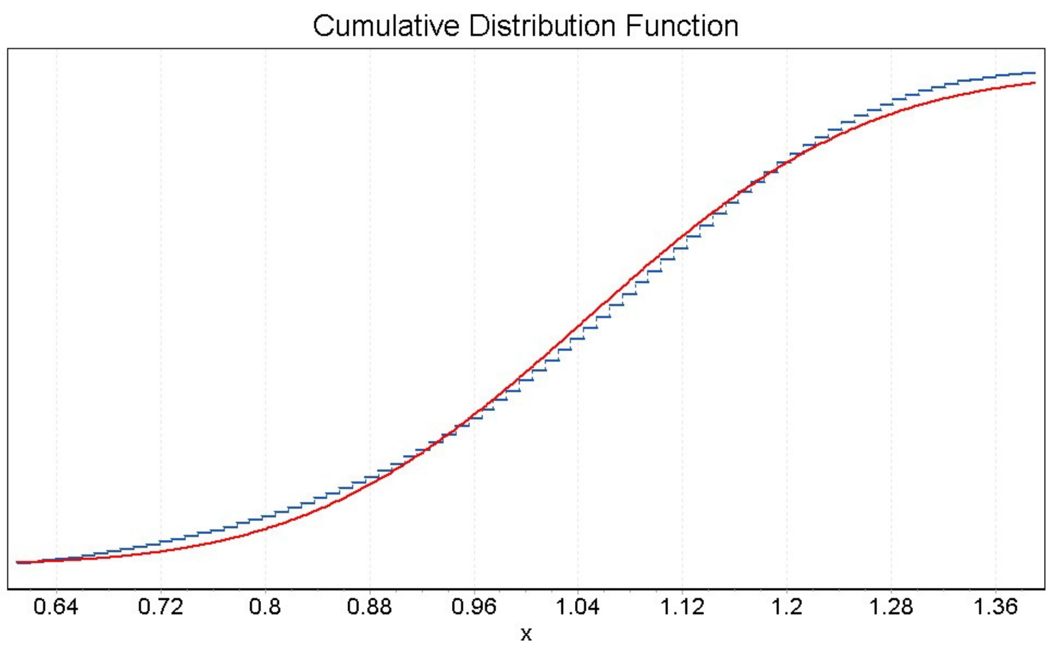


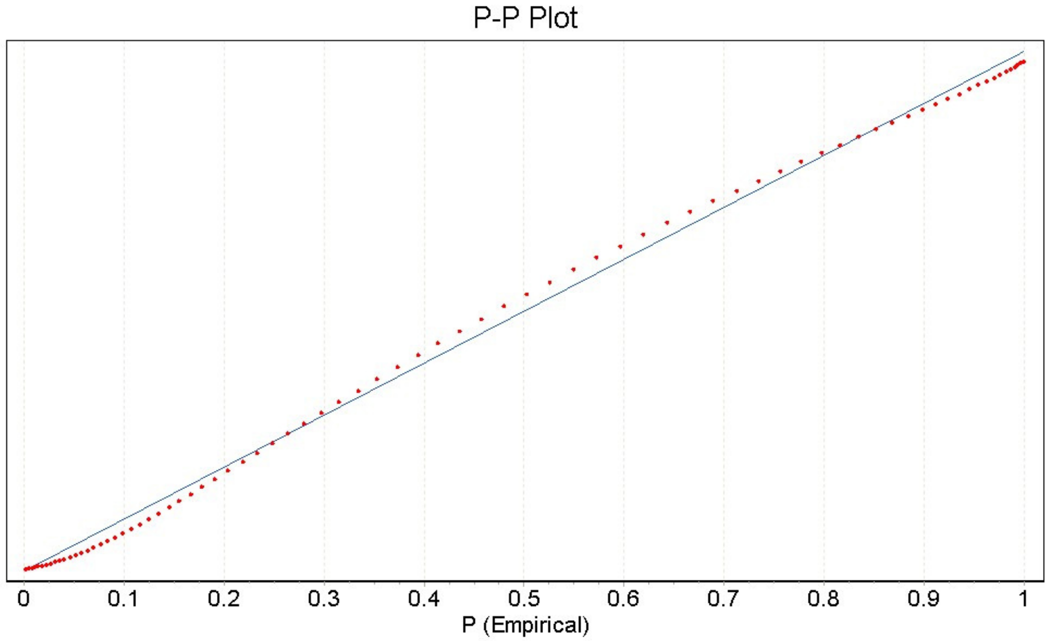
Cumulative Distribution Function
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
x



