Lappeenranta University of Technology
School of Engineering Science
Master’s Programme in Computational Engineering and Technical Physics
Technomathematics Major

Master’s Thesis

Abdulazeez Afolabi

REVISION OF THE ENSEMBLE COMPUTATIONAL MARKET DYNAMICS MODEL WITH BURGERS’ TYPE INTERACTION FOR MODELLING EXTREME EVENTS IN FINANCIAL MARKETS.

Examiners: Dr. Matylda Jabłońska-Sabuka
Professor Tuomo Kauranne

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ABSTRACT

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There are features which make electricity spot prices an exception from other forms of commodities. These are, for instance, the non-storability of the commodity and unpredictability of its prices. These features attained by this commodity have made it an interesting subject for so many years. The major focus of most of the studies is to find out how the commodity-related timeseries, like prices, consumption, etc. can be predicted. This research studies the New Zealand electricity spot price to see if there are possibilities for its prediction. The major challenge is the spikiness that occurs in the electricity spot prices. These rapid price swings that represent extreme volatility are a major hindrance in the forecasting of electricity spot market prices. Putting these sudden jumps into consideration, we propose a model which incorporates a type of Lévy process in modelling electricity spot market prices. This process, generalized hyperbolic distribution, was com-

bined with the Capasso-Morale approach. The resulting model is in a form of a system of interrelated stochastic differential equations.

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Lappeenranta, May 23, 2018

CONTENTS

LIST OF ABBREVIATIONS

ACF	Autocorrelation Function
ARMA	Autoregressive Moving Average
EMH	Efficient Market Hypothesis
GARCH	Generalized Autoregressive Conditional Heteroscadasticity
GH	Generalized Hyperbolic Distribution
GIG	Generalized Inverse Gaussian
hyp	Hyperbolic Distribution
MCMC	Markov Chain Monte Carlo
MSP	Marketing Schedule and Pricing
NI	North Island
NIG	Normal Inverse Gaussian Distribution
nsim	Number of simulation
NZ	New Zealand
PACF	Partial Autocorrelation Function
SDE	Stochastic Differential Equation
SI	South Island
SMP	System Marginal Price
$\stackrel{d}{=}$	Equality in distribution

1 INTRODUCTION

It is well known that when it comes to mathematical modelling, it encompasses different areas which can be data-driven and also incuding those having their root from the first principles of natural sciences, that is modelling with differential equations. In our case, both are combined when we study electricity spot prices with the available data combined with stochastic differential equations. Commodity markets, where electricity is the commodity, is an area that has been under research for so many years and this is due to the fact that the data, electricity spot prices, are challenging to analyse. The commodity in this case is very unique compared to other stocks and commodities traded in stock exchanges, mainly because of our incapacity to store the commodity. Hence, it is difficult to apply the usual methods used for stock management to power exchanges. Due to non-storability, the prices are not stable. Yet, electricity is such a specific utility, that even when the prices are high, customers will still keep using it but the prices are reasonably stable if the transmission is not limiting the supply of electricity [1]. In some cases the consumers who pay more for the commodity are at an advantage if it is noticed that the situation in some region seems congested which activates the marginal congestion cost [2]. One of the behaviours of these prices that cannot be underestimated is its spikiness which is the subject of much published research.

Different techniques have been employed in analysing this commodity in order to understand its behaviour better and to be able to make price predictions. Jabłońska [1] analyses this commodity by well-known time series analysis methods but due to some noticed outliers, predicting the prices may be difficult. Also, considering the volatility of the prices, Ptak et al [3] dive into an approach that was used in the estimation of volatility in the financial markets. In addition, Jabłońska [4] made research on stochastic processes as a supplement to existing methods. With the approach employed, she was able to get some interesting results but still left with some challenges in the predictability of the electricity spot prices.

The purpose of this work is to verify whether the model proposed by Jabłońska [4] can be modified while still being able to reproduce the behaviour of electricity spot markets. This is done by replacing the Wiener process in the original model with a generalized hyperbolic distribution and then eliminating the nonlinear components of the model one at a time. Our study focuses on using one of the generalized hyperbolic families which is the hyperbolic model combined with the existing model developed by Jabłońska [4]. We consider the inclusion of the local interaction, momentum and the hyperbolic distribution. The second situation is neglecting the local interaction while the last case is the removal of the momentum. Our study uses one of the New Zealand nodes (Stratford) from the North Island. The data set runs from 2001 to 2008 and was provided by the Centralized Data Set, New Zealand Electricity Commission which in 2010 changed to Electricity Authority.

The structure of the thesis is as follows. Section 2 elaborates on the theoretical background of the study, introducing the commodity market with its challenges with the proposed remedy. Section 3 gives the basis by reviewing literature in the area of study. The area of stochastic processes is looked into in section 4 which introduces the methods in the work. Section 5 presents the modelling aspect, it discusses the models involved and also gives information about the data with their descriptive statistics. In addition, it looks into the market by comparing those without jumps and those that exhibit jumps and the simulation results are discussed in the section. Finally, section 6 presents the conclusions.

2 THEORETICAL BACKGROUND

Studies and research have been carried out on electricity spot markets mainly on the Nord Pool prices and other markets like the Irish electricity spot market. In this work we consider the New Zealand electricity spot market. The geographic view of New Zealand is given in Figure 1.

Figure 1. The map of New Zealand [5].

Electricity in New Zealand is mostly generated from renewable energy sources which is known to take a percentage of around 70%. Sources like hydropower, geothermal power and wind energy make the country one of the most sustainable in the world of energy generation. In addtion, it is noticed that the demand for electricity is increasing on an average rate of 2.4% per-annum as far back as 1974 and between $1997 - 2007$, it is of 1.7% per year [4].

Figure 2 presents a map with marked trading nodes that shall be used in analysing the New Zealand electricity spot market.

Figure 2. Location of the nodes in the New Zealand grid used in the analysis [4].

The modelling of electricity spot market prices, which is a type of commodity market, is not something new in the world of finance. It is as popular as any other financial time series. There has been slow advancement in the modelling of electricity spot market prices, because the same problem still persists in most of the research that has been carried out. This problem is due to the difficulty in modelling the commodity market which leads to challenges in forecasting. In financial markets it seems earning profits is much easier if there is a possibility to know the situation of the market in the nearest future. Since the predictability of the model seems difficult, trading in such platform is risky. This is due to the known high volatility of financial markets which may occur without any notice. This property of price spikes, sudden jumps, is known as extreme events. These events are somewhat different from the known crash in any other financial market. The spikes present the ups and downs of prices which may jump with a magnitude of 10 to 100 times from its initial value before returning back to its original level. Market crash in the financial markets, is contrast, is the collapse, that is a sudden drop of prices that have been built for months or probably years which also take months or years to regain it original values.

Explicit estimation of future behaviour of the values of economic indicators is indeterministic complicated and not instinctive due to the complex interconnections between these indicators. The correlation or relationship occurring between the current and future values of economic indicators can be approximated by mathematical modelling. There are different mathematical models based on quantitative forecasting which aid in providing valuable estimates of future market trends. In spite of that, some researchers do not agree with forecasting since they believe that future events can not be predicted. There are possibilities for financial volatility to exhibit clustering or pooling which leads to occurrence of autocorrelation, the dependency of future values on past values. These attributes justify the building of sophisticated mathematical models for predicting volatility.

Empirical finance has been an area of finance that has touched different types of sophisticated mathematical models that can be used in modelling prices. Different models have been built by authorities or researchers in this area of finance. There focus is always how can the model built explain the factors affecting the markets such as the change in both supply and demand of specified products in the market. Such changes could lead to pricing signals and, this phenomenon is generally known as market dynamics and also aims at predicting future prices. The Efficient Market Hypothesis (EMH) is a common assumption in financial markets. The assumption is that markets are said to be efficient, that is, they are a reflection of all pertinent or relevant information in the market. This assumption was developed by the economist Eugene Fama in 1970 [6], with the theory that EHM tells us that there are no possibilities for investors to surpass the market, that is, generating a higher return than a particular benchmark in the market due to its efficiency. This means having past information of the performance of the market at hand should not justify the results in the future. The effect of momentum in the market is also a recently studied aspect which says what happened in the history of a market will probably follow the same trend in the nearest future.

The volatility that occurs in the prices of electricity spot markets tends to be high due to the large imbalances between the demand and supply of the commodity which tends unpredictable in deregulated markets. The difficulty in storing the electricity differentiates electricity markets from other markets. Difficulty of storing this commodity on a large scale leads to the need of immediate consumption when electricity is generated or produced. Sudden jumps which occur in the form of price spikes are the outcome of such consumption, this is known as extreme events. The research done on the electricity spot price shows that the regular behaviour of electricity spot price is modelled including both the strong intraday and weekly periodicity and such models are used for predicting the regular price evolution for a short term. In addition, modelling price volatility and sudden jumps in the prices has been done by researchers. They have used different known econometric models for modelling the spiky behaviour of electricity prices but the output of those models shows that they lack predictability power for extreme events.

The existing dynamics of electricity spot prices proposed in recent studies is reviewed.

This is done by merging models originating in population dynamics with fluid dynamics. There exist interactions among the population of individuals, such interaction occurs in three scales. The population in this sense is known to be the traders in the market and their interaction is presented via a system of stochastic differential equations. These three scales are treated in [7]. The scales are as follows:

- **Macroscale:** It directs the entire population.
- **Microscale:** It directs each individual separately.
- **Mesoscale:** It gives the opportunity for each individual to relate to its closest neighbourhood.

The momentum component in relation to the one-dimensional Navier stokes equation (Burgers' equation) from fluid dynamics aids in the generation of global interaction in the research.

The description done by John Maynard Keynes' *Animal Spirits* (1936) helps describe market psychology. His description in relation to some existing models is based on merging the jump components with the mean reverting process. It was argued by Jabłońska [4] that the extreme events occuring in price dynamics are a result of the market momentum and inclusion of the traders' psychology [8].

In this work, Lévy processes will be used to study the differences between the work done by Jabłońska [4], where she introduced Brownian motion in her proposed model based on the Capasso-Morale approach [9]. The type of Lévy process we shall be considering in this research is the hyperbolic distribution.

Lévy processes have been known in financial markets to give opportunity for sudden jumps, that is spikes, which was the weakness of Brownian motion. The process has been known to work well for option pricing after comparing some processes with the Black Scholes model which uses a Wiener process [10]. Using this process we study the behaviour of market spot prices and see if it succeeds in predicting the prices for the nearest future. It will also be verified whether replacement of Wiener process with a Lévy process would allow simplification of some of the non-linear terms in the model. The elimination of the Brownian motion and inclusion of Lévy process differentiates this work from earlier studies.

2.1 Objectives

The main aim of this work is to verify through simulations whether replacement of the Wiener process in the model proposed by Jabłońska [4] with a Lévy process would cause generation of spikes in the trajectories and therefore would allow us to simplify the non-linear components of the equation. The results will be verified through predictions and their statistical and pointwise accuracy.

3 LITERATURE REVIEW

This section gives an overview of literature related to the research to be carried out. In this section, we review some of the models for modelling electricity spot price in electricity markets of some countries like the Nordic electricity market and others. We look into different approaches that have been used in modelling of electricity markets and also used in predicting electricity spot prices.

Ptak et al. [3], is an article that looks into well-known models, this approach was used in estimating of volatility in the financial markets. The reliability of two well known econometric models, both Autoregressive Moving Average (ARMA) and Generalized Autoregressive Conditional Heteroscadasticity (GARCH) were the models considered for electricity spot prices. This was done in order to know the behaviour of the electricity spot market price. Reliability of an optimally chosen GARCH accompanying ARMA model was studied in the article. They used the Markov Chain Monte Carlo (MCMC) method to verify reliability of an optimally chosen GARCH accompany with ARMA model of two electricity spot market price time series.

In their work, the authors looked into the electricity spot markets where Nordic power suppliers are known to have generated approximately 3976 TWh where the highest supply is from Sweden. Due to the difficulty in the predicting of the electricity spot price, they were trying to make the market as perfect as possible by conducting different analyses on spot markets. Their major challenge was that the procedures used in calculating the electricity prices in different countries were crucially different. Setting the prices based on day-ahead and hours ahead orders to strike a balance between supply and demand was their focus. In their prediction, they looked at the underlying assumption of ergodicity over some time scale, and on linear dynamics. This was done under the time series forecasting where they considered the classical Box-Jenkins methods, such as ARMA and ARIMA forecasting. They studied the validity of the assumptions by Monte Carlo simulation. The appropriate approach used in their work was the Markov Chain Monte Carlo (MCMC) which was used in the estimation of unknown parameters of ARMA(P,Q) and GARCH(P,Q) models. These models were constructed for both the NORDPool and NEPool spot markets. They also used the GARCH technique in estimating and forecasting both the NORDPool and the NEPool return series. The occurrence of clusters were said to have been noticed with different variation of amplitude in both sets of data.

They noticed in their result that for the case of NEPool the length of the forecasting horizon was inversely proportional to the predictive values that is, the longer the forecasting horizon was, the more uncertainty was setting in to the predicted values. Although, they found out that both ARMA(1,1) and GARCH(2,1) models can be used for predicting short-term horizon ahead. Also, in the case of NORDPool spot market, where 10 values was predicted they came to the conclusion that GARCH(2,1) model can also be used for predicting returns for short-term horizon. Based on there forecast it was noticed that their must be some essential features in the electricity spot market not captured by the by the models.

Jabłońska [11], considered another electricity market, the Irish Electricity Market. The article was mainly on simulating the uplift process for the Irish Electricity market, analysing the stochastic features of the uplift and reconstructing the original data by simulation. Based on the nature of the uplift wait-jump structure, two alternative algorithms were suggested in their work. The suggested algorithms depend on either the uplift was said to be daily or seasonal. We were made to know in their work that electricity prices were calculated differently, these differences vary among markets. The Irish All Island Market for Electricity, the System Marginal Price (SMP) were considered in their work and was noted that it was calculated on a half hourly basis using Marketing Scheduling and Pricing (MSP) Software which was said to consist of two components. Reaching the demand in a particular half-hour trading period was the first component and this was know to be the Shadow Price which represents the marginal cost per 1 MW of the power. The half-hourly SMP values were complimented with Uplift which was added to the Shadow Price, this was done in order to recover the total costs of all generators. The calculation of this uplift was related to that discussed in [3] where they talked about the problem of modelling the prices due to its spikes. This same problem occurs in the models for calculating the uplift. They treated a question which was postulated by Bord Gáis company during the 70th European Study Group with Industry, that was the possibility of describing and simulating the uplift process as an individual stochastic process, with no background or constraining variables.

