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CONSTITUTIVE MODELING OF HYPERELASTIC FIBROUS MATERIAL

Master’s Thesis

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ABSTRACT

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This thesis will concentrate on the deformation of bodies from incompressible and compressible materials. The analysis is based on several approaches, the first approach is an analytical way. Here we will plan to consider the behavior of cylindrical samples from isotropic and anisotropic materials under inflation and uniaxial stretching loads. The deformation will be considered with semi-inverse method in the framework of the non-linear theory of elasticity, equilibrium equations which describe the behavior of a three-dimensional body will be derived. Further, we will consider the deformation of hyperelastic solids from finite element methods point of view. The numerical results will be received with proven and reliable algorithms, then the results will be checked with softwares. The algorithms will be done for the samples from isotropic and then it will be extended to the cases of anisotropic materials. In the last part, all named ways of the solution will be compared for the solid cylinder case.
PREFACE

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Lappeenranta, August 26, 2018

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1 INTRODUCTION

1.1 Motivation

Cardiovascular diseases are still the main reason of the disability factor in Western countries and around the world. In spite of all treatments, which have been done in recent years, illnesses of heart, tendons and arteries, as well as the whole mechanism behind, are not clearly understood [1]. It should be noted that the task of constructing a general mathematical model of the cardiovascular system has not been solved at the moment as well as computational methods of its investigation. First of all, this is due to the extreme complexity of the biological system under consideration, the functioning of which depends on a huge number of factors, practically from every element of the living organism, for instance, location, age and even physiological conditions [2]. These dependencies still remain largely unformalized. Secondly, the real objects are difficult to describe in the terms of simple geometric forms. Furthermore, under tensile loading, arteries show a non-linear response [3] due to the gradual straightening of fibers from a crimped configuration assumed at rest. From here, the significance of including of the non-linear theory of elasticity in consideration is obvious. In addition, the tissues have viscoelastic properties [4], like creep and stress-relaxation. So, including in models the parameters describing these features can give more possibilities for the correct behavior descriptions. However, the complexity of models after that will significantly increase. The precise understanding of the biomechanical behavior is still necessary. To reveal that mechanism numerous experiments have been done [5, 6] and continue until now. Based on that data a large number of constitutive models of the biological tissue were constructed to capture as much as possible the features observed in experimental tests [7]. These models can give a useful information through the numerical and analytical simulation. However, the key problem of mathematical modeling of the cardiovascular system and obtaining reliable numerical values are not solved [2]. The possible solution of this problem, receiving of detail knowledge about the response of biological tissues to the pressure and other types of mechanicals loadings can be very useful and be applied to the wide range of different types of problems. As an example, one of the various applications is the optimal design of prostheses devices and better choice of vascular transplants, also to discover the mechanism of arterial failure, which damaging processes are not clearly understood yet [8], as well as for description or prediction of some causes of diseases like stenos or aneurysms. Moreover, the results which can be obtained may be useful for body examination.
All of that facts became a good motivation for that work. In this thesis, it will be studied the behavior of hyperelastic solids from continuum mechanics and finite element methods point of view. The most part of the work will be devoted to the description with finite element methods. From that point of view, it will be considered deformations of simple bodies or structures such as beams with rectangular and circular cross-sections based on different types of material laws from simple ones Blatz & Ko and Neo-Hookean to difficult ones such as two and five constants Mooney-Rivlin model. The main part is receiving the correct solution of sample deformation for the special exponential model offered by Gasser, Ogden and Holzapfel (GOH model) in [9] with analytically and numerically. Nowadays, it is one of the models which includes incompressibility, anisotropy and a good description of non-linear response, which are assumed as the main features of living tissues. This material model is often used for the descriptions of arterial walls, tendons, and other biological tissues behaviors.

Similar works have been done before. The use of the finite element method (FEM) and analytical approaches can be found in references in [3, 10], for example. The aim of this study is to develop a finite element based on the absolute nodal coordinate formulation (ANCF). The ANCF is nonlinear finite element approach which is proposed for dynamic analysis in multibody applications by Shabana [11]. It is assumed that ANCF elements based on fibrous material law can be used promisingly for analysis of human motions in future. The most likely candidates for the developed ANCF-based element which can be subjected to large deformations are tendons. In this study, parameters for that are taken from the description of another type of biological object, arteries. The verification of the developed element under uniaxial loading is accomplished via the static analysis based on analytical and numerical solutions.

As it was already mentioned a number of different type of materials will be considered. Firstly, it will be compressible material because the implementation into finite element computation does not need to introduce some additional hypothesis such in case of incompressible materials. In the last case, to describe incompressibility, the so-called additive split of the strain energy is used [13]. The deformation of the isotropic incompressible structure will be considered with Neo-Hookean material and two Mooney-Rivlin models represented two and five constants ones. The final point will be the description of the body response from anisotropic incompressible GOH model. Each step of different material implementation from the beginning to the end will be checked using commercial software ANSYS and ABAQUS.
For some types of named materials, the analytic solutions will be also derived in cases of hollow and solid tubes. To do that one method, namely the semi-inverse method, will be introduced, which is commonly used to describe the behavior of three-dimensional bodies of simple forms like a tube, a cylinder and a sphere. Obtained three-dimensional solutions for radial inflation of cylinders will be checked with known two-dimensional one [9] in solid cylinder cases it will be verified with finite element methods. In the first case, it will be taken very thin object where the coincidence of physical properties in the two-dimensional and three-dimensional theory takes place. From that analytical-numerical experiences, it will be possible to build the diagrams of loading. Furthermore, some additional features can be also derived directly from there, for example, the existence of dropping on the loading diagram reveals the buckling effect of constructions.

During the work, it will be considered several specific questions.

Is it possible to develop ANCF element for beams with rectangular and circular cross section under uniaxial loading? Is it possible to find analytical solution for cylinders based on the incompressible anisotropic material under inflation and uniaxial loading? Does analytical and ANCF solution agree with solutions found by commercial finite element software?

1.2 Background

Non-linear elasticity theory has been developed from the beginning of 40’s of the 20th century. This was primarily motivated by the needs of science and industry. Firstly, for the science, it was necessary to find explanations for some phenomena which were obtained during the experiments, but which could not be explained in the framework of the linear theory. One of the examples of such phenomena is Poynting’s effect. Change of the cylinder length under the torsion. This effect was experimentally detected and described by Poynting at the beginning of the previous century [14]. It was successfully done in the framework of the non-linear theory. Secondly, there were results from many experiments describing the physical behavior of materials including some artificial, which had been already actively used at that time. This firstly referred to the so-called rubber-like materials. They are polymers, rubber, synthetic rubber etc. The behavior of these materials could not be longer described by classical linear theory [15]. The studying of the behavior of that materials was challenging tasks because during the work scientists encountered many difficulties. For instance, in some cases, during the determination of the material properties, they still forced to use standard mechanical experiments: tension, compression, torsion,
bending. And it is important to do several types of experiments. The matter of fact that one type, for instance, uniaxial stretching is not enough as it was shown [3] to reveal all material features. Moreover, during the process of upper described types of deformation of the samples from rubber-like materials non-linear properties appear and these properties lead to difficulties in processes of calculations even in case of simple-geometry bodies such as for example, a cylinder or a cube. Thus, the problems, which led to the development of the non-linear theory, were connected with the description of materials behavior under the deformations, which sometimes achieve 100% and in some cases even exceed several hundreds of percentages from the initial size of samples. Also, it is important to note that in that time possibilities for numerical calculation were not highly developed and there was a need in models which provided analytical solutions. To deal with those difficulties some models were created which are so-called incompressible model materials. That means that the volume of the object from these materials does not change under any type of load. Nowadays it is known a lot of incompressible materials such as Neo-Hookean, Mooney-Rivlin, Knowles, Gent materials and van der Waals. They helped to overcome some difficulties, for instance, to receive the solution in analytical forms, gave the possibility to describe not only non-linear stress-strain behavior as well as simulate very large deformations, also, that materials can describe the strain-hardening effect when the samples have high deformability in the low-stress range and decreasing under increasing load. Despite the benefits the exactly incompressible materials do not really exist in the real world nevertheless these models can be considered as the approximate solution and preliminary step forward of modeling for some tasks.