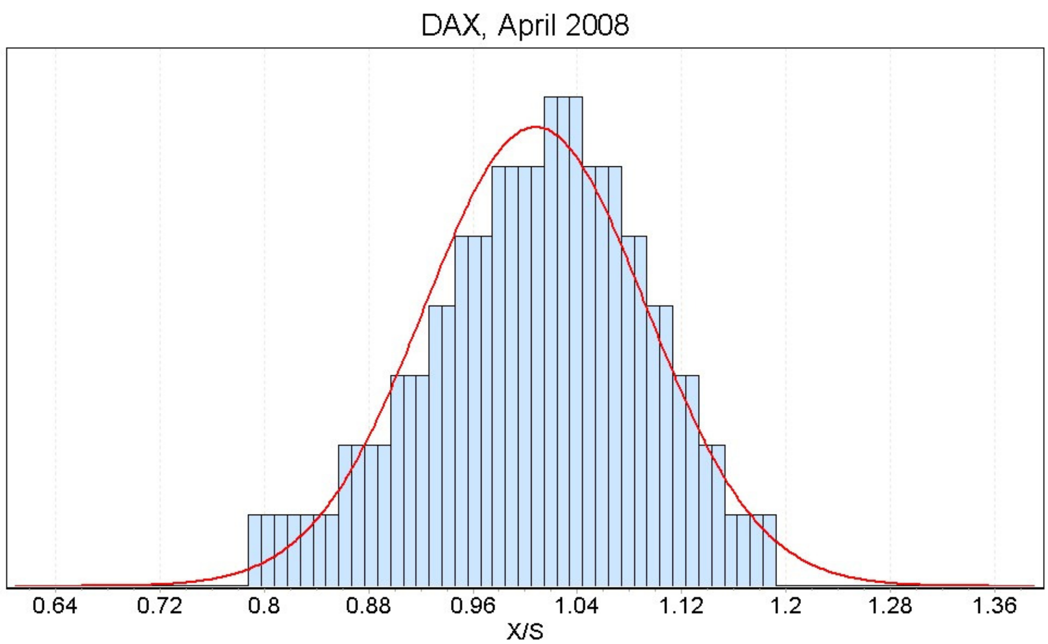
P-P Plot
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
P (Empirical)