The Irish uplift time series was analysed in the article as a purely stochastic process. This was done by elucidating the statistical features and suggesting a reasonable logical approaches for the simulation. Reconstructing a series in a synthetic way in which the behaviour of such series will be similar to the original uplift series by presenting and comparing the statistical parameters which were used, the mean, standard deviation, skewness, kurtosis and autocorrelation.

From the simulation constructed for the first algorithm they were able to compare the behaviour of the synthetic process with the original data and were able to take note of some important features like noting the similarity in the autocorrelation function (ACF) and the partial autocorrelation (PACF) structure of the synthetic data with that of the original data. Also, the statistical features like mean value, standard deviation, skewness and kurtosis were compared with the simulated uplift. The second simulation included the monthly dependency which was absent in the first simulation. The absence shows that there was a good correlation with the original data. The model was now improved by putting into consideration the monthly dependency which actually presents a drastic change, that is there was an improvement noticed in the performance in the model. With these two approaches used in their simulation, they were able to get a good result. The challenge faced was that the known ARMA and GARCH or the mean reverting jump diffusion models failed to work.

Jabłonska [4], has reviewed a number of classical econometric methods, including SDEs, for modelling electricity spot prices from various markets. However, due to the failure of finding models that would accurately predict price spikes, it was argued that econometrics should be bridged with human psychology. This gives the opportunity to bridge up the econometrics with the simulation of human emotions. In this work, terms like ensemble methods for nonlinear stochastic differential equations were used in the mathematical representation of Keynes' Animal Spirits. The relationship between fluid dynamics and collective market behaviour presented the terms used. Price series of the Nordic electricity spot market Nordpool was their focus in the research.

The inability to forecast the emergence of the asset bubble in the U.S. housing marketin 2008 lead to a lack of trust in econometricians. Due to this, econometricians tried gaining back the trust by reconstructing mathematical and econometrics models. This makes them to find ways of explaining the theory behind the psychological element in market traders' actions. This leads to influencing of the human behaviour by what is known as emotions which was first introduced by Keynes [12] where emotions were also called animal spirits. This kind of behaviour has been noticed in the deregulated electricity spot markets, this behaviour was known as the extreme event which were known to be one of the most volatile financial markets. The behaviour, that is the extreme event presents the appearance of spikes known as sudden jumps in price changes within couple of hours or days. Based on this article, econometricians have not been able to construct models which will be able to forecast this extreme event. This prompt the authors to look into the possible origin of the extreme events or price spikes in animal spirits that seems to govern the traders behaviour.

They explored the classical Ornstein-Uhlenbeck type mean-reverting econometric models in relation with new non-linear terms emulating the impact of a distinct animal spirit each. From the view point of the author, the modelling approach used in the research can be extended to simulating other commodity markets after appropriate normalizations. Therefore, they view the model beyond spot market of electricity where sudden jumps occur due to inability to store the commodity. The electricity market has also been studied by making trials in analysing specific auction theories. The common ones are supply function equilibrium and multi-unit independent private value. The occurrence of competition by the traders in the electricity market is a well known influence in the model that was constructed. It is as a couple system of mathematical programs with equilibrium constraints but without explicit numerical result.

Different methodologies have been explored in modelling the electricity spot price but the performance of those models were able to explain local trends and part of the volatility from the historical information on factors affecting the price of electricity. It was attested that the trader's psychology influences the occurrence of jumps in the price but unfortunately it is the most challenging part in modelling of dynamics since these influences are yet to be explained. The emotion of humans in financial markets has been studied by different researchers, affirming that the dynamics that occur in real finance are based on irrational, emotional and often intuitive decisions by human agents. Addressing the human emotion in financial markets, multi-agent models have become a well known approach which are applied to macroeconomy where agents learn from their mistakes. Based on the spikes which were argued to have happened due to human psychology, this triggers the authors to look into ensemble model which accounts for some of the animal spirits in the spot markets.

These features of the animal spirit, that is the attitude of the traders influencing the financial market makes the model introduced by Capasso-Morale known as Capasso-Morale system of stochastic differential equations for modelling animal population dynamics to be adopted. The equation has the form

$$dX_N^k(t) = \big[\gamma_1 \nabla U(X_N^k(t)) + \gamma_2(\nabla(G - V_N) * X_N)(X_N^k(t))\big]dt + \sigma dW^k(t), \quad (1)$$

for $k = 1, ...N$. The physical herding of animal populations are described by the Capasso-Morale equation. Jabłońska [4] also used an equation known as Burgers' equation to build a link between markets and fluids. The equation is as follows

$$u_t + \theta u u_x + \alpha u_x x = f(x,t). \quad (2)$$

An ensemble which represents the individual spot prices bid by traders was proposed in this article. The following system of stochastic differential equations helps us describe the price realization of all traders. This equation is the Lagrangian representation given as follows

$$dX_N^k(t) = \gamma_t[(X_t^* - X_t^k) + (f(k, \mathbf{X}_t) - X_t^k)]dt + \sigma_t dW_t^k +^+ J_t^k dN_t +^- J_t^k dN_t, \quad (3)$$

for $k = 1, ...N$, where X_t^k represents the price of trader k at time t, X_t^* is the global price reversion level at time t, γ_t stands for the mean reversion rate at time t, $\mathbf{X}_t$ is the vector of all traders' prices at time t, $f(k, \mathbf{X}_t)$ is a function describing local interaction of the trader k, W_t^k is the Wiener process value for trader k at time t, σ_t is the standard deviation for the Wiener increment at time t, ${}^+J_t^k$ is the positive jump for trader k at time t, ${}^-J_t^k$ is the negative jump for trader k at time t, N_t represents the counting process for jumps at time t. It was noticed from the simulation that the ensemble model (3) proposed shows a good representation of the real price dynamics. Due to some challenges such as the superimposition of jump components, the author proposed another model linking up with Equation (2) and eliminating the jumps, the model is given as follows

$$dX_N^k(t) = [\gamma_t(X_t^* - X_t^k) + \theta_t(h(k, \mathbf{X}_t) - X_t^k)]dt + \sigma_t dW_t^k, \quad (4)$$

they replace the $f(k, \mathbf{X}_t)$ with $\theta_t(h(k, \mathbf{X}_t))$ and $h(k, \mathbf{X}_t)$ is defined as follows

$$h(k, \mathbf{X}_t) = \mathbf{M}(X_t) \cdot [\mathbf{E}(X_t) - \mathbf{M}(X_t)],$$

where $\mathbf{M}(X_t)$ stands for the mode of random variable X. θ is the strength of local interaction at time t. The given Equation (4) has no separate jump component.

There results for the simulations show that the methods the author proposed replicate well statistical features of the real spot price time series and also the price spikes were well replicated based only on price dynamics and ensemble behaviour.

Jabłońska and Kauranne [13], focus on two things, the representation of the Couzin et al. [14] in a quantitative mathematical form which was amendable to simulation. The result to this non-linear dynamic equation was the other part that was considered in their study. They interpreted this equation as a mathematical interpretation of John Keynes' *Animal Spirit* (1936) [12], that was often stimulated to describe market psychology. Their study takes inspiration from the recent study in the financial market modelling. This inspiration was in the animal behaviour which was described in Couzin et al. [14]. The article Jabłońska [4] has really explored this behaviour where they made connection with the Keynes' animal spirits. Jabłońska and Kauranne [13] is the continuation of Jablońska

[4] where the model used was a one-dimensional model of population dynamics which was simulated to show the behaviour of an ensemble of traders in the electricity spot markets. In continuation of their work they explore a two-dimensional form of the model used in Jabloński [4]. They used this system of stochastic differential equations to know the movement of individuals and how groups are influenced by bigger groups which was what they termed as animal spirits. This was done by simulating the two-dimensional system of stochastic differential equations. Their main aim is to confirm the natural fact that 5% of a population can divert the whole group towards a specfic direction. This study was also based on the Capasso-Morale approach. During the construction of their model in Jabłońska [4] they consider the Ornstein-Uhlenbeck mean reverting process which was also put into consideration in Jabłońska and Kauranne [13]. But the difference is that there are three components representing some type of force acting on separate individuals and on the whole population. These components replaced the single constant mean reversion level. The following are the main components of the model proposed:

- **Global mean:**
  It is a component that represents herding phenomenon. This means when individuals are willing to stay within a bigger group. It was an effect on the whole population, that is the entire population were expected to sway around its center of mass X_t^*. They relate this to the aggregation forces proposed by Morel et al. [7].

- **Momentum:**
  This was said to have been noticed in studies carried out by [14]. This effect denoted as $h(k, X_t)$, shows the significantly different behaviour of an adequate degree of a big subgroup which makes it deviate from the entire population triggers the effect of momentum.

- **Local interaction:**
  It was known naturally that for a big population, each individual has the ability to sense its neighbours. This was what happens in this interaction but to a limited extent. Each member of the population follows $g(k, X_t)$, which gives information about the furthermost neighbour within a range that caters for the closest $p\%$ of the entire population. This prevents sudden overcrowding to occur in any point in space.

- **Randomness:**
  Wiener increment was included in each individual's move. This increment allows randomness in the system.

The two dimensional model is given as follows

$$dX_N^k(t) = [\gamma_t(X_t^* - X_t^k) + \theta_t(h(k, \mathbf{X}_t) - X_t^k) + \xi_t(g(k, \mathbf{X}_t))]dt + \sigma_t dW_t^k, \qquad (5)$$

where

$$h(k, X_t) = \mathbf{M}(X_t) \cdot [\mathbf{E}(X_t) - \mathbf{M}(X_t)],$$

where $M(X)$ represents the mode of a random variable X and $E(X)$ is the classical expected value. Also,

$$g(k, X_t) = \max_{k \in I}\{X_t^k - X_t\}, \text{where } I = \{k|_{X^k \in N_{p\%}^k}\},$$

where X_t^k are continuous stochastic processes representing the movement of each particle, $N_{p\%}^k$ means the neighbourhood of the $k-th$ individual formed by the closest $p\%$ of the population. X_t^* stands for the mean of the whole population at time t, and parameters $\gamma_t,\ \theta_t$ and ξ_t represent the forces with which each of the interactions takes place.

Based on their simulation of the model the authors are able to conclude that it is a good interpretation of Keynes' animal spirits. Since the model was not set up for financial reality, it was noticed that the model presents how forces of confidence in ones own knowledge and trust in other sources of information can form population dynamics.

4 APPROACH AND METHODOLOGY

In this section, a concise introdution is provided on what stochastic processes entail with some examples of the processes. It describes the well known stochastic process, Brownian motion linking up with Lévy processes. Some basic concepts on Lévy processes are looked into and the processes with their characteristic functions are discussed. Some particular interest is also considered, like activities and variations, subordination and measures. The generalized hyperbolic (GH) distribution is also covered and some special cases are considered under the distribution. Definitions and theorems are gotten from Cont and Tankov [15, 16] and Øksendal [17] and some other literature.

4.1 Stochastic processes

Let τ be a subset of $[0, \infty)$. A family of random variables $\{X_t\}_{t \in \tau}$ is indexed by τ. The time parameter t may either be discrete or continuous. When $\tau = \mathbf{N}$ or $\tau = \mathbf{N}_0$, the process $\{X_t\}_{t \in \tau}$ is said to be a discrete time process and when $\tau \in [0, \infty)$, is a continuous time process. For each realization of randomness ω the trajectory $X(\omega) : t \to X_t(\omega)$ defines a function of time called the sample path of the process. This brings the conclusion that a stochastic process is a function of variables t representing time and ω randomness, Tankov [16]. These processes are used in the representation of a group of models which are commonly used in financial markets. Some of the stochastic processes are Brownian motion and Lévy processes.

4.2 Brownian Motion

A stochastic process $W = \{W_t\}_{0 \leq t \leq T}$ is said to be a Brownian motion or Wiener process if the following conditions are satisfied:

(i) $W_0 = 0$.

(ii) W has independent increments, that is for an increasing sequence of times $t_0, t_1, \cdots, t_n$, the random variables $W_{t_0}, W_{t_1} - W_{t_0}, \cdots, W_{t_n} - W_{t_{n-1}}$ are independent.

(iii) If $0 \leq s < t$ the increments $W_t - W_s$ has a normal distribution with mean and standard deviation of 0 and $t - s$, respectively.

(iv) W is a process with continuous sample paths.

This process is common in many stochastic models which take the form of stochastic differential equations (SDE).

4.3 Lévy processes

4.3.1 Definition and Properties

Given a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_{t\geq 0}$, where Ω is the sample space, $\mathcal{F}$ is the $\sigma-$ algebra, $\mathcal{F}_{t\geq 0}$ is a right continuous filtration, P is the probability measure. Let $X = \{X_t\}_{0\leq t\leq T}$ be a continuous time stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$ and a Lévy process defined as follow;

Definition 3.1 (Lévy Process)
A càdlàg (Right Continuous Left Limit), adapted real valued stochastic process $X = \{X_t\}_{0\leq t\leq T}$ with $X_0 = 0$ almost surely is called a Lévy process if it satisfies the following conditions:

(i) X has independent increments, that is for increasing sequence of times $t_0, t_1, \cdots, t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.

(ii) X has stationary increments, that is the distribution of $X_{t+\delta} - X_t$ for any $t \leq T,\ 0 \leq \delta$ which does not depend on t.

(iii) X is stochastically continuous, that is $\forall\ \epsilon > 0$,

$$\lim_{\delta \longrightarrow 0} \mathbf{P}(|X_{t+\delta} - X_t| \geq \epsilon) = 0.$$

(iv) càdlàg property, if there exists a subset $\Omega_0 \in \mathcal{F},\ P(\Omega_0) = 1$, such that, for every $\omega \in \Omega_0,\ \ X(t, \omega)$ is right continuous in t and has a left Yoshio [18].