However, it was a great success which had several significant consequences. Firstly, numerous interdisciplinary publications in other important areas of physical science appeared, which include thermomechanics, electromechanics etc. Secondly, non-linear elasticity theory helped to expand new theories, for example, mechanics of micropolar continuum or Cosserat’s continuum [16]. Thirdly, it prepared the appearance of another relatively new section of mechanics, biomechanics. The next period of active development of the non-linear theory of elasticity is associated with this science. The matter of fact that arterial walls and other biological tissues can be considered like mostly incompressible material. And the whole legacy which was received at the previous stage was very useful. Here non-linear elasticity theory was used to model the behavior of soft biological tissues, vascular walls, and cell membranes. The understanding of the behavior and response of biological tissues to the pressure and other types of mechanical loadings still plays an important role and can be applied to the wide range of problems. As an example, one of the applications is the optimal design of prostheses devices and better choice of vascular transplants, also to understand better the mechanism of arterial failure,
as well as for description or prediction of some diseases such as stenos or aneurysms. Moreover, the results may be useful for body examination. Here the growth of the significance of the non-linear theory of elasticity is associated with the development of such areas as minimally or non-invasive methods of surgery, the solution of the problems of elastography. For example, in the some of the applications minimally invasive surgery deformations can reach high values, so the non-linearity must be taken into account. This is also connected to another fact that different tumors and pathologies exhibit different non-linear properties.

However, there was one limitation. In the previous decades, research efforts were devoted to the understanding of behaviors of isotropic elastic materials. In some cases, when the fibers are distributed randomly it is possible to take into account that material is isotropic. But the obtained results from the experiments with veins, aortas and arteries of humans, rats etc. showed the fact that arterial walls are structurally inhomogeneous, anisotropic, multilayer and viscoelastic [7, 17]. They are also initially stressed [18]. To illustrate in detail some of the above-written features let’s consider, as an example, the human aorta. It consists of three layers: the intima, media, and adventitia, the detail view can be seen in Figure 1.

![Figure 1. The model of artery [19]](image-url)
Furthermore, extensive experiment researches revealed the fact that anisotropy of biological tissues is caused with the fibers which have structural orientations in the several layers. The number of family fibers ranges in different parts of bodies as well as angels between them. That is reason why current researches mostly pay attention on the developing material models with anisotropic features. For example, the simplest form of anisotropy, i.e. transversely isotropic, has only one principle direction that means that material response is isotropic to the rotation around one axis and this is actually fiber direction. However, today it is a known model which includes even four specific directions of fiber reinforcement. Despite that, it is important to note that the constitutive theory of the behavior of strongly anisotropic material at finite strains are far from completion. Now to deal with anisotropy there were created several new models which are based on Fung, Knowels, Takamizava–Hayashi, Horgan–Saccomandi, Arruda, Boyce etc. ideas. It is hard to choose the most appropriate model among them because they have some advantages and disadvantages. But for today the best known and commonly used the model which was proposed initially by Fung [20]. This model represents exponential strain energy function. The model was modified by Holzaphel for the behavior of constructions with perfectly aligned fibers. Then to take into account the distribution of fiber orientations it was modified by Gasser. Many authors in their researches often use also the decomposition of the deformation into two parts. They consider the combination of several models, for instance, Neo-Hookean and Fung-type strain-energy functions, the first one describes the isotropic deformation and the second one reveals anisotropic response [21].

As it was mentioned besides of anisotropy arteries have the feature to change their physical behavior with age. So, the response of them depends on time. Which leads to including viscoelasticity into the model. And the first three-dimensional model which included anisotropy and viscoelasticity was proposed only in the end of 80’s of the last century by Simo [22]. And now it is known some forms of models, which are based on the phenomenological approach, describing viscoelasticity including until 10 invariants.

Another property of biological tissues is the existence of initial stress conditions [4]. This thing is well-known and was considered in the number of works devoted to the biomechanics. Initial stress can be often a cause of premature failure of critical components of any construction because it encourages crack growth. There are several ways to take into account the initial stress. The most common way to do is to include into the model some imperfections. There are ways to deal with, for instance, including dislocations or disclination parameter. Theory of dislocations and their influence on solids behavior can be found in [23] and [24]. To show what is a disclination and how to regulate the initial stress-strain state with it, let’s consider a cylinder and cut it along its generator line,
remove or add some part from the same material and then to glue the edge of the cylinder. This way is often used to include initial stresses and can be found, for example, in works [6, 25].

For the last two decades, there was a new experiencing a growing research activity in biomechanics. This was after the analytical theory of deformation of inhomogeneous, anisotropic, multilayer and viscoelastic model on the basis of potential energy had been built. It turned out that it was possible to describe the behavior of such kind of solids using finite element method. Separately, it is important to say several words about the role which were played by finite element methods in the modeling of biological tissues. Today it is the most common way of solution of different tasks in biomechanics beginning from the analysis and modeling of the behavior of bones, skin and other samples of very soft human tissues such as brain, liver, and kidney. Using this technique, it is possible, for instance, to estimate the influence of boundary conditions with great accuracy impact of which on the experiments and as a result on the ending critical of failure value can be huge. All aspects of modern biomechanics are solved in that way now. Even as it was shown in one of the works of Holzapfel and Gasser mentioned above, it is possible to modulate residual stress behavior in the framework of FEM. From the most recent works, it is interesting to mention including in modeling of some small damage phenomena – local scratches. The including the damage mechanism can explain some biomechanical phenomena among them degradation of material or loss of stiffness. Also, it extended the areas of applications, for instance, at the simulation of the balloon angioplasty operation or the research of biomechanical features of the saphenous vein. The last study has crucial meaning because the one is often used as coronary artery bypass grafts and mismatch in the biomechanical behavior between a coronary artery and a graft will reduce graft patency and acceleration of disease development in the graft. Thus, in the past few decades, the finite element modeling has been developed as an effective tool for modeling and simulation of the biomedical engineering system [26]. The more detail analysis of modeling application in FEM framework is given in articles [10, 27, 28, 29, 30].

To sum up, for the last 80 years, the non-linear theory of elasticity has come a long way from solving and explaining particular problems that the linear theory of elasticity cannot deal with, to an independent science that has found wide practical application even became a starting point for other sciences and helped increase the understanding in some interdisciplinary studies. The accumulation of extensive experience allowed to apply the acquired knowledge to human body researches. Using the non-linear theory in biomechanics opened up the wide opportunities for improving diagnosis, transplantation, and prediction of diseases. An additional incentive for the development in this area of
biomechanics is the application of finite element methods. Which allows to modulate experiments and check some critical points and their influence, boundary conditions or some imperfections in the constructions.

1.3 Objectives and Restrictions

The objective of this thesis is to construct the models which can describe isotropic and anisotropic deformations. For that aim, we will use analytical and finite element methods approaches. From the analytical point of view, we will mostly concentrate on the non-linear behavior of cylindrical form bodies under large inflation and elongation loadings. In other cases, we concentrate our attention on deformation of bricks and also cylinders from hyperelastic anisotropic materials using finite element approach. In this case, we intend to test the possibility to describe the hyperelastic response using a special finite element based on the absolute nodal coordinate formulation.

All these steps need to describe in the future the deformation of real biological tissues. It is a challenging task and requires a lot of preparation which we will do in this work. To simplify the work we will consider several restrictions. Firstly, viscoelasticity will not be studied here. We restrict our attention only on studying of elastic behavior of solids, however, non-linearity will be taken into account. Problems with incompressibility and in one case compressibility will be considered. Secondly, pre-stressed or initial stress behavior will be excluded from the consideration. The third restriction is we will consider the body with simple shapes like rectangular bricks or cylinders. This is due to the fact that in this work we only test the new ways of solving laborious tasks. That is why here we use such simple geometric figures. The cylinder is considered as the approximation of geometry of arteries and tendons, however, the real form is more complex. But this form is commonly used for that aim [31, 32]. However, it was mentioned biological tissues are multilayer construction, here we do not concentrate our attention on it, which can be considered as the next restriction to our work.