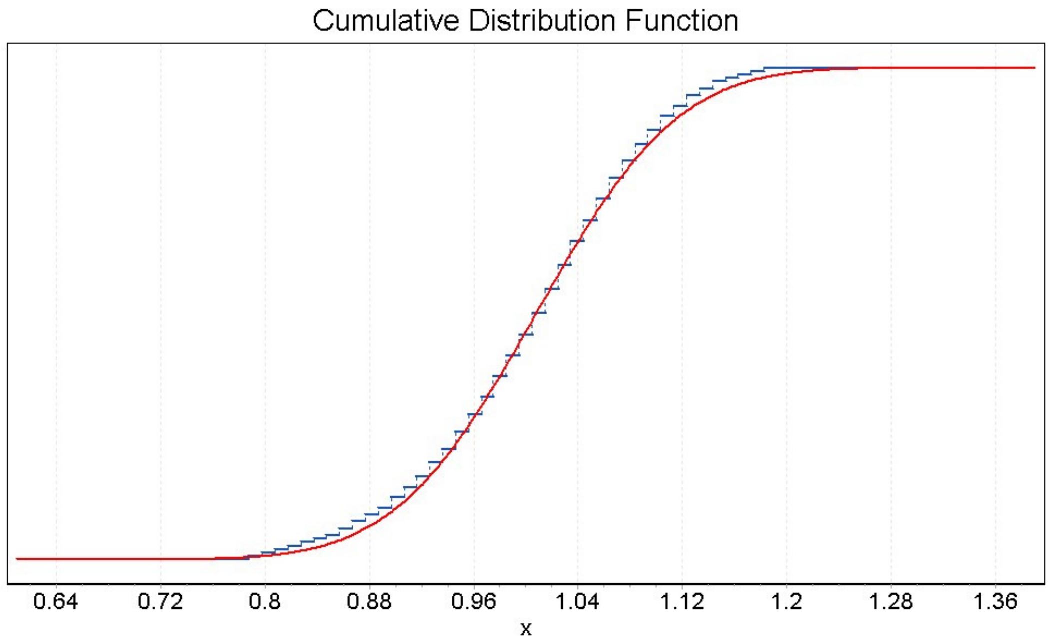

S&P 500, Dec 2008
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
X/S


Cumulative Distribution Function
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
x