The third condition shows the occurrence of a jump at a fixed time t. It has a probability of zero meaning that the occurrence of discontinuities is at random times. This condition differentiates the process from Wiener process. It can be understood from the definition of Lévy processes that they are a general form of stochastic processes. these processes can be viewed easily as continuous time random walks. From the definition, it is noticed that Brownian motion is also a Lévy process. Lévy processes serve as building blocks for Markov processes and semimartigales. In addition, the interpretation of the process having a càdlàg property means the left limit exits

$$X_{t-} = \lim_{s \to t-} X_s$$

There are different Lévy processes which are as follows: linear drift (deterministic Lévy process), Brownian motion (the only non-deterministic Lévy process) with continuous sample path, Poisson and Compound processes. Others are the hyperbolic distribution, generalized hyperbolic (GH) distribution to mention a few. Literature like Eberlein [19], Eberlein [20] looked at the generalized hyperbolic and hyperbolic distributions and these models have been used in modelling financial data by authors like Bibby [21].

Definition 3.2 (Martingales) A stochastic process $X = \{X_t\}_{0 \leq t \leq T}$ where t is continuous or $X = \{X_t\}_{t \in [0,T]}$ where it is discrete is adapted to a filtration $\mathbf{F} = \mathcal{F}_t$ is said to be a martingale if it satisfies the conditions below

(i) integrability property that is $\mathbf{E}[|X_t|] < \infty$

(ii) $\mathbf{E}[X_t|\mathcal{F}_s] = X_s$ for $s < t$ a.s which is the martingale property. Therefore, X_t is a martingale on $[0, \infty)$ if the integrability and the martingale property are satisfied for any $s < t$.

Martingales play a central role in the modern theory of stochastic processes and stochastic calculus. They have a constant expectation, which remains the same under random stopping and converge almost surely. Stochastic integrals are martingales and The main ingredient in the definition of a martingale is the concept of conditional expectation.

Definition 3.3 (Submartingales, Supermartingales) A stochastic process X_t, $t \leq 0$ adapted to a filtration $\mathbf{F}$ is a submartingale if $\mathbf{E}[X_t|\mathcal{F}_s] \geq X_s$ while for a supermartingale $\mathbf{E}[X_t|\mathcal{F}_s] \leq X_s$ if it is integrable for any t and s.

Definition 3.4 (Semimartigale) A semi-martingale is a stochastic process $X = \{X_t\}_{0 \leq t \leq T}$ which admit the decomposition

$$X_t = X_0 + M_t + A_t$$

where X_0 is finite and $\mathcal{F}_0$ measurable, M_t is a local martingale with $M_0 = 0$ and A_t is a finite variation process with $A_0 = 0$. The following are examples of semimartingales

(i) Let $X_t = B_t^2$ where B_t is a Brownian motion is a semimartingale. $X_t = M_t + t$ where $M(t) = B_t^2 - t$ is a martingale and and $A_t = t$ is a finite variation.

(ii) $X_t = N_t$, where N_t is a Poisson process with rate λ, is a semimartingale, as it is a finite variation process.

(iii) A diffusion, that is, a solution to a stochastic differential equation with respect to Brownian motion, is a semimartingale. Indeed, the Ito integral with respect to dB_t is a local martingale and the integral with respect to d_t is a process of finite variation.

For a semimartingale X, the process of jumps $\triangle X$ is defined by

$$\triangle X_t = X_t - X_{t_-}$$

and represents the jump at point t. If X is continuous, then of course, $\triangle X = 0$.

Definition 3.5 (Local Martingale) An adapted process M_t is a local martingale if there exists a sequence of stopping times $\mathcal{T}_n$, such that $\mathcal{T}_n \uparrow \infty$ and for each n the stopped processes $M(t \wedge \mathcal{T}_n)$ is a uniformly integrable martingale in t. The sequence $\mathcal{T}_n$ is called a localizing sequence.

Definition 3.6 (Lévy Measure) Let $\{X_t\}_{t \leq 0}$ be a levy process on $\mathbf{R}^d$. The measured ν on $\mathbf{R}^d$ defined by

$$\nu(A) = \mathbf{E}[\sharp\{t \in [0,1] : \triangle X_t \neq 0, \triangle X_t \in A\}], \quad A \in \mathcal{B}(\mathbf{R}^d)$$

is called the Lévy measure of X: $\nu(A)$ is the expected number of jump of a specific size belonging to A per unit of time. It tells us the behaviour of the jumps of a Lévy process and also gives information about its intensity. In addition, the distribution of the jumps is known via the measure. This can be described with Examples 1 and 2 that shall be presented in the next section.

4.3.2 Lévy Khintchine formula

In this stage, we present the celebrated Lévy Khintchine formula, it is a formula that bridges processes to distributions. Vice versa, that is linking distributions with processes is of one by the Lévy Ito decomposition. Knowing that Lévy processes are a general form of stochastic processes, it is difficult to derive a general expression for the probability density. However, their characteristic function can be used easily to describe the stochastic variables. Therefore, if the characteristic function of a Lévy process can be described, hence the process can be described. This is presented in the following general theorem known as the Lévy-Khintchine formula.

Definition 3.7 (Infinite divisibility) Suppose P_Y is the law of a random variable X, the random variable X is said to possess an *infinitely divisible distribution, if for all* $n \in \mathbf{N}$ *there exits iid random variables* $Y_1, ..., Y_n$ such that $Y \overset{d}{=} \sum_{i=1}^{n} Y_i$.

On the other hand, the law P_Y of a random variable Y is infinitely divisible if for any $n \in \mathbf{N}$ their exist another law $P_{Y^{\frac{1}{n}}}$ of a random variable $Y^{\frac{1}{n}}$ such that

$$P_Y = \underbrace{P_{Y^{\frac{1}{n}}} * ... * P_{Y^{\frac{1}{n}}}}_{n \quad times}$$

which can be given as follows

$$P_Y = \left(P_{Y^{\frac{1}{n}}} * ... * P_{Y^{\frac{1}{n}}} \right)^n$$

Some common examples of infinitely divisible laws which are Normal distribution, Gamma distribution, Poisson distribution, Compound Poisson distribution and Geometric distribution. While a distribution which is not infinitely divisible is the Uniform law on an interval.

Definition 3.8 (Characteristic function of a Lévy process) Let $\{X_t\}_{t \geq 0}$ be a Lévy process on $\mathbf{R}^d$. There exist a continuous function $\varphi : \mathbf{R}^d \to \mathbf{R}$ known as the Lévy exponent or characteristic exponent of X, such that

$$\mathbf{E}[e^{i.uX_t}] = e^{t\varphi(u)}, \ \ u \in \mathbf{R}^d.$$

See Appendix 1 for details of the characteristic function.

Theorem 1 *(Lévy-Khintchine formula).*
Let $\{X_t\}_{t \geq 0}$ *be a Lévy process on* $\mathbf{R}^d$ *with a characteristic triplet* (ζ, Ψ, ν)*, where* ζ *is a* $d \times 1$ *matrix,* Ψ *is a positive definite* $d \times d$ *matrix and* ν *is a positive measure on* $\mathbf{R}^d$ *such*

that the Lévy measure satisfies the integrability condition

$$\int_{\mathbf{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty. \tag{6}$$

Then

$$\mathbf{E}[e^{iuX_t}] = \exp(t\varphi(u)), \ \ u \in \mathbf{R}$$

where

$$\varphi(u) = i\zeta.u - \frac{1}{2}u.\Psi u + \int_{\mathbf{R}^d} (e^{iu.x} - 1 - iu.x\mathbf{1}_{|x|\leq 1})\nu(dx) \tag{7}$$

where $\varphi(u)$ is known as Lévy or characteristic exponent.

Considering a Poisson process and how it is integrated with the Khintchine formula.

Example 1 Consider a Poisson process N_t with intensity λ. The characteristic function for this process is derived as follow

$$\begin{aligned}
\phi_{N_t}(u) &= \mathbf{E}\big[e^{iuN_t}\big], \\
&= \sum_{x=0}^{\infty} e^{iux}\frac{e^{-\lambda t}(\lambda t)^x}{x!}, \\
&= e^{-\lambda t}\sum_{x=0}^{\infty} \frac{(\lambda t e^{iu})^x}{x!}, \\
&= e^{\lambda t(e^{iu}-1)}.
\end{aligned}$$

Expanding $e^{\lambda t(e^{iu}-1)}$ using the expansion procedure for exponential function, the characteristic function is presented as

$$\varphi(u) = \lambda(e^{iu} - 1).$$

We arrive at a formula after comparing with the Lévy Khintchine formula. The formula arrived at is a case where the measure is of jump size 1.

$$\varphi(u) = \int_{R_0} (e^{iux} - 1)\lambda\delta(x-1)dx,$$

where δ is known to be Dirac delta function and whenever the indicator function in Lévy

Khintchine formula is zero the measure ν is positive. This cancelled out the last term in the Lévy Khintchine formula. Where we have the following $(0, 0, \lambda\delta(x-1)dx)$ to be the characteristic triplet for the Poisson process.

Example 2 Consider a compound Poisson process. This process is known to be the generalisation of a Poisson process where the waiting times between jumps are exponential but the jump sizes can have an arbitrary distribution. The compound Poisson process is known with its intensity $\lambda > 0$ and jumps size distribution f is a stochastic process X_t defined as

$$X_t = \sum_{i=1}^{N_t} Y_i$$

where jumps size are i.i.d with distribution f and N_t is a Poisson process with intensity λ, independent from $\{Y_i\}_{i-1}$. The characteristic function is given as

$$\begin{aligned}
\phi_{X_t}(u) = \mathbf{E}[e^{iuX_t}] &= \mathbf{E}\Big[\mathbf{E}\big[e^{iu\sum_{i=1}^{N_t} Y_i}\big|N_t = n\big]\Big]P(N_t = n) \\
&= \sum_{x\geq 0} \mathbf{E}\Big[e^{iu\sum_{i=1}^{n} Y_i}\Big]e^{-\lambda}\frac{\lambda^n}{n!} \\
&= \sum_{x\geq 0}\Big(\int\limits_R e^{iux}F(dx)\Big)^n e^{-\lambda}\frac{\lambda^n}{n!}, \\
&= \Big(\lambda\int\limits_R (e^{iux}-1)F(dx)\Big).
\end{aligned}$$

The formula can now be written as

$$\varphi(u) = \Big(\int\limits_R (e^{iux}-1)\lambda F(dx)\Big).$$

The Lévy measure $\nu(dx)$ in this case is given as $\lambda F(dx)$ where λ denotes the jump intensity and F is the jump size distribution.

4.3.3 Activities and Variation of Lévy processes

These are generally used in the characterization of the behaviour of jumps in a Lévy process. The Lévy measure is an important measure which moves with useful information about the structure of the process. It relates the expected number of jumps of a particular height in a time interval of length 1. It occurred that the measure has 0 mass at the origin while some jumps can occur around the origin. Away from the origin it is said to be bounded with only a finite number of jumps. From Example 2 above, note that the Lévy measure is given as $\nu(dx) = \lambda F(dx)$. On one hand, if ν is considered to be a finite measure, meaning $\lambda := \nu(\mathbf{R}) = \int_{\mathbf{R}} \nu(dx) < \infty$, the probability measure can now be defined as $F(dx) = \frac{\nu(dx)}{\lambda}$ where λ and $F(dx)$ is the expected numbers of jumps and distribution of jumps size respectively. Meaning, the process has a finite number of jumps on every compact interval, in this case the process has a finite activity. On the other hand, if $\nu(\mathbf{R}) = \infty$, infinite number of jumps should be expected in every compact interval, in this case the process has infinite activity Cont and Tankov [15, 16].

For the case of the finite and infinite variation, both are also governed by a proposition, Cont and Tankov [15,16]. Lévy process is said to possess finite variation if $\int_{|x|\leq 1} |x|\nu(dx) < \infty$ while for infinite variation if $\int_{|x|\leq 1} |x|\nu(dx) = \infty$. Both activities and variations can be exhibited by different Lévy processes.

4.3.4 Stochastic Calculus with Itô formula

Stochastic calculus exits in the form of stochastic differential equations or integrals. Considering a stochastic process $X_t(\omega)$ with the stochastic differential equation

$$\frac{dX_t}{dt} = \alpha(t, X_t) + \sigma(t, X_t)W_t, \quad \alpha(t, x) \in \mathbf{R}, \sigma(t, x) \in \mathbf{R}$$

where W_t is one-dimensional white noise. The process X_t satisfies the following stochastic integral equation

$$X_t = X_0 + \int_0^t \alpha(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

and in differential form which is known to be the general form of a stochastic differential equations is given as

$$dX_t = \alpha(t, X_t)dt + \sigma(t, X_t)dB_t, \tag{8}$$

where B_t is the Brownian or Wiener process, α denotes the drift function while σ presents the diffusion part of the equation Øksendal [17]. Equation (9) has been a popularly used SDE in financial markets. It is mostly used in the option pricing in the Black-Scholes frame work. In our case, where we are considering the electricity market, lots of work has been done using the Wiener process in the equation but as of present we shall be considering the Lévy process in place of the Brownian motion and the process shall be governing the random part. We have the SDE to be written as follows

$$dX_t = \alpha(t, X_t)dt + \sigma(t, X_t)dL_t, \tag{9}$$

where L_t is representing the Lévy process in this case.

Proposition 1 (Itô formula for jump-diffusion process) Let X be a diffusion process with jumps, defined as the sum of a drift term, a Brownian stochastic integral and a compound Poisson process

$$X_t = X_0 + \int_0^t \alpha_s + \int_0^t \sigma_s dW_s + \sum_{j=1}^{N_t} \triangle X_j,$$

where α_t and σ_t are continuous non-anticipating processes Cont [15] with

$$\mathbf{E}\left[\int_0^T \sigma_t^2 dt\right] < \infty.$$

For any $\mathbf{C}^{1,2}$ function $f : [0,T] \times \mathbf{R} \to \mathbf{R}$, the process $Y_t = f(t, X_t)$ can be represented as

$$\begin{aligned} f(t, X_t) = f(0, X_0) &+ \int_0^t \left[\frac{\partial f}{\partial s}(s, X_s) + \alpha_s \frac{\partial f}{\partial x}(s, X_s)\right] ds \\ &+ \frac{1}{2}\int_0^t \sigma_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\sigma_s dW_s \\ &+ \sum_{i \geq 1, T_i \leq t} \left[f(X_{T_i} + \triangle X_i) - f(X_{T_i})\right]. \end{aligned}$$

In differential form

$$\begin{aligned} df(f, X_t) = \frac{\partial f}{\partial t}(t, X_t)dt &+ \alpha_t \frac{\partial f}{\partial x}(t, X_t)dt \\ &+ \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)\sigma_t dW_t \\ &+ \left[f(X_t + \triangle X_t) - f(X_t)\right]. \end{aligned}$$

See Cont and Tankov [15, 16] for details.