This job will consist of three parts. The first part will be the implementation of a finite element approach in a framework of the absolute nodal coordinate formulation to the brick deformation. Here we will use this geometric form such as it has some advantages, for example, it easy to generate mesh for beam-like structures and, as a result, check the convergence of the methods.
The second part of the work will be devoted to the derivation of the analytical solution of cylindrical anisotropic solid under the stretching and inflation loads.

And the next part will be devoted to a comparison of samples deformation from several materials but described with different methods such as analytical and finite element one. Furthermore, in the last case, it will be used commercial finite element software and ANCF approach.
2 FINITE ELEMENT METHODS

Firstly, some words about the development of FEM. The finite element technique appeared as one of the solution methods of various problems in Mechanics. The first known descriptions of FEM is given in [33], however, before that, there were already some attempts to consider bodies consisting of some small objects, for instance, bar assembly [34]. The modern name of this approach was received only in 1960 after the article [35]. More detail information about the early history of the development of finite element methods is possible to find in [36]. At present, it is universally recognized as a general way of solving a wide range of problems in various fields such as Mechanics, Fluid Dynamics etc. The rapid growth of popularity and widespread using of FEM, its development as the leading method for the solution of physical problems were facilitated by a number of advantages of finite element analysis. For example, the objects under study can have different physical nature, it might be either solid body or liquid, gases etc. Furthermore, the finite element can have the different shape and different size, that facilitates work within contact zones. It is possible to investigate homogeneous and inhomogeneous, isotropic and anisotropic objects with linear and non-linear properties and to consider any type of boundary conditions. The advantages of this method become more obvious if we compare it with limitations imposed on the semi-inverse method described in subsection 3.2.1.

Analysis with the finite element method consists in an approximation of continuous medium with infinitely large numbers of degrees of freedom by a set of elements that has finite numbers of degrees. Then the relationships between these elements are established. The behavior in the inner region of elements is defined with so-called form functions. They determine the movement in the inner region of the element with nodes movement. Nodes are the points where the finite elements are connected. Unknowns in finite element models are possible and independent displacements of nodes of finite element model. Thus, the FEM is designed as a system of nodes. In general, the finite element model is the same as a basic system for the displacement method. Sometimes, especially in the place where stress concentration appears, to achieve a good quality of results there is a need to increase the number of nodes or, which is the same, to decrease the size of the elements in that places. In the large constructions, finite element models can have millions of nodes with a great number degrees of freedom. So, the realization is possible only through computer calculation. To apply the FEM in practice it is necessary to understand not only the theory of mechanics because the application of the finite element method is often based on the variational principles, but also have skills in programming. To ensure the convenience of programming the ratios between unknown and known variables are
written in a compact matrix or tensor form. This way of description of physical problems is fully mathematically justified and is implemented in software products such as COSMOS, MSC.NASTRAN, ABAQUS, and ANSYS which are constantly being improved along with programming tools. So, the recognition of FEM is explained by the simplicity of its mathematical form and physical interpretation.

Although some materials and models under consideration are implemented in such software systems as ABAQUS and ANSYS we want to offer our own possible solution which will be based on the absolute nodal coordinate formulation (ANCF) which was developed by Shabana in [11] for the large deformations in multi-body dynamics problems. There are some reasons to pay a lot of attention to this element. It is dictated by the fact that it has some advantages [29], if it compares with other elements. Especially, when we deal with flexible bodies.

2.1 Absolute nodal coordinate formulation

The idea behind of the absolute nodal coordinate formulation is using for the nodes global position vector and gradient vector to describe the nodal position and orientations, respectively. Which are defined with respect to a fixed global references coordinate system. It leads to some benefits which should be mentioned. The ANCF element describes the deformation using the node coordinates without bringing in additional degrees of freedom such as the rotation degrees. Instead of the rotation degrees of freedom, the slopes of coordinates are used. For instance, for the plane elements, it allows to avoid the proposals about rotation in element and allows to calculate the complex shape of the body after deformation using only a small number of elements [37]. In addition, absolute nodal coordinate formulation automatically describes non-linear effects [38]. This idea relaxes some assumptions which are used in Euler-Bernouli, Timoshenko, Reissner and Mindlin theories [37, 39]. That finite element leads to saving mass matrix as a constant. In a case of three-dimensional implementation, it will be proved below. As a result, it leads to exclusion of Coriolis inertia and centrifugal forces from the equation of motion that could be a big calculation advantage [29, 37, 40].

But it also causes some troubles like the slow convergence of solutions, locking problem [41]. Furthermore, if the element has some initial distortions or curvature it should be modeled with care to exclude initial strain which can have the influence on the results. As a result the accuracy of numerical outputs can reduce [42]. But there are some possible solutions to deal with that problems. To avoid influence curvature in the element it
is possible to use additional components like higher order derivatives [43] or using local coordinate system [44].

In general, it is distinguished several types of ANCF elements which are based on the number of gradient vectors per node:

1. **Fully parameterized.** In that case ANCF element has for each node one position vector and three position gradient vectors.

2. **Gradient deficient.** Here there is no full set of gradient vectors, in 3D cases usually 2 and 1 vector in 2D cases, that enough to calculate the volume and to use a continuum mechanics approach.

3. **Higher-order coordinates.** Sometimes it is allocated to another group. Where nodal coordinates are used position vector higher-order derivatives [40].

About last one it should be said that higher-order coordinates is only used for continuum mechanics based elements, because higher-order terms in beam theories causes ill-conditioned stiffness matrix.

![Diagram](image.png)

**Figure 2.** difference between fully parameterized and gradient deficient elements for rectangular cross-section in 2D and 3D theories [29]
In this study, we will use three nodes gradient deficient ANSF element [45] to describe the beam deformation. The whole implementation with the number of functions has done in Matlab and was based on functions and ideas of Marko Matikainen which were written for the beam deformation from St. Venant – Kirchhoff isotropic hyperelastic model in a number above mentioned works, for example [45] etc.

Here we will give some brief explanation of mechanics behind. Let, firstly, define vector $\mathbf{p} = \mathbf{p}(x)$ position of some particle $x$ in current configuration. From here and further we will assume that the all processes are performed in Cartesian systems. The vector $\mathbf{p}_0$ defines position of the same particle in referent configuration.

![Image](image.png)

**Figure 3.** ANCF three node element with vectors $\mathbf{p}$ and $\mathbf{p}_0$ of particle $x$ in current and referent configurations, respectively. Denote through $I, II, III$ the nodes. Here we follow the [45].

The position of particle $x$ with vector $\mathbf{p}$ in current configuration can be described

$$\mathbf{p}(x) = \mathbf{S}(x) \mathbf{e},$$

where $\mathbf{S}$ is a shape function matrix and $\mathbf{e}$ is nodal coordinate vector which contains the positions and derivatives of its positions respect to some of coordinates, where, for instance, for the node $i$ it can be written as

$$e^i = \left[ p^i_x p^i_y p^i_z \right]^T,$$
in (2) the following notation for the partial derivatives is used

\[
p_{i,y} = \begin{bmatrix}
  p_{i,1,y} \\
  p_{i,2,y} \\
  p_{i,3,y}
\end{bmatrix} = \frac{\partial p_i}{\partial y}.
\]  

(3)

The nodal coordinate vector \( e \) have displacement and rotational coordinates [43]. To derive shape function matrix \( S \) we will use some basis functions, such it was done in [45, 46] to approximate the location of vector \( p \).

The set of basis function for interpolation of vector \( p \) is

\[
[1, x, y, z, xy, xz, x^2, y^2, z^2].
\]  

(4)

But such we assume that elements are isoparametic we will introduce for the elements local coordinate system \( \{\xi, \eta, \zeta\} \). Furthermore, the range of local coordinates varies \([-1, 1]\).