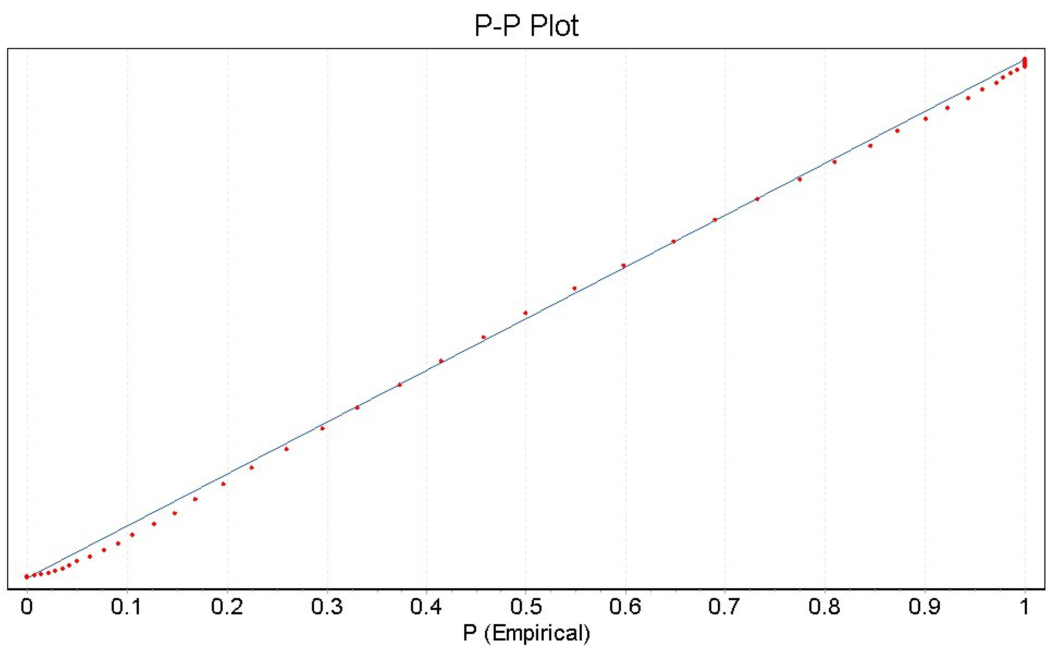


- **DAX Implied Probability Density Functions**

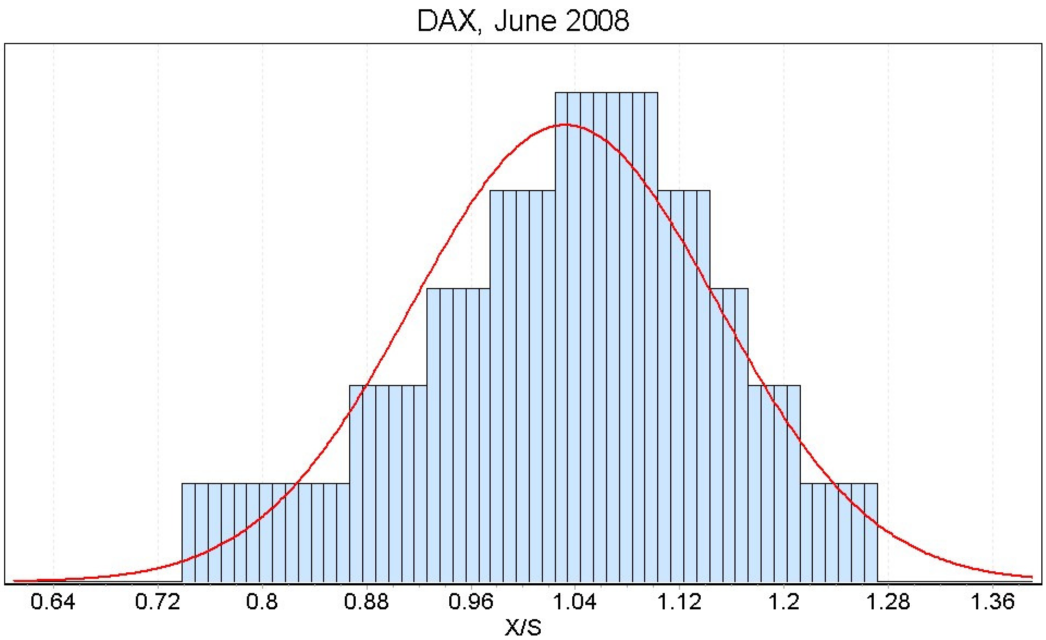