4.3.5 Subordination

The term subordination is a process in which transformation takes place on a stochastic process to generate a new stochastic process via stochastic time changed by a subordinator, an increasing Lévy process which is independent of the original process Sato [22]. The time change could be considered in two directions that is considering a change in time by looking into the original clock and a random clock. The calendar time is termed the original clock while the random clock is termed the business time. If a business time is noticed to be fast, this means there is strong activeness on such a business day. Randomness in the business activity leads to randomness in volatility. Normal and non-normal innovations are generated by Brownian motion and pure jumps by a Lévy process. Justifying the use of a subordinator is by considering the bidding of electricity which leads to the fluctuation of the prices in the electricity market, this can be delineated by the normal innovation. Therefore, random time change in a sense helps in capturing stochastic volatility Carr & Wu [23]. Bochner [24] introduced the main idea behind the subordinator in 1949 and gives a detail explanation in his book [25]. The process of obtaining a new Lévy process from the original or existing Lévy process, one of the procedures is to create a density on the probability space of the original process on every discrete interval of time. This procedure is known as a density transformation (see Sato [22]).

Theorem 2 *Let $\{Z_t\}_{t\geq 0}$ be a subordinator (an increasing Lévy process on $\mathbf{R}$) with Lévy measure ρ, drift β_0 and $P_{Z_1} = \lambda$.*
That is

$$\mathbf{E}[e^{-uZ_t}] = \int_{[0,\infty)} e^{-us}\lambda^t(ds) = e^{t\Psi(-u)}, \quad u \geq 0,$$

where, for any complex ω with $Re\ \omega \leq 0$

$$\Psi(\omega) = \beta_0\omega + \int_{[0,\infty)} (e^{ws} - 1)\rho(ds).$$

with

$$\beta_0 \geq 0 \quad and \int_{[0,\infty)} (1 \wedge s)\rho(ds) < \infty.$$

Let $\{X_t\}$ be a Lévy process on $\mathbf{R}^d$ with generating triplet (A, ν, γ) and let $\mu = P_{X_1}$.

Suppose that $\{X_t\}$ and $\{Z_t\}$ are independent. Define

$$Y_t(\omega) = X_{Z_t(\omega)}(\omega), \quad t \geq 0.$$

Then $\{Y_t\}$ is a Lévy process on $\mathbf{R}^d$ and

$$P[Y_t \in B] = \int_{[0,\infty)} \mu^s(B)\lambda^t(ds), B \in \mathbf{B}(\mathbf{R}^d),$$
$$\mathbf{E}[e^{i\langle z,Y_t\rangle}] = e^{t\Psi}(\log \hat{\mu}(z)), \quad z \in \mathbf{R}^d.$$

The generating triplet $(A^\star, \nu^\star, \gamma^\star)$ of Y_t is as follows

$$\begin{aligned}
A^\star &= \beta_0 A, \\
\nu^\star &= \beta_0 \nu(B) + \int_{0,\infty} \mu^s(B)\rho(ds), \quad B \in \mathbf{B}(\mathbf{R}^d \setminus \{0\}), \\
\gamma^\star &= \beta_0 \gamma + \int_{0,\infty} \rho(ds) \int_{|x|\leq 1} x\mu^s(dx).
\end{aligned}$$

If $\beta_0 = 0$ and $\int_{(0,1]} s^{\frac{1}{2}}\rho(ds) < \infty$, then $\{Y_t\}$ is of the type A or B has drift of 0 Sato [22].

Theorem 2 shows how the transformation of a Lévy process takes place. The transformation is done from the Lévy process $\{X_t\}$ to another process $\{Y_t\}$ by the help of subordinator Z_t is called subordination. It is said that any Lévy process identical in law with $\{Y_t\}$ is said to be subordinate to $\{X_t\}$. The subordination $\{Z_t\}$ is sometimes known as a directing process Sato [22].

4.4 Generalized Hyperbolic Distribution

The discovery of stochastic processes in modelling of financial assets is done by distributional assumptions on the increments and the dependency structure. It is not strange when the returns of financial assets have semi-heavy tails, meaning there is a different between kurtosis of the normal distribution and the actual kurtosis. In this sense, the actual kurtosis is said to be higher compared to the kurtosis of the normal distribution Mandelbrot [26]. This highly flexible distribution, the generalized hyperbolic distribution (GH) is known for its heavy tails.

The generalized hyperbolic distribution was known for its flexibility for some time now, these highly flexible distributions were introduced by Barndorff [27] to model grain-size distribution of wind-blown sand. The first application of these distributions in finance was by Eberlin and Keller [19]. The interesting thing about the GH distribution is that, it is a family with different subclasses embedded in it. These subclasses in GH are known based on values associated with the parameters that come with the distributions. Some of these subclasses are as follows, hyperbolic, variance gamma, normal inverse Gaussian (NIG), normal distributions, to mention a few. Figure 3 presents these families of distributions.

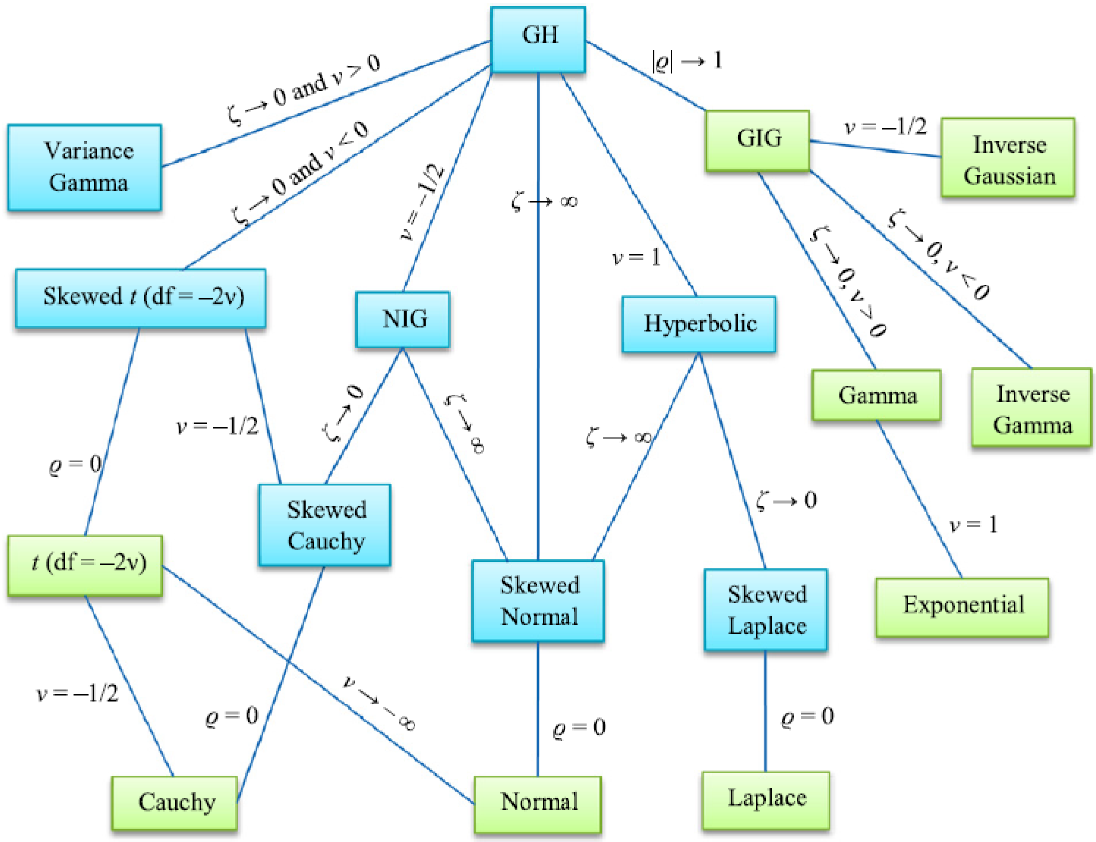

Figure 3. Generalized hyperbolic tree, BenSaïda [28].

However, there are some in the subclasses that are termed as special cases, this is because they have been used in modelling financial returns in financial markets. These special cases are: hyperbolic, normal inverse Gaussian (NIG), student-t, normal and variance gamma distributions. Barndorff-Nielsen [29] and Blæsild [30], present the mathematical properties of the distributions that are well known.

Recently, these distributions are categorised under Lévy processes and have been used for pricing movement in financial markets. Due to failure of Brownian motion in some aspects of modelling, these highly flexible distributions are now used in place of the classical Brownian motion. It was noticed by Ernst Eberlein [31], that among the subclasses of the distributions transpire by providing an excellent fit to market data. In our case, the subclass to consider is the hyperbolic distribution. This is done by replacing the Brownian motion in an existing model created by Jabłońska [32].

The moment generating function and characteristic function of the distribution are both presented. The definition of the distribution focuses more on the characteristic function. Appendix 1 and Appendix 2 present the general form of both the characteristic function and the moment generating function.

Definition 3.9 (Generalized hyperbolic distribution)

By Barndoff [27], the GH distribution with the parameters GH$(\alpha, \beta, \delta, \nu)$ is defined via its characteristics function Schoutens [33]. The characteristic function $\varphi(u)$ defined as $\mathbf{E}[e^{iuX}]$ is given as

$$\varphi_{GH}(u : \alpha, \beta, \delta, \nu) = e^{iu\mu}\left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2}\right)^{\frac{\nu}{2}} \frac{K_\nu(\delta\sqrt{\alpha^2 - (\beta + iu)^2})}{K_\nu(\delta\sqrt{\alpha^2 - \beta^2})}$$

where K_ν is the modified Bessel function. For $x \in \mathbf{R}$ the GH distribution has its density function defined as follows

$$\begin{aligned} f_{GH}(x; \alpha, \beta, \delta, \nu) &= a(\alpha, \beta, \delta, \nu)(\delta^2 + x^2)^{\frac{\nu}{2} - \frac{1}{4}} \\ &\quad \times K_{\nu - \frac{1}{2}}(\alpha\sqrt{\delta^2 + x^2})e^{(\beta x)}, \\ a(\alpha, \beta, \delta, \nu) &= \frac{(\alpha^2 - \beta^2)^{\frac{\nu}{2}}}{\sqrt{2\pi}\alpha^{\nu - \frac{1}{2}}\delta^\nu K_\nu(\delta\sqrt{\alpha^2 - \beta^2})}, \end{aligned} \tag{10}$$

and K_ν is a modified Bessel function of the third kind with the index ν and $\alpha, \beta, \delta, \nu$ are the tail, skewness, scale and subfamily Schoutens [33]. It satisfies the differential equation

$$x^2 y^{''} + xy - (x^2 + y^2)y = 0.$$

See [34] for the differential equation. See Appendix 3 and Appendix 4 for the Bessel function and the modified Bessel function. The following condition should be satisfied by the parameters.

$$\begin{aligned} &\delta \geq 0, \ \ |\beta| < \alpha \quad \text{if } \lambda > 0, \\ &\delta > 0, \ \ |\beta| < \alpha \quad \text{if } \lambda = 0, \\ &\delta \geq 0, \ \ |\beta| \leq \alpha \quad \text{if } \lambda < 0. \end{aligned} \tag{11}$$

In the GH density presented in Equation 10, it is noticed that the location parameter is not included. If the location parameter μ is included then Equation 12 presents the GH density,

$$\begin{aligned} f_{GH}(x;\alpha,\beta,\delta,\mu,\nu) &= a(\alpha,\beta,\delta,\nu)(\delta^2+(x-\mu)^2)^{\frac{\nu}{2}-\frac{1}{4}} \\ &\quad \times K_{\nu-\frac{1}{2}}(\alpha\sqrt{\delta^2+(x-\mu)^2})e^{(\beta(x-\mu))}, \\ a(\alpha,\beta,\delta,\nu) &= \frac{(\alpha^2-\beta^2)^{\frac{\nu}{2}}}{\sqrt{2\pi}\alpha^{\nu-\frac{1}{2}}\delta^{\nu}K_{\nu}(\delta\sqrt{\alpha^2-\beta^2})}. \end{aligned} \tag{12}$$

(See Scott et-al [35]).

The parameters μ and δ describe the location and the scale of the distribution while the index ν defines the subclasses of the generalised hyperbolic distribution and is directly related to tail fatness Barndorff [29]. The GH distribution may be represented as a normal variance mean mixture with Generalized Inverse Gaussian (GIG) as mixing distribution. The case where $\delta \to \infty$ and $\delta/\alpha \to \sigma^2$, we obtain the normal distribution. The generation of Lévy processes in GH model is that the GH distribution is infinitely divisible. Bessel function K_ν with its properties aid in simplifying the function GH with the respective values of ν ranging from -0.5 to 1 Prause [36].