It is done to deal with standard integration procedures, e.g. Gaussian quadrature integration. The substitution will have the next form

\[
\xi = \frac{2x}{l_x}, \eta = \frac{2y}{l_y}, \zeta = \frac{2z}{l_z},
\]

where the expressions \( l_x, l_y \) and \( l_z \) are characteristic dimensions of a rectangular element.

![Figure 4. Three-node beam element in local and physical coordinates [45].](image-url)
And the expression (1) can be rewrite in terms

\[ p(x) = S(\xi) e, \]  

(5)

Now to describe some features of ANCF element we give the derivation of motion equation. The motion equation can be derived using Lagrangian of the system through Hamilton’s principle

\[ \delta \int_{t_1}^{t_2} (W_1 + W_2 + W_3) \, dt = 0, \]  

(6)

where \( W_1 \) is the work which is done with external forces, \( W_2 \) is internal potential energy, \( W_3 \) is kinetic energy. Variation of the last term in nodal coordinates can be written

\[ \delta \int_{t_1}^{t_2} W_3 \, dt = - \int_{t_1}^{t_2} \dot{e}^T \int_{V} \rho S(\xi)^T S(\xi) \, dV \delta dt. \]  

(7)

\( \rho \) is the mass density, \( V \) is the volume of element in the referent configuration. The expression under the second sign of integral is usually called mass matrix and defined in the form

\[ M = \int_{V} \rho S(\xi)^T S(\xi) \, dV. \]  

(8)

Now as it was said above in listing advantages of ANCF element and it can be seen from (8) that mass matrix is constant, such as it is not function of nodal coordinate. Also we can consider the variation of elastic potential energy \( W_2 \) in the nodal coordinate might be described in the next form

\[ \int_{V} S : \delta E dV = \int_{V} S : \frac{\partial E}{\partial e} dV \delta e. \]  

(9)

In (9) \( S \) is the second Piola–Kirchhoff stress tensor and \( E \) is the Green strain tensor which has the form

\[ E = \frac{1}{2} (C - I). \]  

(10)

\( I \) is identity tensor and \( C \) is Cauchy-Green tensor

\[ C = F^T \cdot F. \]  

(11)

Here we also give the derivation of gradient tensor which will be described for that aim
like proportion between vector $p$ in initial and actual configuration

$$\mathbf{F} = \frac{\partial \mathbf{p}}{\partial \mathbf{p}_0} = \frac{\partial \mathbf{p}}{\partial x} \left( \frac{\partial \mathbf{p}_0}{\partial x} \right)^{-1}. \quad (12)$$

It is also possible seen from (10) another feature of ANCF element, namely geometric non-linearity. The matter of fact that $\mathbf{E}$ captures non-linearity features such as the $\mathbf{F}$ includes in the second power. The vector of inner forces, in our case just elastic ones, derived directly from the expression (9)

$$F^{\text{elastic}} = \int_{V} \mathbf{S} : \frac{\partial \mathbf{E}}{\partial \mathbf{e}} dV. \quad (13)$$

The expressions for $\mathbf{S}$ will be given in each case separately for different types of materials incompressible, compressible and anisotropic below. Note that $: \mathbf{S}$ indicates the double inner product between the tensors.

The virtual work of the external force is described in the next form

$$\delta W_1 = \int_{V} \mathbf{b}^T \delta \mathbf{p} dV = \int_{V} \mathbf{b}^T \mathbf{S}_m dV \delta \mathbf{e}, \quad (14)$$

from here we can derived the expression for the total vector of external force

$$F^{\text{ext}} = \int_{V} \mathbf{b}^T \mathbf{S}_m dV. \quad (15)$$

The vector $\mathbf{b}$ is the vector of body forces. Take equations (7), (9), (13), (14), (15) and substituting these expressions into (6) we can derive based on Hamilton’s principle equations of motion in discrete form. More detail description and presentation of mechanism of ANCF element it is possible to find in references [43, 45, 46, 47, 48].

2.2 Materials models implementation

Here we describe all materials which will be used in our investigation. All materials will be divided into two groups: compressible and incompressible.
2.2.1 Compressible material

Let’s firstly consider compressible one because of simplicity of implementation that type of material. The deformation formulation of solid compressible material in the terms of the second Piola – Kirchhoff stress tensor is not a difficult task because there is no need to take into account some additional function. As a result, the compressible material derivation of the second Piola – Kirchhoff stress tensor just follow the formula

$$ S = 2 \frac{\partial \Psi}{\partial C}. $$

$\Psi$ is potential density function per unit volume.

2.2.2 Incompressible material

Here, we define some relations which will help us to describe the deformation of incompressible materials. Firstly, some words about computational modeling of incompressible materials. The modeling of incompressible or nearly incompressible materials has a quite long history starting from the 70’s with the work of Ogden [49]. We consider this type separately from 2.2.1 because of some problems arise during the computational work with this type of materials. For example, the well-known locking problem in incompressible solid [50] and there are several ways to deal with it. The most promising ways are to change the equations adding the new pressure parameter. One way, for example, is adding so-called Herrmann pressure which is used in mathematical literature as treated method [51]. But in engineering, the most common way is to use the function of hydrostatic pressure, due to the incompressibility is given through the so-called penalty function which describes the hydrostatic pressure or fluid phase [30]. This function will be defined through, for example, ANSYS or ABAQUS predefined functions.

So, let’s consider a continuum body $\Omega_0$ in referent configuration and take some point such as $X \in \Omega_0$ which will come after deformation $x$ in actual configuration with motion $\chi$. Now we describe some geometric relations. $F$ is deformation gradient $\frac{\partial x}{\partial X}$, positive defined right Cauchy-Green defined by expression (11). In addition, define volume ratio, i.e. the Jacobian $J$

$$ J = \det(F) > 0. \quad (16) $$

The expression (16) has the connection with (40) but such we use finite element approximation the rule of incompressibility is not very strict and can sometimes be violated, i.e.
J ≠ 1. So, the material is considered as not incompressible, but slightly compressible. And below we will write how the compressibility of the material is regulated in this case. For the later use, we do the multiplicative decomposition of the deformation gradient into dilational (volumetric) and distortion (isochoric) parts.

\[ F = J^{\frac{1}{3}} \bar{F}, \]  

(17)

where \( \bar{F} \) is the distortional part of the deformation and follow expression for that is true

\[ \det(\bar{F}) = 1. \]

The left and right Cauchy–Green tensors in that case are given by

\[ \mathbb{C} = F^T \cdot F, \quad \mathbb{B} = F \cdot F^T. \]

Now we consider the decomposition of the energy function. In general, the energy density function of the elastic continuum, for instance, for isotropic material can be expressed through three invariants of the Cauchy–Green deformation tensor and invariant of the structural tensor, the last one is true when we deal with the anisotropic material. Its explanation will be given in 3.2.1. \( \Psi = \Psi(I_1, I_2, I_3, I_4) \) (39), but such we deal with slightly compressible material it is not possible to drop out from our energy function dependency of \( I_3 \) invariant. In that case, it is possible to employ a penalty method and modify energy function [27] to separate the dependence from the third invariant. It will have the following form

\[ \Psi = \Psi(T_1, T_2, T_3) + \Psi_{vol}(J), \]

(18)

where \( T_1 \) and \( T_2 \) are invariants of \( \mathbb{C} \). Usually the form of \( \Psi_{vol}(J) \) are defined from experiments, but we will take from ANSYS and ABAQUS predefined functions. Then we describe deformation in terms of the second Piola – Kirchhoff stress tensor. We will follow the way given in [28].

\[ S = pJC^{-1} + 2J^{-\frac{3}{2}} \left[ \frac{\partial \Psi}{\partial \mathbb{C}} - \frac{1}{3} \left( \frac{\partial \Psi}{\partial \mathbb{C}} : \mathbb{C} \right) \mathbb{C}^{-1} \right], \]