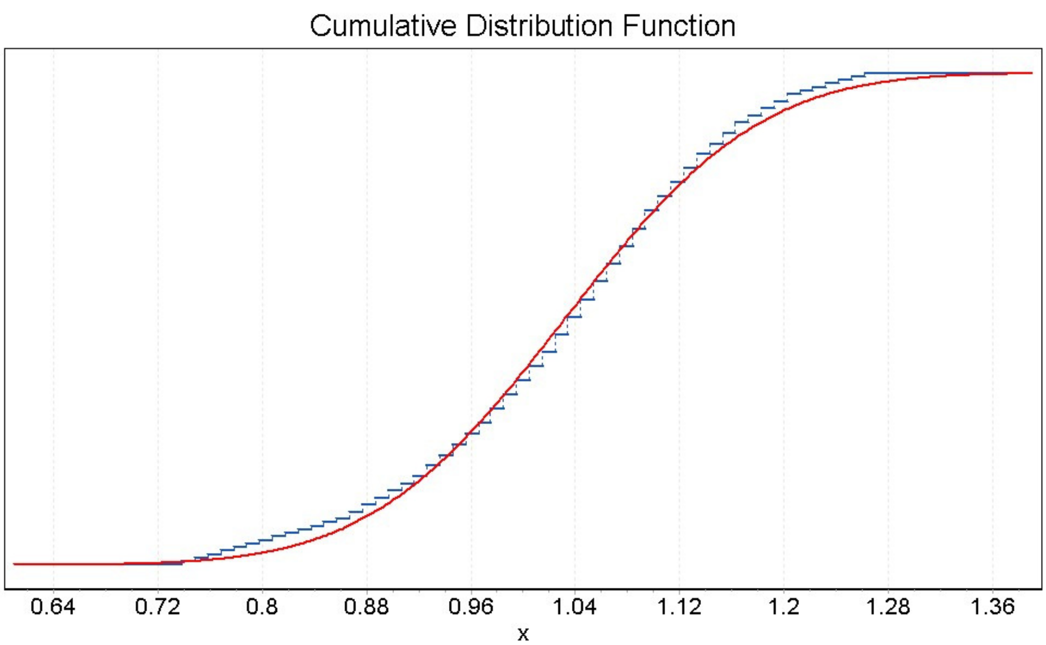


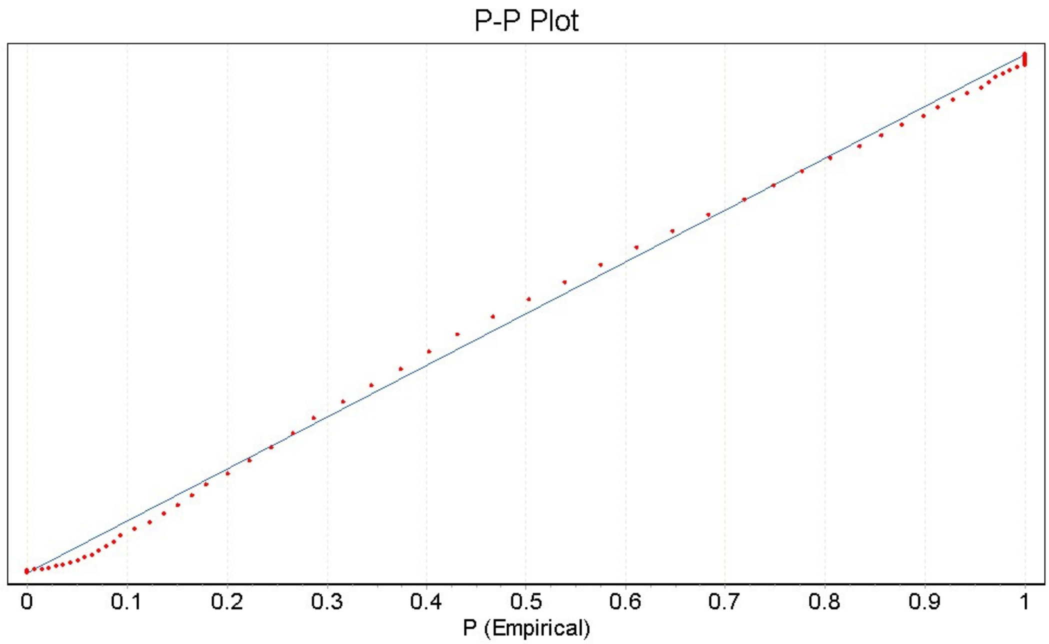
Cumulative Distribution Function
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
x