Literature have proposed different parametrizations for GH density. The first parametrization which is also known as (α, β) parametrizations provided in Equation (11). The following are the other parametrization of both location and scale parameters, which are 2nd, 3rd and 4th parametrizations which are (ζ, ϱ), (ξ, χ), $(\bar{\alpha}, \bar{\beta})$ parametrizations respectively Scott et-al. [35].

$$\text{2nd parametrization} \qquad \zeta = \delta\sqrt{\alpha^2-\beta^2}, \quad \varrho = \frac{\beta}{\alpha}, \tag{13}$$

$$\text{3rd parametrization} \qquad \xi = (1+\zeta)^{-\frac{1}{2}}, \quad \chi = \xi\varrho, \tag{14}$$

$$\text{4th parametrization} \qquad \bar{\alpha} = \alpha\delta, \quad \bar{\beta} = \beta\delta. \tag{15}$$

Since the current form of the GH density in Equation (12) is less important with its present parameters, this leads to the parametrization of the parameters. BenSaïda [28] present a new form of the GH density by making a modification on (ζ, ϱ), that is second parametrization of Equation 13 by including both mean and variance in the new density given in Equation (16).

$$f_{GH}(x;\nu,\zeta,\varrho,\mu,\sigma) = \frac{\sqrt{\zeta}(1-\varrho^2)^{\frac{\nu}{2}-\frac{1}{4}}}{\sqrt{2\pi}\delta\sigma K_\nu(\zeta)}\left[\left(\frac{x-\mu}{\delta\sigma}\right)^2+1\right]^{\frac{\nu}{2}-\frac{1}{4}} K_{\nu-\frac{1}{2}}\Theta(x;\zeta,\varrho,\mu,\sigma,\delta), \tag{16}$$

$$\Theta(x;\zeta,\varrho,\mu,\sigma,\delta) = \left(\frac{\zeta}{\sqrt{1-\varrho^2}}\sqrt{\left(\frac{x-\mu}{\sigma\delta}\right)^2+1}\right)e^{\frac{\zeta\varrho}{\sqrt{1-\varrho^2}}\left(\frac{x-\mu}{\delta\sigma}\right)}.$$

where K_ν is the modified Bessel function of the third kind, ζ is the shape parameter, $|\varrho|$ is the skewness parameter, μ is the location parameter and the scale parameter is denoted as δ. In this case, GH becomes the normal distribution when $\zeta \to \infty$ and $\varrho = 0$. In addition, when $\varrho < 0$ and $\varrho > 0$ the GH becomes skewed to the left and right respectively. Equation 17 presents another form in which GH distributions can be presented. It is written as a normal variance-mean mixture of the generalized inverse Gaussian (GIG) distribution, which is known as the most useful representation of GH distribution.

$$f_{GH}(x;\nu,\alpha,\beta,\delta,\mu) = \int_0^\infty N(x;\mu+\beta w, w)\text{gig}(w;\nu,,\delta^2,\alpha^2-\beta^2)dw \tag{17}$$

where N and gig$(x;\nu,\chi,\psi)$ denote the normal density function with respect to mean and variance and generalized inverse Gaussian distribution respectively [10].

4.4.1 Special cases of the GH distribution

The special cases of the GH distribution are also known as some of the subclasses of the distribution. Some of these subclasses are presented as follow;

Definition 3.10 (Generalized Inverse Gaussian). The generalized inverse Gaussian (GIG) distribution is given by

$$f_{\text{gig}}(x;\nu,\chi,\psi) = \left(\frac{\psi}{\chi}\right)^{\frac{\nu}{2}}\left[2K_\nu(\sqrt{\chi\psi})\right]^{-1}x^{\nu-1}e^{-\frac{(\frac{\chi}{x}+\psi x)}{2}}, \tag{18}$$

where K_ν is the modified Bessel function of the third kind and x is positive. (See Scott et-al. [35] for more details.)

Definition 3.11 (Normal Inverse Gaussian (NIG) Distribution). The density for the NIG distribution is obtained from the GH distribution when $\nu = -\frac{1}{2}$. The density is given as follows:

$$\begin{aligned} f_{\text{nig}}(x;\alpha,\beta,\delta,\mu) =& \pi^{-1}\delta\alpha e^{\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\left[\delta^2+(x-\mu)^2\right]^{-\frac{1}{2}} \\ & \times K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)e^{[\beta(x-\mu)]} \end{aligned} \tag{19}$$

The parameters satisfy the following $\mu \in \mathbf{R},\ \delta > 0$ and $0 \leq |\beta| < \alpha$. The density in Equation 19 is also referred to as GH skew Student's t-distribution. The density was able to be obtained based on the properties of the modified Bessel function. (See Scott et-al. [35] and Aas et-al [37] for more details.)

Definition 3.12 (Hyperbolic Distribution). The density for the hyp distribution is obtained from the GH distribution when $\nu = 1$.

$$f_{\text{hyp}}(x;\alpha,\beta,\delta,\mu) = \frac{\sqrt{\alpha^2-\beta^2}}{2\delta\alpha K_1(\delta\sqrt{\alpha^2-\beta^2})}e^{(-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu))} \tag{20}$$

where $x,\ \mu \in \mathbf{R},\ \ 0 \geq \delta,\ \ |\beta| < \alpha$.

5 MODELLING ELECTRICITY SPOT PRICES

In this section, the existing model is introduced briefly, these models are of two types which give direction to the proposed model. The model proposed was also given consideration and the main focus of the model is the aspect of the simulation of the model.

5.1 Existing Models

These models are of one and two dimensions. Both models are based on the approach of Capasso-Morale where this model is used for modelling herding of animals. These have been talked about extensively in section 3.

5.1.1 One-Dimensional Model

The one dimensional model built by Jabłońska [4] was based on the Capasso-Morale approach and this can be seen in section 3, in the section the given Equation (4) is the one-dimensional model. This model is of two components, the global mean and the momentum effect. This model is the foundation for the model proposed.

5.1.2 Two-Dimensional Model

In the case of the two-dimensional model which was built by Jabłońska and Kauranne [13]. This was built upon the one-dimensional model, the only difference was that there was a new component that was introduced to the one-dimensional model that has been in existence. The component included was known to be the local interaction which makes a total of three components for the two-dimensional model. Equation (5) in section 3 provide the form in which the model exist.

5.2 Model Proposed

The model proposed in this work follows closely with the articles [4, 13] where we shall also carry out numerical simulation with different model parameter values. The approach we shall also be employing in this study is the Capasso-Morale approach where we have the information of each individual been followed separately in the population and with this, a system of coupled stochastic differential equation were generated. The goal here is to modify the type of randomness and cater for the sudden jumps that may seems to arrive during the life of the electricity spot market. This randomness shall be replaced with the hyperbolic distribution which is a type of Lévy process. The components like global mean and the momentum effect stay the same as was proposed by Jabłońska [4]. The model is of the following form

$$dX_N^k(t) = [\gamma_t(X_t^* - X_t^k) + \theta_t(h(k, \mathbf{X}_t) - X_t^k)]dt + \text{hyp}(x; \alpha, \beta, \delta, \mu). \tag{21}$$

for $k = 1, ..., N$, where

- dX_N^k is the change in prices with respect to the movement of each trader k and in the total population of N individuals,
- X_t^k is the price of the trader k at time t,
- X_t^* is the global reversion level at time t,
- γ_t is the mean reversion rate at time t,
- $\mathbf{X}_t$ is the vector of all traders' prices at time t,
- $\text{hyp}(x; \alpha, \beta, \delta, \mu)$ is the introduced Lévy process, hyperbolic process,
- σ_t is the standard deviation for the hyperbolic process at time t

5.3 Simulation of the Proposed Model

5.3.1 Data Description

The data used in this work are daily electricity prices of New Zealand which run from 2001 to 2008. The data is of 11 nodes which comprises of six nodes from the North Island and five nodes from the South Island. The data is from Centralized Data Set, New Zealand Electricity Commission which in 2010 turned into Electricity Authority. Figure 4 presents the daily data for the electricity prices. It is important to know that a larger share of New Zealand's electricity production takes place in the South Island part while the North Island has higher electricity demand. The higher demand in the North is due to its development. It can be noticed from the data presented in Figure 4 that the North area seems spiky compared to the South Island. The data is also presented in a histogram

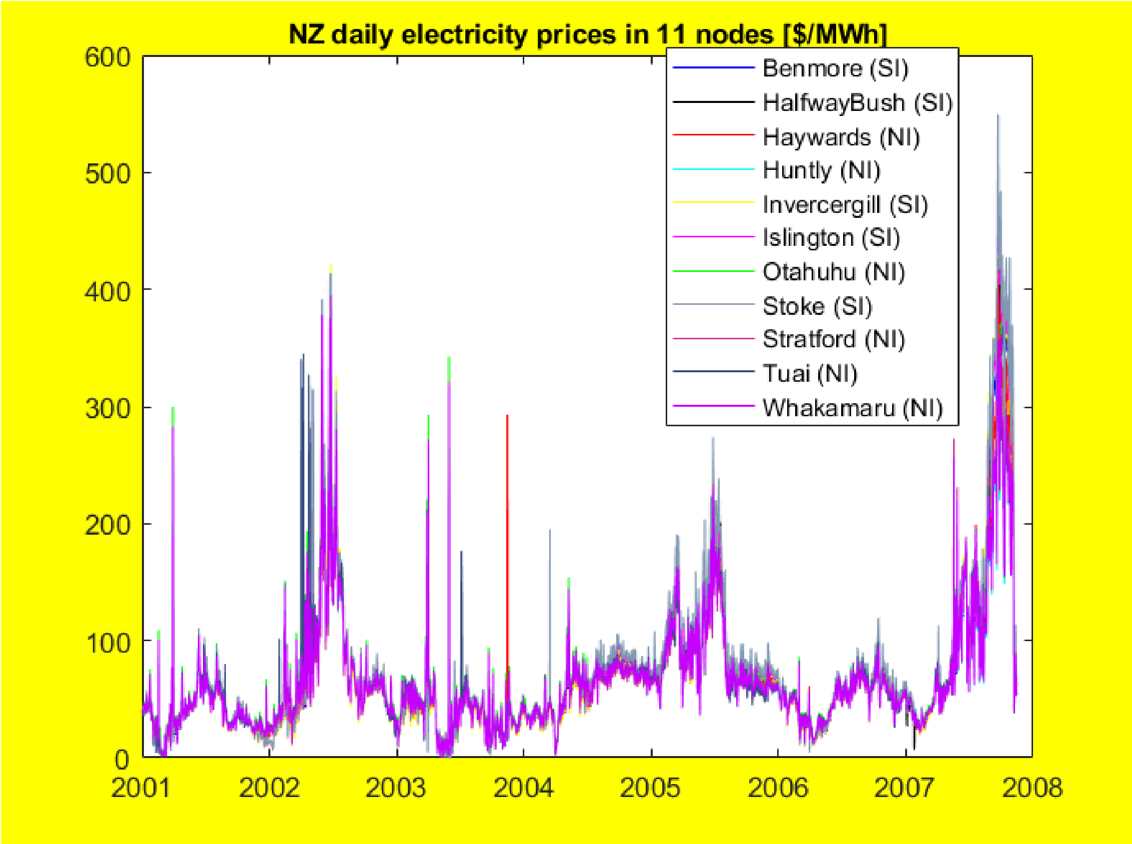

Figure 4. Daily electricity spot prices of New Zealand.

form to better view the distribution and the behaviour of the data. Figure 5 presents the histogram form of the electricity prices of New Zealand. It can be inferred from Figure 5, the data is far from a normal distribution and skewness is positive, that is the spread of the data is to the right.

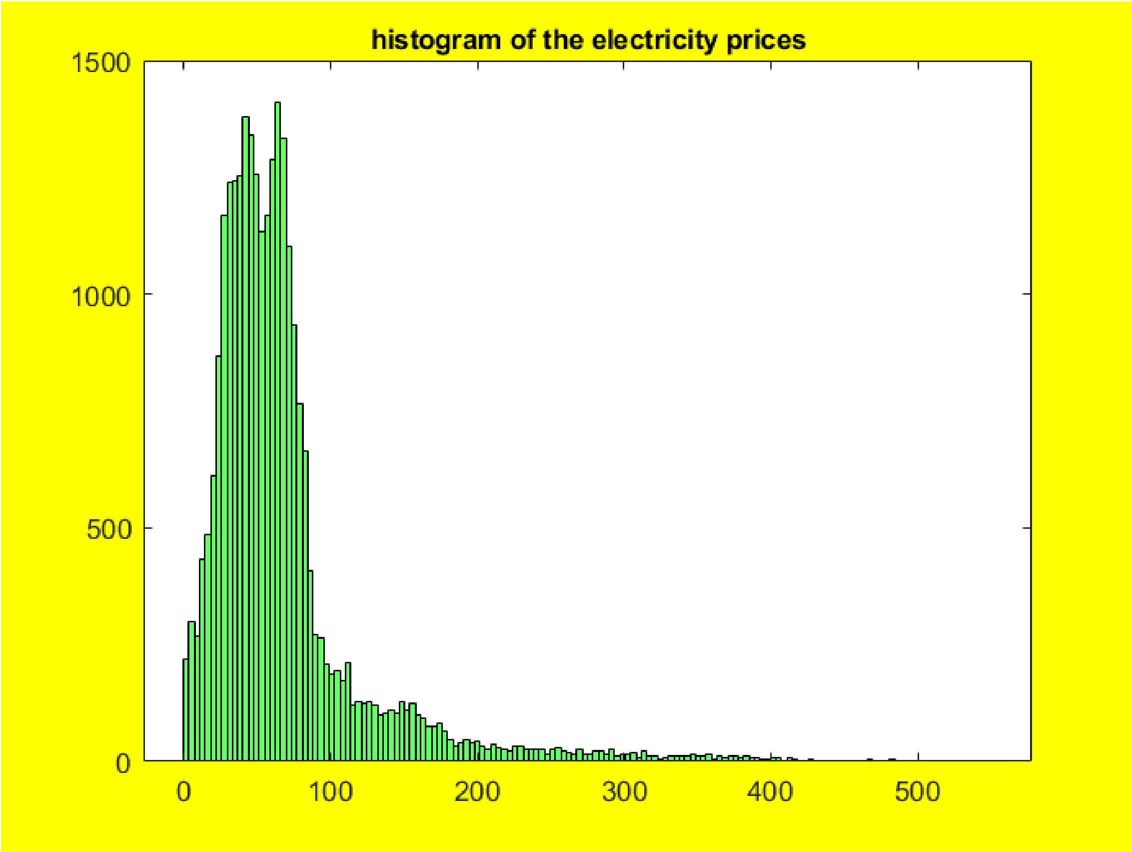

Figure 5. Daily electricty prices of New Zealand

5.3.2 Pricing with jumps

It has been confirmed by different studies that most Brownian based model possesses some problems when it comes to pricing in any kind of markets. For instance, research shows that the popularly known method for option pricing, which is a Brownian based model called Black-Scholes model has some problems which was given in Schoutens [33], Carr et-al [38]. This same problem occurs in electricty markets. In this sense, we can make a toy comparison between the real market with the leptokurtic property and the Brownian motion based market model which gives an introduction to what the result should looks like. Figure 6 shows both real market and the Brownian based model market where it is noticed that the higher peak and heavy like tails are possessed by the real market compared to the other market. The tails of the real market aid in describing the probability of extreme events which the Brownian based model failed to predict. It fails to consider the sudden larger movements in the market. This leads to adding a model which puts into consideration the sudden movement of the real market in the proposed model which follow some analyses in this work. Tankov and Cont [15, 16] present some models containing jumps like compound Poisson processes and pure jump processes.