(19)

where following the chain rule

\[ \frac{\partial \Psi}{\partial \mathbb{C}} = \sum_k \frac{\partial \Psi}{\partial T_k} \frac{\partial T_k}{\partial \mathbb{C}} \]

(20)
and the expression for $p$ has the form

$$p = \frac{\partial \Psi_{vol}(J)}{\partial J}. \quad (21)$$

$$\frac{\partial J}{\partial F} = J F^{-1}. \quad (22)$$

we will also show here some expressions which is possible to be found in [52].

$$\frac{\partial T_1}{\partial C} = I, \quad \frac{\partial T_2}{\partial C} = T_1 I - C, \quad \frac{\partial T_4}{\partial C} = A,$$

2.3 Materials under consideration

2.3.1 Blatz & Co material

Here we consider the most famous model of compressible rubber material. It is Blatz & Ko model, three-constant function of the specific potential energy is given by the relation [53]:

$$\Psi = \frac{1}{2} \mu \left(1 - \beta \right) \left[ I_2 I_3^{-1} + \frac{1}{\alpha} (I_3^\alpha - 1) - 3 \right] + \frac{1}{2} \mu \beta \left[ I_1 + \frac{1}{\alpha} (I_3^{-\alpha} - 1) - 3 \right]. \quad (23)$$

Where $I_1, I_2$ and $I_3$ invariants defined by (39). For the small deformations, the parameter $\mu$ has the meaning of a shear modulus, the parameter $\alpha$ is related to the Poisson’s ratio by the expression $\alpha = \frac{\nu}{1 - 2\nu}$. The material parameter $\beta \in [0, 1]$ characterizes the non-linearity. It affects the deformability of the material for extremely large deformations.

We will consider the simplest version of material, when $\alpha = \frac{1}{2}, \beta = 0$, such as only that formulation is implemented in ANSYS for the checking our result received with absolute nodal coordinate function elements. Then the expression (23) is reduced to the following form:

$$\Psi = \frac{1}{2} \mu \left[ I_2 I_3^{-1} + 2\sqrt{I_3} - 5 \right]. \quad (24)$$

So, in our case we will have the next form of the second Piola – Kirchhoff tensor

$$S = \mu \left[ I_3 I_1 I - C + \left( -\frac{I_2}{I_3^2} + \frac{1}{\sqrt{I_3}} \right) I_3 C^{-1} \right], \quad (25)$$
2.3.2 Neo-Hookean material

Now we are going to deal with the behaviour of incompressible materials. Let’s consider, firstly, Neo-Hookean material. It is the simplest incompressible isotropic one. There are some advice of usage of this type of material, for instance, it is recommended to apply it when the deformation does not exceed 30%. The potential energy function has the form

$$\Psi = \frac{\mu}{2} (T_1 - 3) + \frac{1}{d} (J - 1)^2,$$

where $\frac{1}{d} (J - 1)^2$ is penalty function which can be found in ANSYS or ABAQUS manuals. From (26) and (19) if we follow the way given in [28] the second Piola – Kirchhoff stress tensor has form

$$S = \frac{2}{d} (J - 1) J C^{-1} + \mu J^{-\frac{3}{2}} \left[ I - \frac{1}{3} \text{tr}(C) C^{-1} \right].$$

The parameter $d$ should be chosen small enough to describe the behavior of the incompressible solid. The idea is to give the second part of the right side of equation (27) big values in violation cases of the incompressibility conditions (40). In perfect conditions, $d$ should be equal to zero, but in this case, this part has infinite value, and obviously, that is not allowed for machine calculation. To avoid that, for instance, in ANSYS the meaning of $d = 0$ is rewritten in the possible smallest meaning. The same procedure we will implement in ANCF element functions, but it is important to notice that even quite small value does not guarantee that the obtained results will be correct. To illustrate that fact for the ANCF and ANSYS elements we will take several small values of compressible parameter $d$, where the difference in results will be quite noticeable. From another side, the very small $d$ meaning also affects on convergence of the solution.

2.3.3 Mooney-Rivlin material

In this subsection we consider more difficult isotropic material so-called Mooney-Rivlin material, which is also related to the isotropic material. Here we introduce two types of Mooney-Rivlin potential energy functions. In contrast to the previous case, these materials can be used when the deformation exceeds 30 percent of initial samples. The first one has the form

$$\Psi = c_{10} (T_1 - 3) + c_{01} (T_2 - 3) + \frac{1}{d} (J - 1)^2.$$
From here follow the rule (19) we received $S$

$$S = \frac{2}{d} (J - 1) J C^{-1} + 2 J^{-\frac{3}{2}} \left[ c_{10} I + c_{01} (I_1 I - \bar{C}) - \frac{1}{3} (c_{10} \text{tr} (\bar{C}) + c_{01} (I_1 I - \bar{C}) : \bar{C}) \bar{C}^{-1} \right],$$

Also to demonstrate the possibility to catch non-linear effects with ANCF element we introduce a more difficult type of Mooney-Rivlin material. It is five constant model, where the strain energy function the form

$$\Psi = c_{10} (T_1 - 3) + c_{01} (T_2 - 3) + c_{20} (T_1 - 3)^2 + c_{11} (T_1 - 3) (T_2 - 3) + c_{02} (T_2 - 3)^2 + \Psi_{\text{vol}},$$

$$\Psi_{\text{vol}} = \frac{1}{d} (J - 1)^2.$$ 

Follow formulas (19), (22) and (20) it is possible to derive the expression for $S$.

### 2.3.4 Incompressible anisotropic material

The most interesting and important part is deformation of anisotropic material. We want to describe using ANCF element the deformation of beam from the anisotropic material, where the anisotropy is given in the exponential form, offered by Holzapfel, Ogden and Gasser in work [9] which has the form

$$\Psi = A_1 (T_1 - 3) + c_{12} \left( e^{c_2 \left( \kappa T_1 + (1 - 3 \kappa) T_4^{-1} \right)} - 1 \right) + \Psi_{\text{vol}},$$

where $I_4$ is invariant depends on fiber direction and similar to (44)

$$I_4 = \bar{C} : A_0.$$ 

It should be also said some words about $\Psi_{\text{vol}}$. This function has different forms in different softwares, for instance, in ANSYS it is $\frac{1}{d} (J - 1)^2$ and $\frac{1}{d} \left( \frac{J^2 - 1}{2} - \ln J \right)$ in ABAQUS, respectively. The part of $\Psi_{\text{vol}}$ in the brackets does not have significant influence on the results, but the function $\frac{1}{d} \left( \frac{J^2 - 1}{2} - \ln J \right)$ showed faster and better convergence.

Such in the case of Blatz&Ko material it is often used the simplified model of the material, when the distribution of fibers parameter is equal to zero, i.e $\kappa = 0$. We will also use this model to check obtained results from ANCF element because only this model is implemented in ANSYS software. This case means all fibers have one direction. In that
case, the expression (28) will reduce to

$$\Psi = A_1 (T_1 - 3) + \frac{c_1}{2c_2} (e^{c_2(T_1-1)} - 1) + \Psi_{\text{vol}}. \quad (29)$$

The second Piola – Kirchhoff $S$ in the case of full form (28) has form

$$S = \frac{1}{d} \left( J - \frac{1}{J} \right) J C^{-1} +$$

$$+ 2J^{-\frac{2}{3}} \left[ \left( A_1 + \kappa \frac{c_1}{2} e^{c_2(\kappa T_1+(1-3\kappa)T_1-1)} \right) I + (1 - 3\kappa) \frac{c_1}{2} e^{c_2(\kappa T_1+(1-3\kappa)T_1-1)} A_0 \right]$$

$$- \frac{2}{3} J^{-\frac{2}{3}} \left[ \left( A_1 + \kappa \frac{c_1}{2} e^{c_2(\kappa T_1+(1-3\kappa)T_1-1)} \right) \text{tr} (\bar{C}) \right]$$

$$- \frac{2}{3} J^{-\frac{2}{3}} \left[ (1 - 3\kappa) \frac{c_1}{2} e^{c_2(\kappa T_1+(1-3\kappa)T_1-1)} A_0 : \bar{C} \right] \bar{C}^{-1} \quad (30)$$

To receive the expression for $S$ in particular case (29) is enough to put $\kappa = 0$ in (30).