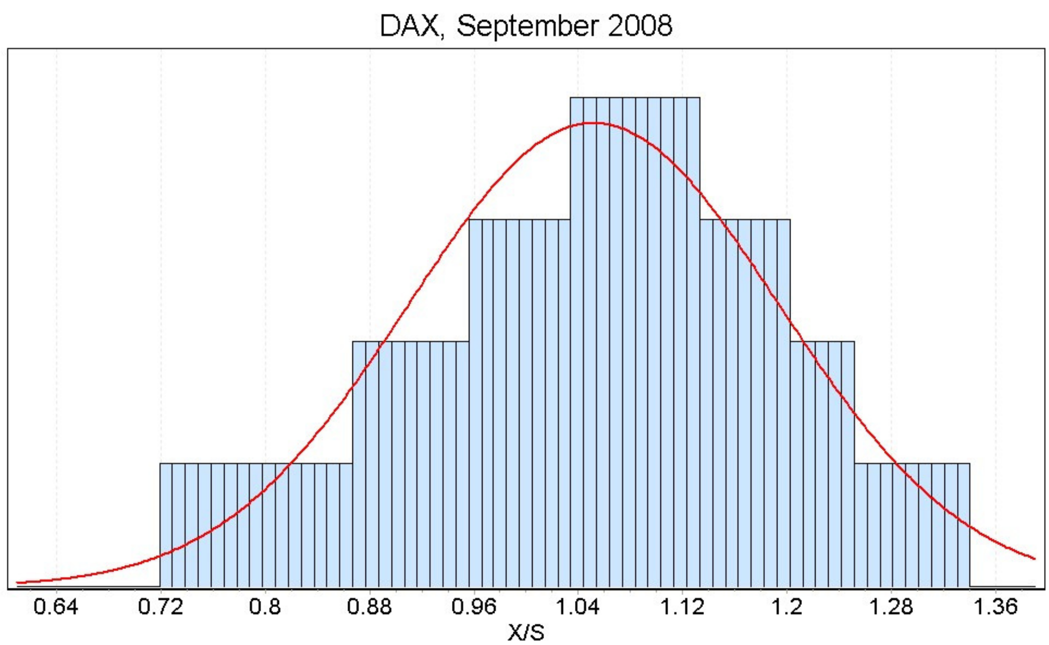


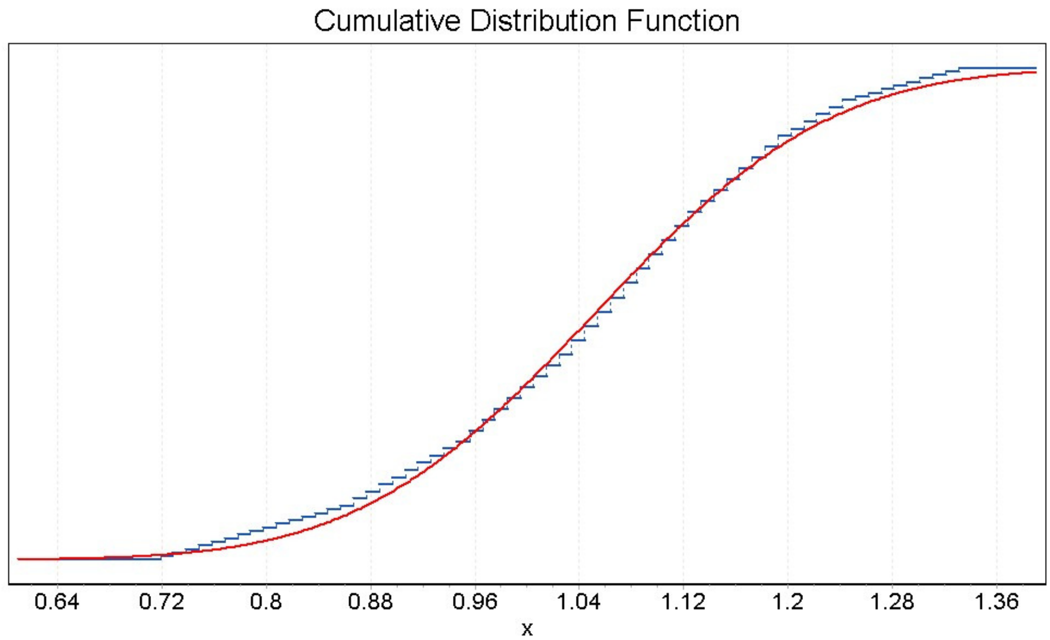
P-P Plot
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
P (Empirical)