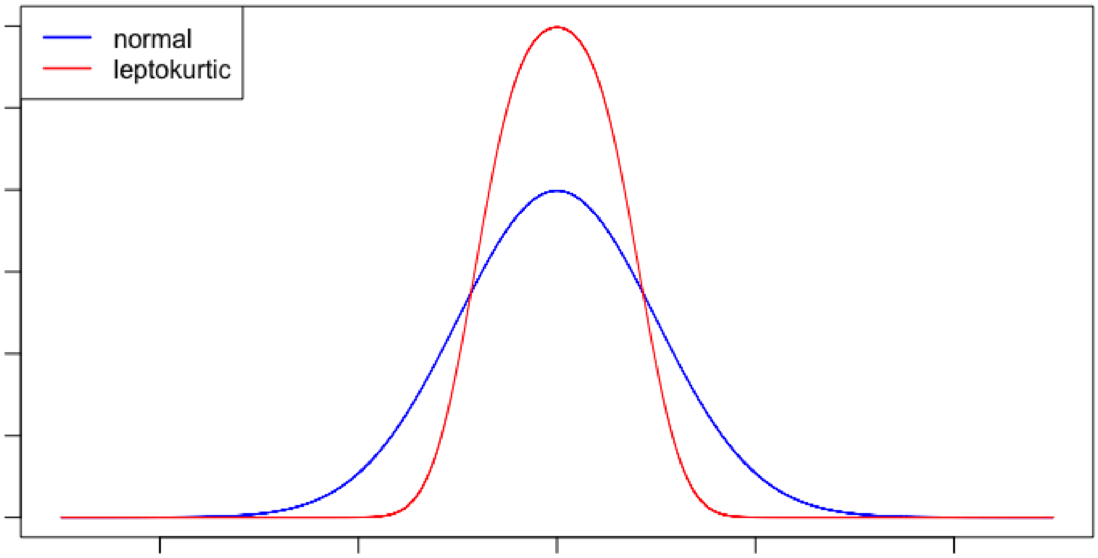

Figure 6. Comparing the real market with the Brownian based market.

5.3.3 Descriptive statistics

In this section, we explore the major descriptive statistics of the data in order to have more certainty on the behaviour of the data. Table 1 presents the basic statistics for further assessment of the data. It can be inferred from Table 1 that both the mean and the standard deviation are not significantly different and the skewness acts as a support to Figure 5 that the data is not normal and positively skewed to the right. In addition, it is noticed that the kurtosis is higher when compared to that of the normal distribution which is known to be 3. This means, it can be confidently said, that the data possesses the leptokurtic property.

5.3.4 Simulation results

In this section, the results are grouped into three subdivisions. In this subdivision, the analysis of the results is done by considering the whole model which includes all the components which are the global mean, momentum, local interaction and the randomness generated from the hyperbolic distribution. To study the model further, we eliminate the local interaction component to study the model's behaviour. Finally, the momentum component is also excluded from the model to find out how important the component is in the built model. In addition, we present the basic statistics like mean, standard deviation, skewness and the kurtosis for the simulated price for each of the stages. The simulation carried out in this section has its nsim, which is the maximum number of simulations to

Table 1. Basic statistics for New Zealand daily electricity spot prices.

Descriptive statistics of the NZ daily electricity prices.				
North Island				
Nodes	Mean	St dev	Kurtosis	Skewness
Heywards	68.9906	55.9683	11.9915	2.6833
Huntly	66.9990	49.0998	11.2128	2.5131
Otatuhu	68.9241	50.6772	11.2126	2.5155
Stratford	65.4265	50.3376	11.6290	2.6146
Tuai	67.9036	51.7905	10.5775	2.4637
Whakamaru	67.0162	50.1452	11.0142	2.4994
South Island				
Benmore	67.0766	60.8445	14.3118	3.0057
Halfway Bush	68.5894	64.6447	14.6344	3.0530
Invercargill	69.6794	66.3726	14.7042	3.0645
Islington	71.8645	65.9866	14.5879	3.0289
Stoke	74.4452	68.7175	14.2918	3.0003

be 100.

The complete model

In this situation, the model in Equation (21) is considered in its entirety for the first analysis. That is, the model includes all the components which are global mean, momentum, local interaction and the randomness from hyperbolic distribution. The results of the analysis is presented as follows. Figure 7 presents the original price with the simulated price while Figure 8 shows the original price with the average of the simulated price. It can be inferred from Figure 7 that the appearance of spikes is quite frequent in the simulated trajectory but not withstanding some of the spikes can be found where the original price seems to be irregular also negative spikes are noticed from the simulated data. It can be seen that some portion of the simulated data are nicely following with the original. In addition, the situation is different considering Figure 8 which actually follows the trend of the original data nicely. However, quantifying the differences is done by presenting the basic statistics for both the original price in Table 1 where Stratford is considered and the simulated price in Table 2. The basic statistics collected by both tables are mean, standard deviation, kurtosis and skewness. It is observed that the basic statistics for the original data are not well replicated but the values for the kurtosis and skewness are closed which can be found in Table 1 and Table 2 for both original and simulated respectively.

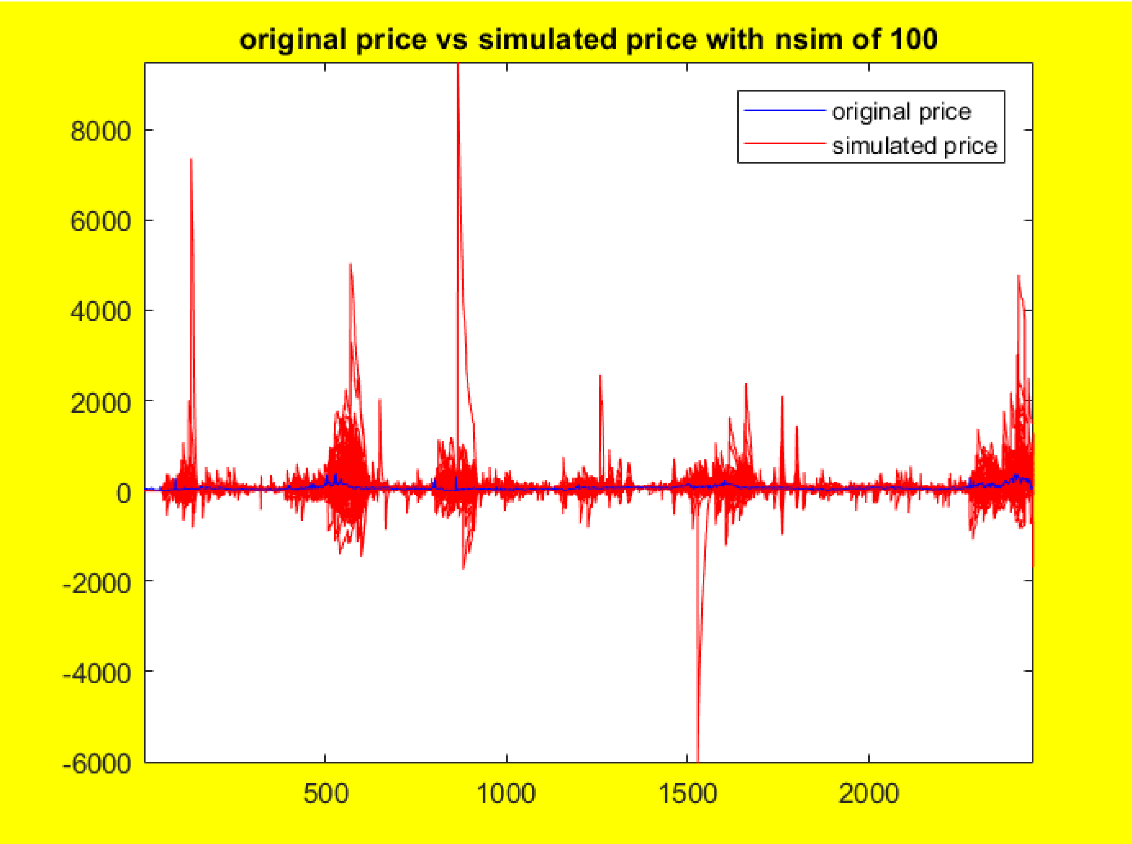

Figure 7. Simulation of the original price and simulated price.

Table 2. Basic statistics for simulated prices with all three components.

Mean	82.0247
St dev	139.5171
Skewness	3.3726
Kurtosis	17.9782

Model without the local interaction component

The same procedure is followed as in the complete model, the only difference is that a component has been excluded from the entire model. The model under analysis in this case is without the local interaction component. The results can be observed from both Figure 9 and Figure 10 that there are no much differences compared to the results presented under the complete model. In Figure 9 it can be inferred that the simulated follows the trend of the original and also generate spikes where the original data seems to be irregular, a negative spikes can also be noticed in the figure. The situation is different when considering the mean of the simulated price with original price presented in Figure 10. It is noticed that the mean follows the trend of the original price quite well. In addition, it can be inferred in Table 2 for the complete model and Table 3 for the model

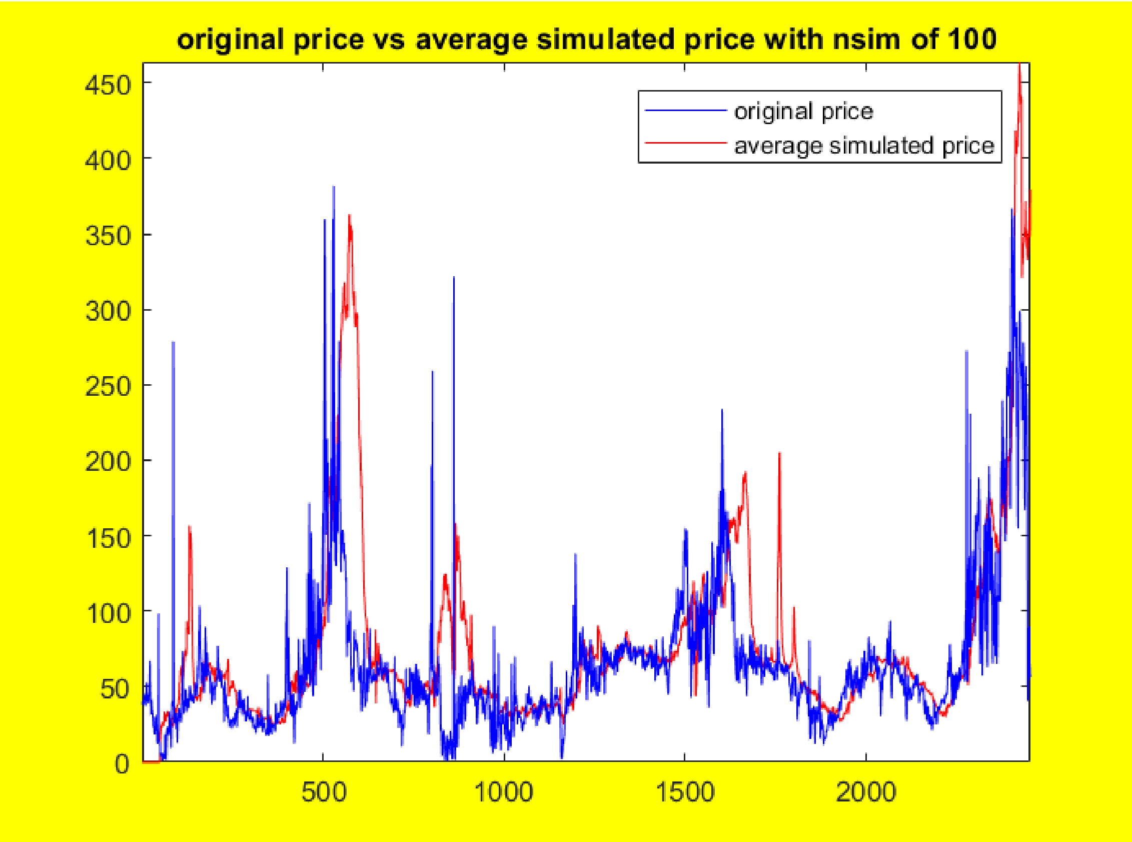

Figure 8. Simulation of the original price and avearge simulated price.

without local interaction component that there are not much differences from the values meaning that the removal of the component does not affect the complete model.

Table 3. Basic statistics for Simulated prices without the local interaction component.