### 2.4 Task formulation

Now, we formulate the tasks, the solution of which will be given further. As it was said we consider hyperelastic beams with rectangular and circular cross sections. Now, let’s describe the boundary conditions. One side of the beam is rigidly clamped, it means that all coordinates are fixed at this end for the ANCF element and solid elements in commercial softwares, for example, where $x = 0$. So, we consider cantilevered objects. The opposite side $x = L$ is subjected to the uniaxial load. In the case of ANCF element the whole load is applied to the last node. In ANSYS and ABAQUS models the force is replaced by the uniform pressure with the value $\frac{F}{HW}$ Pa, where $F$ is the value of total force, $H$ and $W$ are the dimensions of cross-sections of the bar.
The value of applied force will be given in the right column of all tables below. This value is applied to the node of the model of ANCF element with $x = L$ in referent configuration. Other surfaces of objects under consideration are free.

Some more words about the way of implementation in commercial softwares. In ANSYS to describe the deformation in all cases was used SOLID 186 element. It is a quadratic 3D twenty node solid element. In case of ABAQUS it was used C3D8R element, it is linear 3D eight node element.

### 2.5 Numerical calculation

#### 2.5.1 Numerical integration

Here, we reveal some part of the calculation algorithm which was used in code for ANCF element implementation, namely the way of numerical integration and calculation of displacements. In section 2.1 we introduced the local coordinate system for the ANCF element. As a result, our characteristic dimension variables vary in the range $[-1, 1]$. It was done to simplify the integration of areas during deformations, which will be done through Gauss integration. It is one the most used schemes. In general, the formula for integration of any function $f(x, y, z)$ over the entire volume of the body after coming to
local coordinates of the reference body can be computed as

$$\int \int \int_B f(x, y, z) \, dx \, dy \, dz = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta, \zeta) \, J \, d\xi \, d\eta \, d\zeta, \quad (31)$$

where $J$ is multiplication scale factor, $B$ is the entire volume, and $f(x, y, z)$ is a function, integration of which we are interested in. Then after picking some points, so-called Gauss integration points, the integration is approximated as a sum of these points.

$$\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) \, J \, d\xi \, d\eta = \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) \, J \, w_i, \quad (32)$$

where $i$, are the numbers of points in a cross section, $\xi_i$, $\eta_i$ and $\zeta_i$ are the coordinates of each point in local system. $w$ is so-called the weight of each point. In the case of circular cross section we refer to [54] receiving the formula

$$\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) \, J \, d\xi \, d\eta = \pi \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) \, J \, w_i, \quad (33)$$

So, now we just need to choose the points for each type of cross section. It has already told that we operate with a three-node ANCF element, that means along one of the dimensions, namely $\xi$, the number of Gauss points is defined. These points are placed in the geometric center of the cross sections, along the axis through the center of the cube, they are arranged as follows In the case of rectangular cross section along both other axes the points are placed in the same way as in Table 1 with the same weights. The schemes for selecting points are shown in the picture below.
Figure 6. schemes of integration points in different cases for ANCF element: (b) rectangular cross section; (a) circular cross section [54].

Table 2. Coordinates of integration points in circular cross section and their weights

<table>
<thead>
<tr>
<th>η</th>
<th>ζ</th>
<th>w</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$\frac{1}{16}$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{3}}$</td>
<td>0</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>$-\frac{1}{\sqrt{3}}$</td>
<td>0</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{6}}$</td>
<td>$\frac{1}{2}\sqrt{2}$</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{6}}$</td>
<td>$-\frac{1}{2}\sqrt{2}$</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>$-\frac{1}{\sqrt{6}}$</td>
<td>$\frac{1}{2}\sqrt{2}$</td>
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<tr>
<td>$-\frac{1}{\sqrt{6}}$</td>
<td>$-\frac{1}{2}\sqrt{2}$</td>
<td>$\frac{1}{8}$</td>
</tr>
</tbody>
</table>

2.5.2 Displacement calculation

The calculation of displacement in our algorithm Newton-Raphson’s way is used for. In general, it looks

\[ K \Delta u = f - f_{\text{internal}}, \quad (34) \]
\[ u_{i+1} = u_i + \Delta u, \quad (35) \]

where \( K \) is the tangent stiffness matrix, the tangent stiffness matrix defines in our algorithm through finite differences. \( \Delta u \) is the vector of updates to the vector of displacements, \( f \) is a vector of forces, \( f_{\text{internal}} \) is a vector of internal forces. The iteration calculation continues until \( u \) converges, the convergence of the whole system was checked with varying different meshes.
3 RESULTS

3.1 Elongation of beams with rectangular cross sections

Here, the behavior of beams with rectangular cross section, will be considered from the different type of hyperelastic materials: compressible, incompressible, isotropic and anisotropic. All results will be presented in the tables and graphics to illustrate the difference in the results. In all tables, all results will be compared through the elongations of bars along their largest dimension. This parameter we assume as the most significant and notable. All calculations have been done with ANSYS and Matlab, and also in the most of cases, they have been repeated with ABAQUS and Maple. Also here we can study the influence of penalty function and constant $d$ which is related to compressibility and bulk modulus, the introduction for the constant will be given below.

3.1.1 Blatz & Co material

Here, we want to describe the deformation of the compressible material with ANCF element. We do not want to catch some special features, for instance, the existence of a falling section on the loading diagram under the stretching load, which can mean that there is buckling effect [55]. The presence of such phenomenon requires a separate verification and additional efforts to deal with. To avoid that we have considered the material with high-value module $\mu$, which is higher than those usually used with respect to applied loading [56] or [57] and we used quite a narrow load range.

Let’s consider the beam with the next geometrical and physical features: $\mu = 7.9615 \cdot 10^{10}$ Pa, $L = 2$m, $H = 0.5$ m, $W = 0.1$ m, where $L, H, W$ are length, height and width, respectively.

<table>
<thead>
<tr>
<th>Table 3. Deformation of Blatz&amp;Ko material.</th>
</tr>
</thead>
<tbody>
<tr>
<td>ANSYS</td>
</tr>
<tr>
<td>0.0203</td>
</tr>
<tr>
<td>0.0415</td>
</tr>
<tr>
<td>0.0634</td>
</tr>
<tr>
<td>0.0862</td>
</tr>
</tbody>
</table>
3.1.2 Neo-Hookean material

Now we go to the next part of this study, the description of deformation of beams from incompressible materials. Although, it is possible to consider the material (24) like slightly compressible [56], but we put our attention to other materials.

Let’s consider the beam in this theoretical experiments with the geometrical parameters, $W = 0.1, H = 0.5, L = 2$. 
Figure 8. Deformation of brick from Neo-Hookean material with $\mu = 7.64 \cdot 10^6$ Pa

Figure 9. Deformation of brick from Neo-Hookean material with $\mu = 7.64 \cdot 10^3$ Pa
Table 4. Neo-Hookean with $\mu = 7.64 \cdot 10^6$ Pa, $d = 10^{-5}$

<table>
<thead>
<tr>
<th>ANSYS</th>
<th>ANCF</th>
<th>Total force (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0022</td>
<td>0.0023</td>
<td>100</td>
</tr>
<tr>
<td>0.0109</td>
<td>0.0118</td>
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</tr>
<tr>
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<td>0.0176</td>
<td>750</td>
</tr>
<tr>
<td>0.0220</td>
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<td>1000</td>
</tr>
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<td>0.0330</td>
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<td>1500</td>
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</tr>
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<td>0.0666</td>
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</tr>
<tr>
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<td>15000</td>
</tr>
</tbody>
</table>

Table 5. Neo-Hookean with $\mu = 7.64 \cdot 10^3$ Pa, $d = 0$ (for ANSYS and ABAQUS) and $d = 10^{-7}$ (for ANCF)

<table>
<thead>
<tr>
<th>ANSYS</th>
<th>ANCF</th>
<th>ABAQUS</th>
<th>Total force (N)</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.0172</td>
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</tr>
<tr>
<td>0.0344</td>
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<td>0.0342</td>
<td>20</td>
</tr>
<tr>
<td>0.0859</td>
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<td>0.0854</td>
<td>50</td>
</tr>
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</tr>
<tr>
<td>0.171</td>
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</tr>
<tr>
<td>0.3410</td>
<td>0.4167</td>
<td>0.3393</td>
<td>200</td>
</tr>
</tbody>
</table>
3.1.3 Mooney-Rivlin material

As it was mentioned in 2.3.3 we considered two types on Mooney-Rivlin materials. The first one is two-constant model. The constants are equal to \( c_{10} = 33.4 \cdot 10^4 \) Pa, \( c_{01} = -337 \) Pa. And all dimensions of the beam are the same like in the previous cases, so \( W = 0.1, H = 0.5, L = 2 \). That part is given to demonstrate the influence of compressible parameter \( d \). Let’s consider the deformation with \( d = 10^{-5} \).