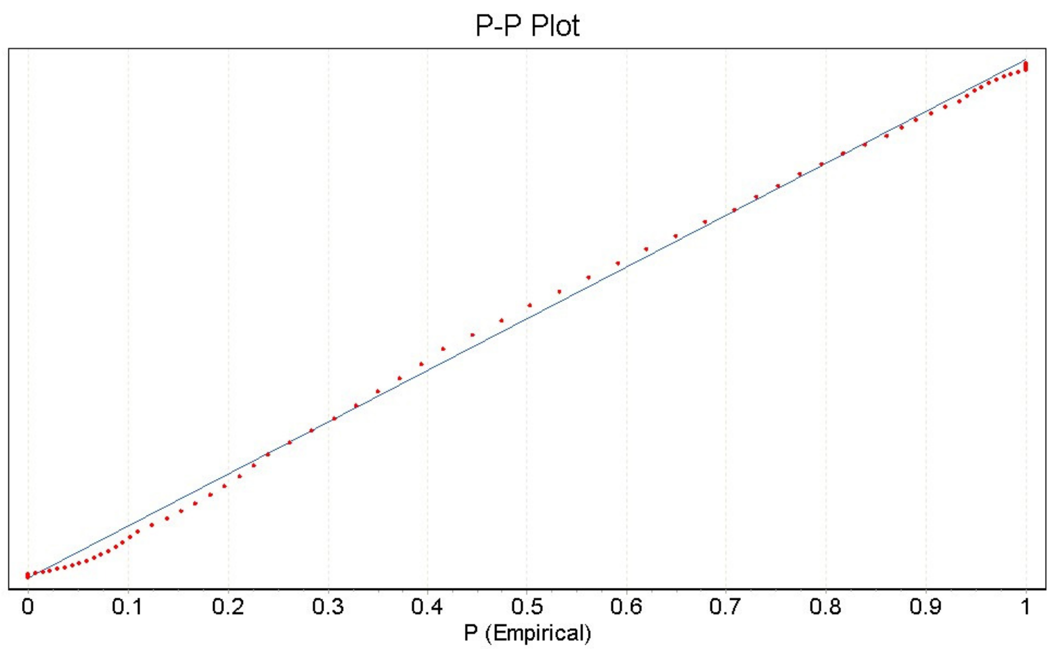

DAX, June 2008
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
X/S

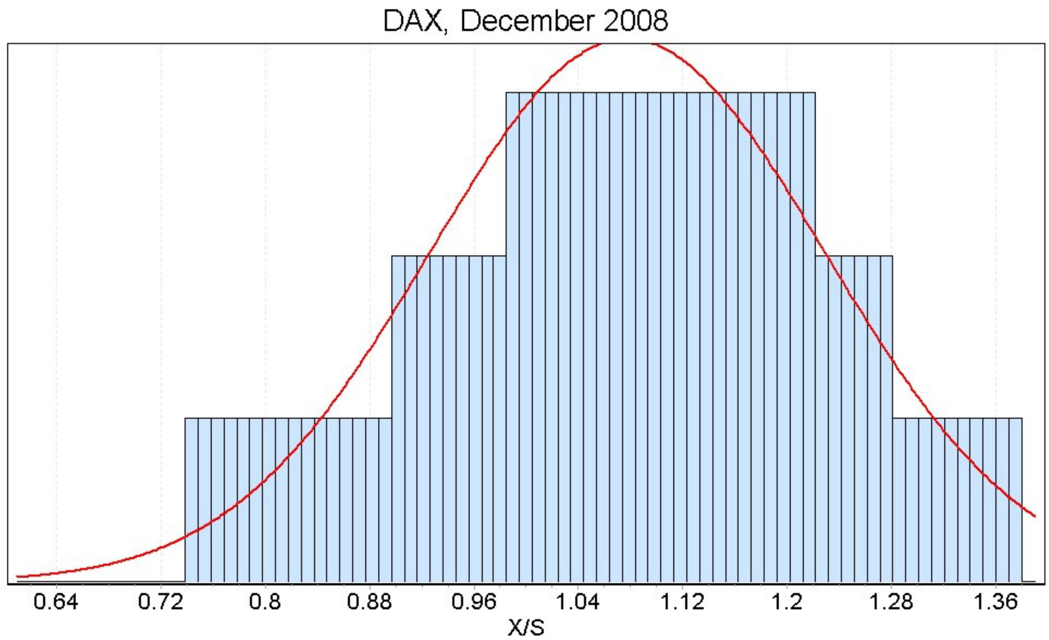

Cumulative Distribution Function
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
x