Mean	70.5738
St dev	136.8707
Skewness	1.9363
Kurtosis	15.3686

Model without the momentum component

As it has been established from past research by Cousin et al [14] that the momentum is a significant component that influences the results in their research. However, the component is also excluded from model in Equation (21) to find out how significant the momentum is in the model and to study the behaviour of the model without the component. It can be seen from Figure 11 presenting the original price and the simulated that their is a huge difference between the original data and the simulated. Also, it can be inferred

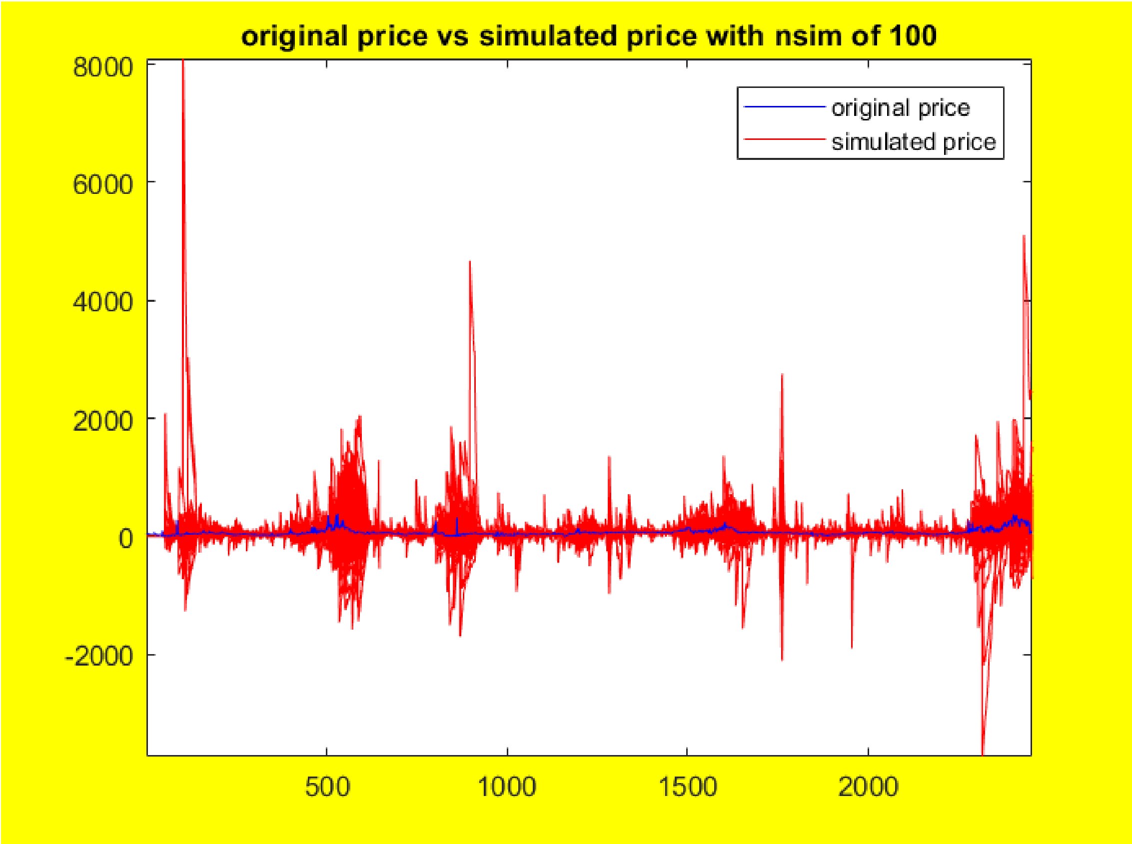

Figure 9. Simulation of the original price and simulated price without the local interaction component.

from Figure 12 that the mean of the simulated trajectory derails completely from that of the original data. In addition, looking at their differences by comparing the basic statistics from Table 1 and Table 4 presenting both the original data and the simulated data respectively, it can be seen that there is a great difference in the values generated in both cases. This informed us that the momentum component is an important component in the model.

Table 4. Basic statistics for Simulated prices without the momentum component.

Mean	-414.1349
St dev	327.5930
Skewness	0.0125
Kurtosis	2.3963

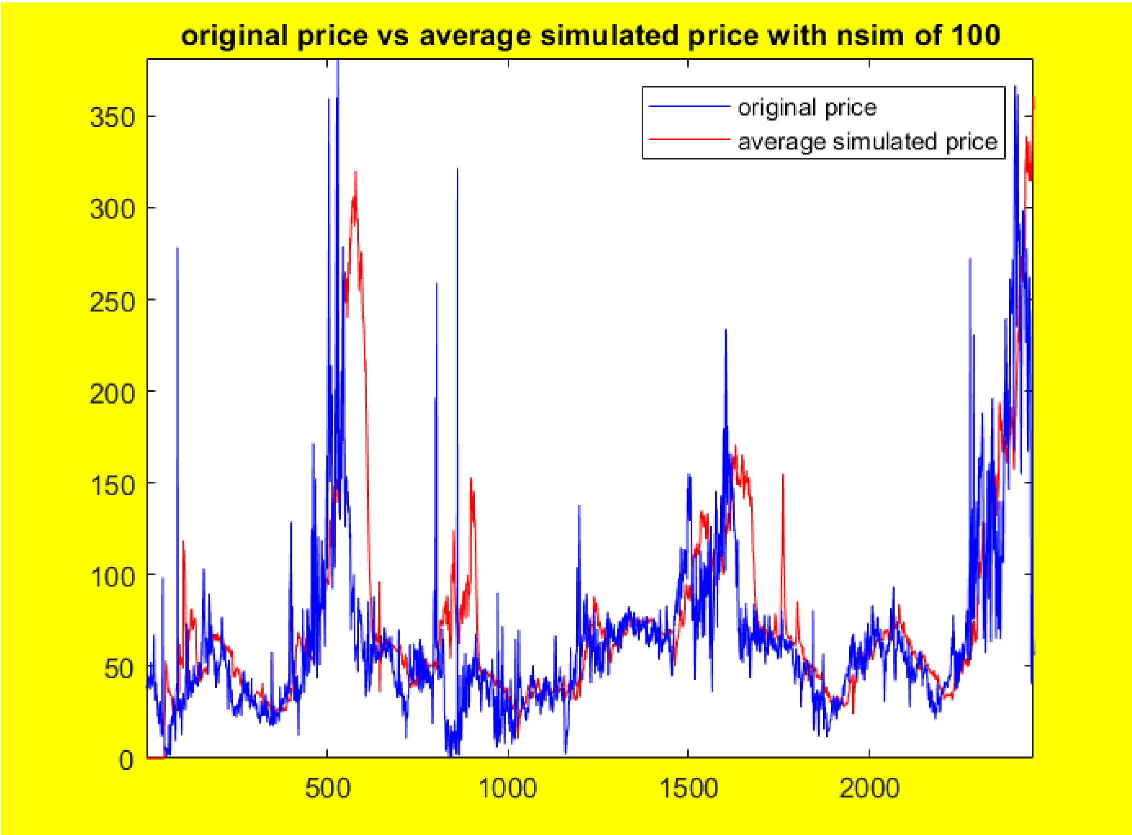

Figure 10. Simulation of the original price and avearge simulated price without the local interaction component.

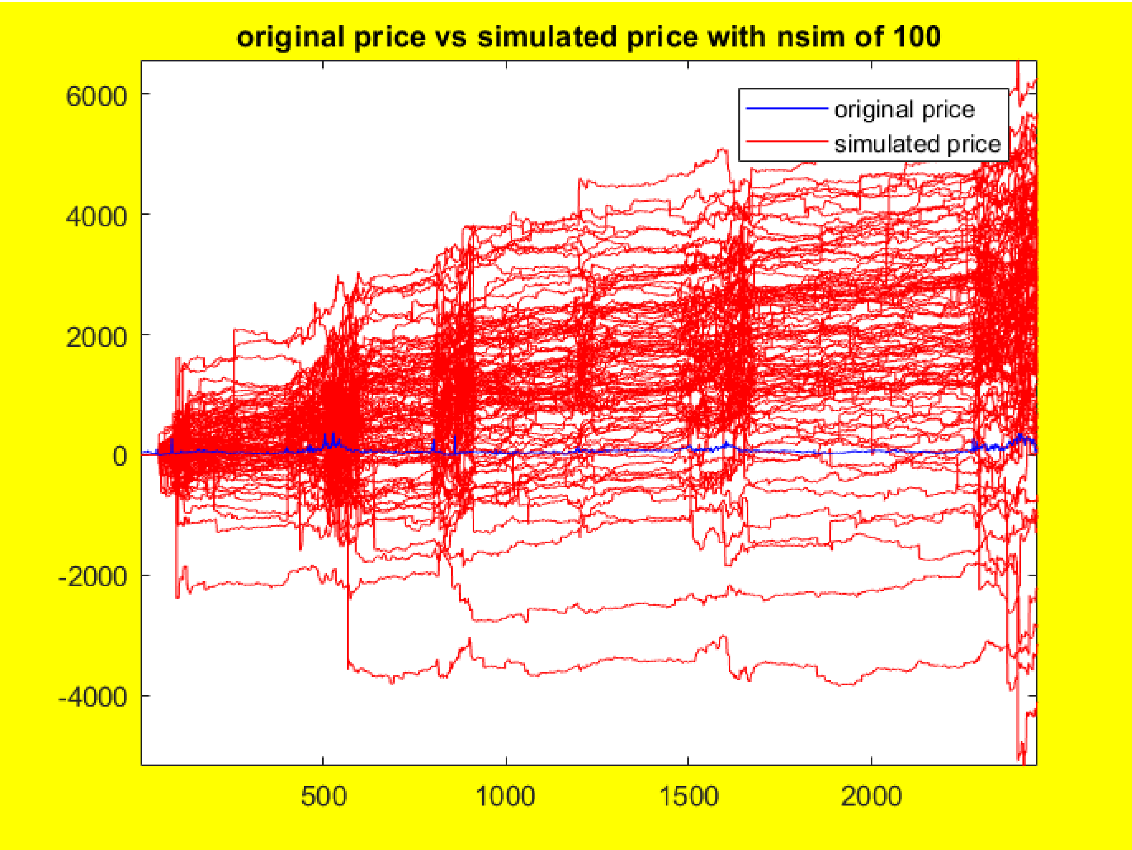

Figure 11. Simulation of the original price and simulated price without the momentum component.

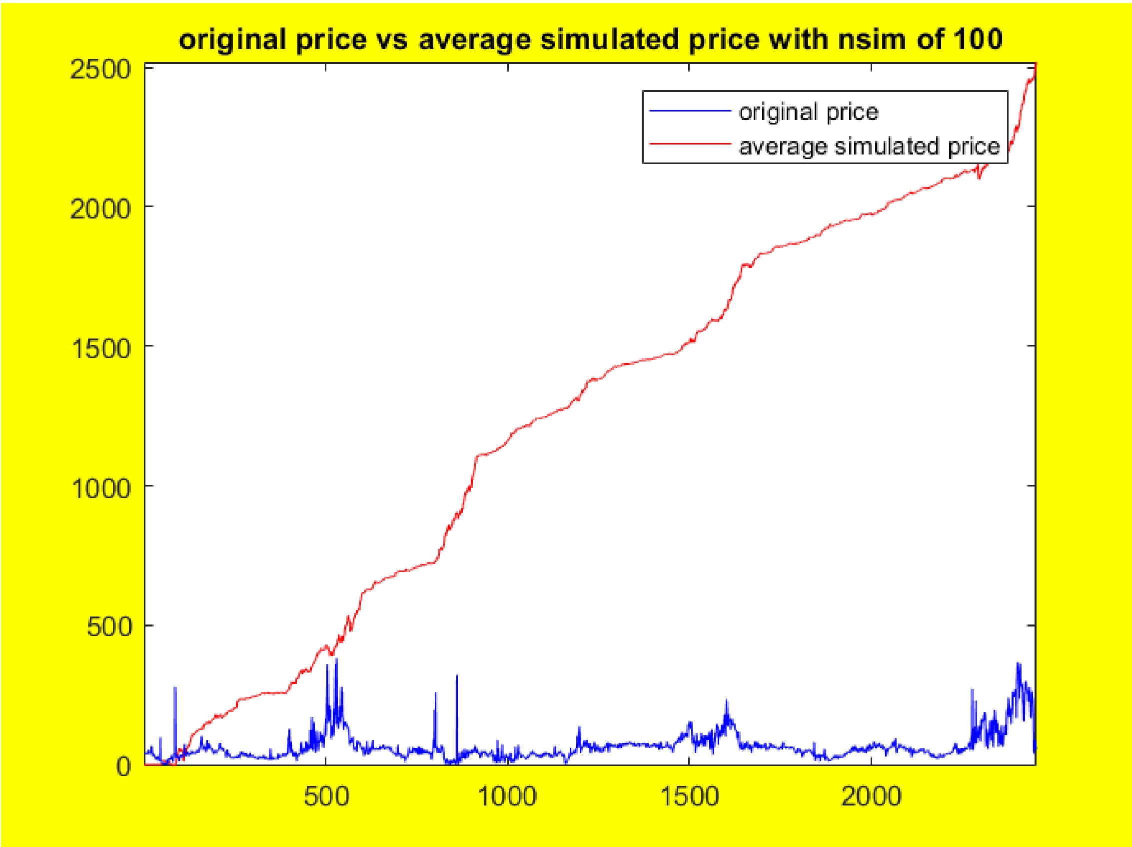

Figure 12. Simulation of the original price and avearge simulated price without the momentum component.

6 CONCLUSION

In this work we study the daily electricity spot prices of New Zealand over an eight years period, running from 2001 to 2008. It comprises prices from 11 nodes from both the South Island and the North Island of the country. These prices followed the same pattern in the plot generated and therefore, one of the prices is chosen for analysis. The prices under our analysis are from Stratford which is from the North Island.

We commenced by looking at Lévy processes which is a type of stochastic process. Definitions and properties of the process was considered which differentiate it from Brownian motion. In the course of the study, the family of generalised hyperbolic distributions was looked at where we single out some of the special cases of the distribution. The hyperbolic distribution is considered in our research. Prior to the analysis, the models for modelling the electricity spot prices are discussed with the proposed model which incorporates the Lévy process.

The analysis commenced by first viewing the plot of the daily electricity spot price of New Zealand with its histogram. Then we looked at the basic statistics like the mean, standard deviation, skewness and kurtosis of the electricity spot prices for both the South Island and North Island. In the analysis, the proposed model is segmented into three sub-classes which are the complete model with all the necessary components, model without the local interaction component and finally a model without the momentum component. Analysis was carried out on these three segments of the proposed model by comparing the original price with the simulated price and also with the average simulated price in the three cases.

In analysing the models, we looked at the basic statistics and presented the simulation results with *nsim* of 100 for each cases. It was noticed in the first two cases that the results seem pretty much the same looking at their basic statistics and the results from the simulation but the case of the third model without the momentum component is completely different, looking at the basic statistics and the generated plots from the simulation. We conclude that the removal of the local component from the model has no effect but excluding the momentum from the proposed model has a negative impact on the model.

The surprising aspect of the results in general is the spikiness from the simulated price which goes into negative whereas there are no negative spikes noticed from the original price. This is caused by the fact that the model is unable to ”learn” the non-negative feature of electricity spot prices. Not withstanding, there are still some unsteadiness noticed from the original price which was also followed by the simulated price.

REFERENCES

[1] Matylda Jabłońska. Analysis of outliers in electricity spot prices with example of New England and New Zealand markets. *Master's thesis, Lappeenranta University of Technology*, 2008.