![Figure 10. Deformation of brick from Mooney-Rivlin material with \( d = 10^{-5} \)](image)

<table>
<thead>
<tr>
<th>ANSYS</th>
<th>ANCF</th>
<th>Total force (N)</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
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</tr>
<tr>
<td>0.1323</td>
<td>0.1284</td>
<td>3000</td>
</tr>
<tr>
<td>0.1795</td>
<td>0.1724</td>
<td>4000</td>
</tr>
<tr>
<td>0.2286</td>
<td>0.2173</td>
<td>5000</td>
</tr>
<tr>
<td>0.2762</td>
<td>0.2630</td>
<td>6000</td>
</tr>
</tbody>
</table>

And now let’s take the bar with the same physical and geometrical features, but we just
changed the value of \( d \). From here we received the next results

![Graph showing deformation of brick from Mooney-Rivlin material with \( d = 0 \) (for ANSYS and ABAQUS) and \( d = 10^{-7} \) (for ANCF)](image)

**Figure 11.** Deformation of brick from Mooney-Rivlin material with \( d = 0 \) (for ANSYS and ABAQUS) and \( d = 10^{-7} \) (for ANCF)

**Table 7.** Mooney-Rivlin with \( d = 0 \) (for ANSYS and ABAQUS) and \( d = 10^{-7} \) (for ANCF)

<table>
<thead>
<tr>
<th>ANSYS</th>
<th>ANCF</th>
<th>ABAQUS</th>
<th>Total force (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00197</td>
<td>0.002</td>
<td>0.00195</td>
<td>100</td>
</tr>
<tr>
<td>0.0039</td>
<td>0.004</td>
<td>0.00392</td>
<td>200</td>
</tr>
<tr>
<td>0.0098</td>
<td>0.0100</td>
<td>0.0098</td>
<td>500</td>
</tr>
<tr>
<td>0.0197</td>
<td>0.0202</td>
<td>0.0196</td>
<td>1000</td>
</tr>
<tr>
<td>0.0394</td>
<td>0.0408</td>
<td>0.0391</td>
<td>2000</td>
</tr>
<tr>
<td>0.0984</td>
<td>0.105</td>
<td>0.0978</td>
<td>5000</td>
</tr>
<tr>
<td>0.1474</td>
<td>0.1615</td>
<td>0.1468</td>
<td>7500</td>
</tr>
<tr>
<td>0.293</td>
<td>0.3489</td>
<td>0.292</td>
<td>15000</td>
</tr>
<tr>
<td>0.341</td>
<td>0.4176</td>
<td>0.3398</td>
<td>17500</td>
</tr>
</tbody>
</table>
We already mentioned some words about the effect of compressible parameter $d$. The previous two tables allows to compare the influence of it. We dealt with two identical bars and also applied forces were the same, however, there was only one differences, it is $d$. As it is possible to notice in case $d = 10^{-5}$ the material differs from $d = 10^{-7}$, so we can conclude that even meaning $10^{-5}$ does not describe incompressibility and it is needed lower one. In literature, it is possible to find some recommendations about how to choose it. For example, in ANSYS manuals it is given the formula for calculation

$$d = \frac{1 - 2\nu}{c_{10} + c_{01}},$$

where $\nu$ is Poison’s coefficient which should be chosen very close to 0.5, for example 0.49, but in considered case, $d$ became too small, around $6 \cdot 10^{-8}$, that it had influence on convergence of the system for ANCF solution. But at the same time then higher value of parameter, than faster calculation of system proceeds.

Now let’s consider five-constant Mooney-Rivlin material with formula

$$\Psi = c_{10} \left( \bar{T}_1 - 3 \right) + c_{01} \left( \bar{T}_2 - 3 \right) + c_{20} \left( \bar{T}_1 - 3 \right)^2 + c_{11} \left( \bar{T}_1 - 3 \right) \left( \bar{T}_2 - 3 \right) + c_{02} \left( \bar{T}_2 - 3 \right)^2 + \Psi_{vol}.$$

For the calculation we take the next meaning of constants (Pa) from [58]

$$c_{10} = -7.7 \cdot 10^5, c_{01} = 9.1 \cdot 10^5, c_{20} = -2.7 \cdot 10^5, c_{11} = 1.03 \cdot 10^6, c_{02} = -5.9 \cdot 10^5,$$

![Figure 12. Visualization of the Table 8](image-url)
Here blue and red lines show the ANCF solution with different $d$, $d = 10^{-5}$ for blue, $d = 10^{-7}$ for red line. As it is seen the high value of compressibility parameter reveals big errors in calculations.

As it is possible to notice ANCF element can describe the non-linear deformation even in case when potential energy deformation is given in quite complicated form of five constant Mooney-Rivlin model.

### Table 8. Five constant Mooney-Rivlin model

<table>
<thead>
<tr>
<th>ANSYS $d = 0$</th>
<th>ANCF with $d = 10^{-5}$</th>
<th>ANCF with $d = 10^{-7}$</th>
<th>Press = Total force/ $W \cdot H$ (Pa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00102921</td>
<td>0.001529</td>
<td>0.001049</td>
<td>436.695</td>
</tr>
<tr>
<td>0.00206608</td>
<td>0.003069</td>
<td>0.002106</td>
<td>873.673</td>
</tr>
<tr>
<td>0.00310952</td>
<td>0.003069</td>
<td>0.003172</td>
<td>1310.42</td>
</tr>
<tr>
<td>0.00483765</td>
<td>0.007179</td>
<td>0.00494</td>
<td>2027.33</td>
</tr>
<tr>
<td>0.00657564</td>
<td>0.009751</td>
<td>0.006721</td>
<td>2739.97</td>
</tr>
<tr>
<td>0.00920094</td>
<td>0.013632</td>
<td>0.009418</td>
<td>3800.98</td>
</tr>
<tr>
<td>0.0131805</td>
<td>0.019502</td>
<td>0.013523</td>
<td>5374</td>
</tr>
<tr>
<td>0.0192451</td>
<td>0.028425</td>
<td>0.019817</td>
<td>7690.39</td>
</tr>
<tr>
<td>0.0285606</td>
<td>0.04209</td>
<td>0.029586</td>
<td>11062.4</td>
</tr>
<tr>
<td>0.0430393</td>
<td>0.063301</td>
<td>0.045048</td>
<td>15869.8</td>
</tr>
<tr>
<td>0.0659284</td>
<td>0.097095</td>
<td>0.070379</td>
<td>22447.2</td>
</tr>
<tr>
<td>0.0982634</td>
<td>0.147366</td>
<td>0.109281</td>
<td>29775.2</td>
</tr>
<tr>
<td>0.113587</td>
<td>0.174751</td>
<td>0.130625</td>
<td>32558.9</td>
</tr>
<tr>
<td>0.121505</td>
<td>0.190853</td>
<td>0.142915</td>
<td>33812.2</td>
</tr>
<tr>
<td>0.129012</td>
<td>0.209529</td>
<td>0.156466</td>
<td>34923</td>
</tr>
<tr>
<td>0.129506</td>
<td>0.210019</td>
<td>0.156804</td>
<td>34947.3</td>
</tr>
<tr>
<td>0.129626</td>
<td>0.210383</td>
<td>0.157055</td>
<td>34965.2</td>
</tr>
<tr>
<td>0.129762</td>
<td>0.2106</td>
<td>0.157271</td>
<td>34980.5</td>
</tr>
<tr>
<td>0.129862</td>
<td>0.221</td>
<td>0.157547</td>
<td>35000</td>
</tr>
</tbody>
</table>

#### 3.1.4 Incompressible anisotropic material

Now we take the beam with square cross-section $L = 2m$, $H = 0.1$ m and $W = 0.1$ m. The fibres will be given through plane vector $\mathbf{a}$ with components $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$. The physical features we take $A_1 = 7640$ Pa, $c_1 = 996600$ Pa, $c_2 = 524.6$. The values were taken from Holzapfel-Gasser-Ogden article, 2006.