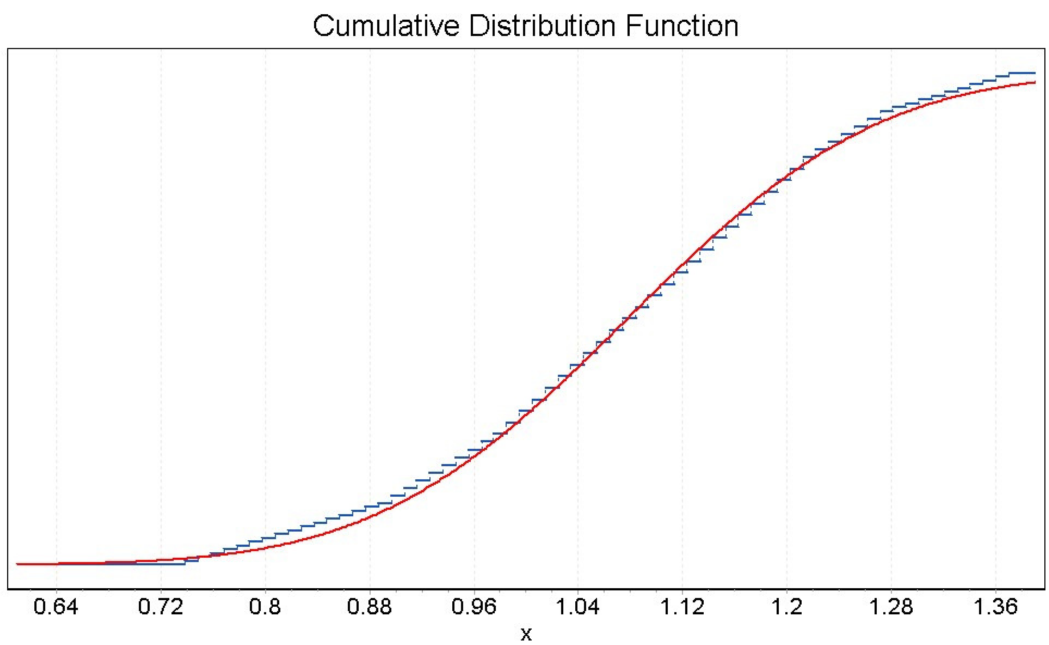

P-P Plot
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
P (Empirical)


DAX, September 2008
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
X/S


Cumulative Distribution Function
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
x


P-P Plot
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
P (Empirical)


DAX, December 2008
0.64
0.72
0.8
0.88
0.96
1.04
1.12
1.2
1.28
1.36
X/S


Cumulative Distribution Function
0.64
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0.8
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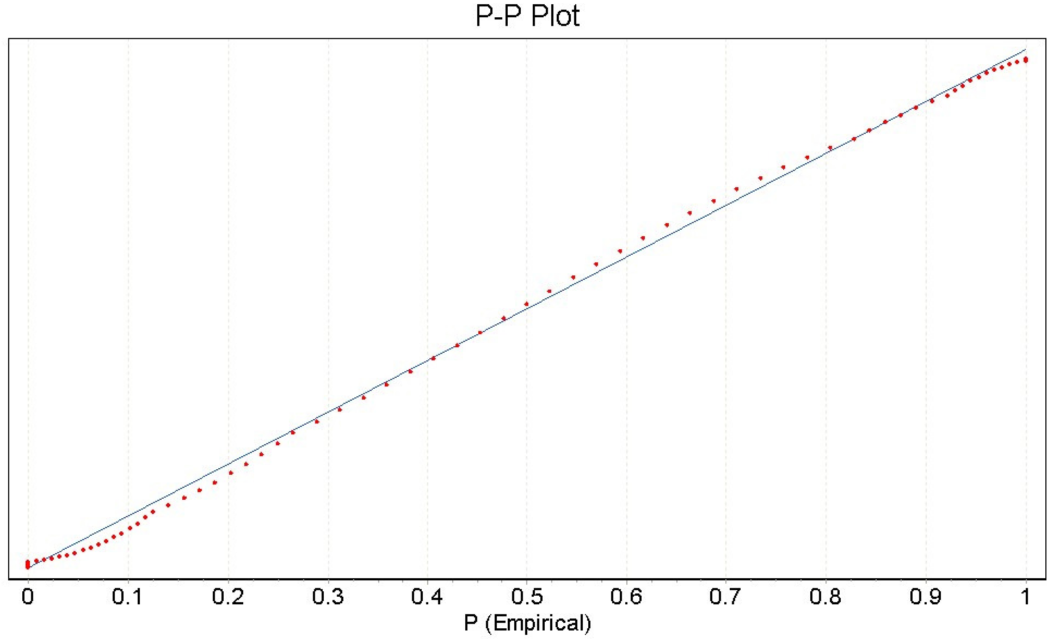
0.96
1.04
1.12
1.2
1.28
1.36
x


P-P Plot
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
P (Empirical)