[2] Lester Hadsell and Hany A Shawky. Electricity price volatility and the marginal cost of congestion: An empirical study of peak hours on the nyiso market, 2001—-2004. *The Energy Journal*, pages 157–179, 2006.

[3] Piotr Ptak, Matylda Jabłońska, Dominique Habimana, and Tuomo Kauranne. Reliability of ARMA and GARCH models of electricity spot market prices. In *European Symposium on Time Series Prediction, Porvoo, Finland, September 17*, volume 19, pages 21–35, 2008.

[4] Matylda Jabłońska. From fluid dynamics to human psychology. What drives financial markets towards extreme events. *Acta Universitatis Lappeenrantaensis*, 2011.

[5] www.worldatlas.com. ACCESSED 3RD APRIL 2018.

[6] Burton G Malkiel and Eugene F Fama. Efficient capital markets: A review of theory and empirical work. *The Journal of Finance*, 25(2):383–417, 1970.

[7] Daniela Morale, Vincenzo Capasso, and Karl Oelschläger. An interacting particle system modelling aggregation behavior: from individuals to populations. *Journal of Mathematical Biology*, 50(1):49–66, 2005.

[8] Matylda Jabłońska, Hasifa Nampala, and Tuomo Kauranne. The multiple-mean-reversion jump-diffusion model for nordic electricity spot prices. *The Journal of Energy Markets*, 4(2):3, 2011.

[9] Annamaria Bianchi, Vincenzo Capasso, and Daniela Morale. Estimation and prediction of a nonlinear model for price herding in. *Complex Models and Intensive Computational Methods for Estimation and Prediction*, pages 365–370, 2005.

[10] Karsten Prause et al. *The generalized hyperbolic model: Estimation, financial derivatives, and risk measures*. PhD thesis, PhD thesis, University of Freiburg, 1999.

[11] Matylda Jabłońska, Arno Mayrhofer, and James P Gleeson. Stochastic simulation of the uplift process for the irish electricity market. *Mathematics-in-Industry Case Studies*, 2, 2010.

[12] John Maynard Keynes. The general theory of employment, investment, and money. *London and New York, quoted from: https://sites. google. com/site/biblioeconomicus/KeynesJohnMaynard-TheGeneralTheoryOf EmploymentInterestAndMoney. pdf [Accessed 07 Jan 2014]*, 1936.

[13] Matylda Jabłońska and Tuomo Kauranne. Animal spirits in population spatial dynamics. *Parallel Problem Solving from Nature-PPSN XII*, pages 205–214, 2012.

[14] Iain D Couzin, Jens Krause, Nigel R Franks, and Simon A Levin. Effective leadership and decision-making in animal groups on the move. *Nature*, 433(7025):513–516, 2005.

[15] Rama Cont. *Financial modelling with jump processes*. Chapman and Hall, 2004.

[16] Peter Tankov. *Financial modelling with jump processes*, volume 2. CRC press, 2003.

[17] Bernt Øksendal. Stochastic differential equations. In *Stochastic differential equations*, pages 65–84. Springer, 2003.

[18] Yoshio Miyahara. *Option pricing in incomplete markets: Modeling based on geometric Lévy processes and minimal entropy martingale measures*, volume 3. World Scientific, 2011.

[19] Ernst Eberlein, Ulrich Keller, et al. Hyperbolic distributions in finance. *Bernoulli*, 1(3):281–299, 1995.

[20] Ernst Eberlein and Karsten Prause. The generalized hyperbolic distributions: Financial derivatives and risk measures. *Mathematical Finance -Bachelier Congress 2000, Springer, New York*, (56):245–267, 2000.

[21] Bo Martin Bibby and Michael Sørensen. A hyperbolic diffusion model for stock prices. *Finance and Stochastics*, 1(1):25–41, 1997.

[22] Ken-iti Sato. *Lévy processes and infinitely divisible distributions*. Cambridge university press, 1999.

[23] Peter Carr and Liuren Wu. Time-changed lévy processes and option pricing. *Journal of Financial economics*, 71(1):113–141, 2004.

[24] Salomon Bochner. Diffusion equation and stochastic processes. *Proceedings of the National Academy of Sciences*, 35(7):368–370, 1949.

[25] Salomon Bochner. *Harmonic analysis and the theory of probability*. Courier Corporation, 2013.

[26] Benoit Mandelbrot. New methods in statistical economics. *Journal of political economy*, 71(5):421–440, 1963.

[27] Ernst Eberlein and Ulrich Keller. Hyperbolic distributions in finance, 1995.

[28] Ahmed BenSaïda and Skander Slim. Highly flexible distributions to fit multiple frequency financial returns. *Physica A: Statistical Mechanics and its Applications*, 442:203–213, 2016.

[29] Ole E Barndorff-Nielsen and Preben Blaesild. Hyperbolic distributions and ramifications: Contributions to theory and application. In *Statistical distributions in scientific work*, pages 19–44. Springer, 1981.

[30] Preben Blæsild. Generalized hyperbolic and generalized inverse gaussian distributions. Technical report, Working Paper, University of Århus, 1999.

[31] Ernst Eberlein and Ernst August v Hammerstein. Generalized hyperbolic and inverse gaussian distributions: limiting cases and approximation of processes. In *Seminar on stochastic analysis, random fields and applications IV*, pages 221–264. Springer, 2004.

[32] Matylda Jabłońska and Tuomo Kauranne. Multi-agent stochastic simulation for the electricity spot market price. *Emergent Results of Artificial Economics*, pages 3–14, 2011.

[33] Wim Schoutens. *Lévy Processes in Finance*, volume 156. 2003.

[34] Milton Abramowitz and Irene A Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*, volume 55. Courier Corporation, 1964.

[35] David J Scott, Diethelm Würtz, Christine Dong, and Thanh Tam Tran. Moments of the generalized hyperbolic distribution. *Computational statistics*, 26(3):459–476, 2011.

[36] Karsten Prause. Modelling financial data using generalized hyperbolic distributions. *fDM Preprint*, 48, 1997.

[37] Kjersti Aas and Ingrid Hobæk Haff. The generalized hyperbolic skew student'st-distribution. *Journal of Financial Econometrics*, 4(2):275–309, 2006.

[38] Peter Carr, Hélyette Geman, Dilip B Madan, and Marc Yor. The fine structure of asset returns: An empirical investigation. *The Journal of Business*, 75(2):305–332, 2002.

Appendix 1. Characteristic Functions

(Characteristic function for n-dimensional random vector). Let $X = (X_1, \cdots X_n)$ be a random vector. The characteristic function $\varphi_X : \mathbf{R}^n \to C$ is given by

$$\begin{aligned}
\varphi_X(t_1, t_2, \cdots, t_n) &= \mathbf{E}[e^{(it^T X)}] = \mathbf{E}[e^{i\sum_{k=1}^{n} t_k X_k}],\\
&= \mathbf{E}(cos(t^T X)) + i\mathbf{E}(sin(t^T X)),\\
&= \int\limits_{\mathbf{R}^n} cos(t^T x)F(x)dx + i\int\limits_{\mathbf{R}^n} sin(t^T x)F(x)dx,
\end{aligned}$$

where i is the imaginary unit, $t \in \mathbf{R}^n$ is the argument of φ and $F(x)$ is the probability distribution.

(Characteristic function). Let $X = (X_1, \cdots X_n)$ be a random vector. The characteristic function $\varphi_X(t) : \mathbf{R} \to C$ is given by

$$\begin{aligned}
\varphi_X(t) &= \mathbf{E}(e^{itX}) = \int\limits_{\mathbf{R}} e^{itx}F(x)dx,\\
&= \mathbf{E}(cos(tX)) + i\mathbf{E}(sin(tX)),\\
&= \int\limits_{\mathbf{R}} cos(tx)F(x)dx + i\int\limits_{\mathbf{R}} sin(tx)F(x)dx,
\end{aligned}$$

where i is the imaginary unit, $t \in \mathbf{R}$ is the argument of φ and $F(x)$ is the probability distribution.

(i) if X is a discrete random variable, it is of the following form

$$\varphi_X(t) = \mathbf{E}(e^{itX}) = \sum_{x=0}^{\infty} e^{itx} f(x),$$

where $f(x)$ is a probability density function.

(ii) if X is said to be continuous random variable, the characteristic function is presented as follow

$$\varphi_X(t) = \mathbf{E}(e^{itX}) = \int\limits_{-\infty}^{\infty} e^{itx} f(x)dx,$$

where $f(x)$ is a probability density function.

Appendix 1.

The characteristic function $\varphi(u)$ of the hyp distribution is defined as $\mathbf{E}[e^{iuX}]$

$$\varphi(u) = e^{iu\mu}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{\lambda/2}\frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+iu)^2})}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}.$$

Appendix 2. Moment Generating Function

(Moment Generating Function). If X is a random variable, then its moment generating function (mgf) is given in the following forms

(i) for discrete random variable X,

$$M_X(t) = \mathbf{E}(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} f(x),$$

(ii) for continuous random variable X,

$$M_X(t) = \mathbf{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

where $f(x)$ is the probability density function.

The moment generating function $M(u)$ of the hyp distribution is defined as $\mathbf{E}[e^{uX}]$.

$$M(u) = e^{u\mu} \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + u)^2} \right)^{\lambda/2} \frac{K_\lambda(\delta\sqrt{\alpha^2 - (\beta + u)^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}, \quad |\beta + u| < \alpha.$$

(See Prause [10] for proof).

Appendix 3. Bessel Functions

Bessel Functions are one of the special functions presented by Schoutens [33]. The differential equation satisfied by the third kind takes its reference from Abramowitz & Stegun [34] which is a standard reference for Bessel functions. The first kind of Bessel functions denoted as $\mathbf{J}_{\pm\nu}(z)$, the second kind as $\mathbf{N}_{\nu}$, while for the third kind, the solutions to the differential equation in Equation (22) are $\mathbf{H}_{\nu}^{(\mathbf{1})}(z)$ and $\mathbf{H}_{\nu}^{(\mathbf{2})}(z)$.

$$z^2\frac{d^2(\omega)}{dz^2} + z\frac{d(\omega)}{dz} + (z^2 - \nu^2)\omega = 0. \tag{22}$$

The following series present the function $\mathbf{J}_{\nu}(z)$

$$\mathbf{J}_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{\left(\frac{-z^2}{4}\right)^k}{k!\Gamma(\nu+k+1)},$$

and $\mathbf{N}_{\nu}(z)$ satisfies

$$\mathbf{N}_{\nu}(z) = \frac{\mathbf{J}_{\nu}(z)\cos(\nu\pi) - \mathbf{J}_{-\nu}(z)}{\sin(\nu\pi)},$$

replacing the right hand side by its limiting value if ν is an integer or zero. Also,

$$\begin{aligned}\mathbf{H}_{\nu}^{(\mathbf{1})}(z) &= \mathbf{J}_{\nu}(z) + i\mathbf{N}_{\nu}(z),\\ \mathbf{H}_{\nu}^{(\mathbf{2})}(z) &= \mathbf{J}_{\nu}(z) + i\mathbf{N}_{\nu}(z).\end{aligned}$$

The following present some useful properties of the Bessel functions

$$\begin{aligned}\mathbf{J}_{\frac{1}{2}}(z) &= \sqrt{\frac{2}{\pi z}}\sin z,\\ \mathbf{J}_{-\frac{1}{2}}(z) &= \sqrt{\frac{2}{\pi z}}\cos z,\\ \mathbf{J}_{\frac{3}{2}}(z) &= \sqrt{\frac{2}{\pi z}}\left(\frac{\sin z}{z} - \cos z\right),\\ \mathbf{J}_{-\frac{3}{2}}(z) &= \sqrt{\frac{2}{\pi z}}\left(\frac{\cos z}{z} + \sin z\right),\\ \mathbf{J}_{\frac{\mathbf{n}+\mathbf{1}}{\mathbf{2}}}(z) &= (-1)^n\mathbf{N}_{\frac{-\mathbf{n}-\mathbf{1}}{\mathbf{2}}}(z), \qquad n = 0, 1, 2, \cdots,\\ \mathbf{J}_{\frac{\mathbf{n}+\mathbf{1}}{\mathbf{2}}}(z) &= (-1)^{n-1}\mathbf{N}_{\frac{\mathbf{n}+\mathbf{1}}{\mathbf{2}}}(z), \qquad n = 0, 1, 2, \cdots.\end{aligned}$$

Appendix 4. Modified Bessel Functions

Both the modified Bessel functions of the first kind $\mathbf{I}_{\pm\nu}(\mathbf{z})$ and the MacDonald function which is the Bessel function of the third kind $\mathbf{K}_\nu(z)$ are solutions to Equation (22) which is a differential equation. The series below expresses how the function $\mathbf{I}_\nu(\mathbf{z})$ can be presented

$$\mathbf{I}_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{-z^2}{4}\right)^k}{k!\Gamma(\nu + k + 1)},$$

and $\mathbf{K}_\nu(z)$ satisfies

$$\mathbf{K}_\nu(z) = \frac{\pi}{2} \frac{\mathbf{I}_\nu(z) - \mathbf{I}_{-\nu}(z)}{\sin(\nu\pi)},$$

replacing the right hand side by its limiting value if ν is an integer or zero. The function can also be presented in an integral form

$$\mathbf{K}_\nu(z) = \frac{1}{2} \int_0^\infty u^{\nu-1} e^{(-\frac{1}{2}z(u+u^{-1}))du}, \qquad x > 0.$$

Some useful properties of the modified Bessel function are as follows

$$\begin{aligned}
\mathbf{K}_\nu(z) &= \mathbf{K}_{-\nu}(z), \\
\mathbf{K}_{\nu+\mathbf{1}}(z) &= \frac{2\nu}{z}\mathbf{K}_\nu(z) + \mathbf{K}_{\nu-\mathbf{1}}(z), \\
\mathbf{K}_{\frac{\mathbf{1}}{\mathbf{2}}}(z) &= \sqrt{\frac{\pi}{2}} z^{-\frac{1}{2}} e^{-z}, \\
\mathbf{K}'_\nu(z) &= -\frac{\nu}{z}\mathbf{K}_\nu(z) - \mathbf{K}_{\nu-\mathbf{1}}(z).
\end{aligned}$$