After some calculations with other $d$ we will receive the next results.
Table 9. Anisotropic material with $d = 0$ (for ANSYS and ABAQUS) and $d = 10^{-7}$ (for ANCF)

<table>
<thead>
<tr>
<th></th>
<th>ANSYS</th>
<th>ANCF</th>
<th>ABAQUS</th>
<th>Total force (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0432</td>
<td>0.044</td>
<td>0.0432</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>0.0865</td>
<td>0.08313</td>
<td>0.08712</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>0.1727</td>
<td>0.168</td>
<td>0.1738</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>0.3431</td>
<td>0.3396</td>
<td>—</td>
<td>80</td>
<td></td>
</tr>
</tbody>
</table>

Figure 13. Deformation of brick from anisotropic material with $d = 0$ (for ANSYS and ABAQUS) and $d = 10^{-7}$ (for ANCF)

Figure 14. Deformation of brick from anisotropic material with $d = 10^{-5}$ with data from Table 10
Table 10. anisotropic material with $d = 10^{-5}$

<table>
<thead>
<tr>
<th>ANSYS</th>
<th>ANCF</th>
<th>Total force (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0088</td>
<td>0.0089</td>
<td>10</td>
</tr>
<tr>
<td>0.0443</td>
<td>0.0455</td>
<td>50</td>
</tr>
<tr>
<td>0.0887</td>
<td>0.0932</td>
<td>100</td>
</tr>
<tr>
<td>0.133</td>
<td>0.143</td>
<td>150</td>
</tr>
<tr>
<td>0.177</td>
<td>0.195</td>
<td>200</td>
</tr>
<tr>
<td>0.222</td>
<td>0.251</td>
<td>250</td>
</tr>
</tbody>
</table>

Also we consider the deformation of beam with $L = 2m$, $H = 0.5$ m and $W = 0.1$ m. The another difference from the previous solution is that fiber direction is given not-plane but by vector $\mathbf{a}$ with components \( \left( -\frac{3}{\sqrt{145}}, \frac{6}{\sqrt{145}}, \frac{10}{\sqrt{145}} \right) \). However, we still take the parameter $\kappa$ is equal to zero, so to receive (29) model.

Table 11. anisotropic material with $d = 0$ (for ANSYS) and $d = 10^{-7}$ (for ANCF)

<table>
<thead>
<tr>
<th>ANSYS</th>
<th>ANCF</th>
<th>Total force (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0086</td>
<td>0.008</td>
<td>10</td>
</tr>
<tr>
<td>0.0172</td>
<td>0.0173</td>
<td>20</td>
</tr>
<tr>
<td>0.0645</td>
<td>0.0668</td>
<td>75</td>
</tr>
<tr>
<td>0.0859</td>
<td>0.0932</td>
<td>100</td>
</tr>
<tr>
<td>0.128</td>
<td>0.1387</td>
<td>150</td>
</tr>
<tr>
<td>0.171</td>
<td>0.1898</td>
<td>200</td>
</tr>
<tr>
<td>0.214</td>
<td>0.2432</td>
<td>250</td>
</tr>
<tr>
<td>0.256</td>
<td>0.299</td>
<td>300</td>
</tr>
</tbody>
</table>
As we can see the ANCF element gives the good solution even in the case of anisotropic material and can be used for the further possible extension, for instance, to describe the deformation of arteries and other biological tissues and even in case of multilayer constructions.

3.2 Elongation of cylinder

In this chapter the deformation of cylinder will be considered. This experiment is often used to modulate the tendons behavior [30]. The same as in the previous cases we concentrate on pure elastic behavior of the biological tissues. Including in models other specific features can be considered as future extension of the present work.

3.2.1 Semi-inverse method

Firstly, it is important to note that this task is possible to solve in the analytic way. For that aim we will use so-called semi-inverse method. This method is often used to describe the deformation in continuum mechanics [25, 59, 57]. The relevance of this method is connected with the fact that direct numerical solutions of many non-linear three-dimensional problems are extremely time-consuming even with using of modern computer power. This technique allows reducing the number of independent variables. However, there are some
limitations imposed on the usage of this method. It can be applied only to a narrow class of problems or the body which have the canonical form, like, a sphere, a cylinder or a sector of a cylinder. Moreover, these bodies should be subjected to only simple loadings, such as axial stretching, compression, torsion or bending.

Now we give a brief explanation of this method. As it was mentioned before the method consists in constructing geometric transformations that can help to reduce the initial problem to the problem with a smaller number of independent variables. In this case, the equilibrium equations must be satisfied in the entire volume of the three-dimensional body, and the boundary conditions on a part of the surface can be satisfied in the integral sense. The method was proposed by Saint-Venant in the 19th century and subsequently generalized to the case of large deformations. The algorithm of the method is as follows:

1. Set the expected deformation linking with the positions of body points in the referent and actual configuration
2. Find the necessary deformation characteristics, for example, deformation and stresses tensors.
3. Set the boundary conditions imposed on the surfaces of the body.
4. Find the distribution of forces assumed by the assumption of deformation from the equilibrium equations.

But even using semi-inverse methods does not guarantee that it will be easy to obtain the solution. There are some difficulties which often arise during the process. The main difficulty with the semi-inverse method is that the defined deformation cannot be random. It also must satisfy differential equilibrium equations of the whole body. That is why it is implemented to the bodies with the simple form because their possible future form is not difficult to predict. The second problem is even with smaller number of variables equilibrium equations could be difficult to solve. To overcome it numerical methods are usually used.

Let’s consider a continuum body in fixed reference configuration $\Omega_0$. Here we use a continuum approach that means we do not take into account molecular construction of object. Then, after a deformation point $X$ from $\Omega_0$ becomes the point $x$ from a new configuration $\Omega$ that is co-called current configuration. The connection between two points and also between configurations can be provided by the semi-inverse method. Using the example of a hollow circular cylinder of height $h$ with inner and outer radii $A$ and $B$ we describe
the deformations in axial and radial directions in cylindrical coordinate systems as it is presented in works [59, 60]:

\[
\begin{align*}
  r &= r(R), \\
  \phi &= \Phi, \\
  z &= \lambda_z z,
\end{align*}
\]  

(36)

where \( R, \Phi, Z \) and \( r, \phi, z \) are the cylindrical coordinates in the reference and current configurations, respectively, with unit vectors \( e_R, e_\phi, e_Z \) and \( e_r, e_\phi, e_z \). \( A \leq R \leq B, 0 \leq Z \leq L \). \( \lambda_z = \frac{1}{L} \) and \( \lambda_r = \frac{2}{R} \) are the deformation parameters, which are the coefficients of elongation of the cylinder length along the axial direction and the azimuthal stretch. The case \( \lambda_z < 1 \) corresponds to the compression of the cylinder along the axial coordinate, and \( \lambda_z > 1 \) corresponds to its extension. The same is for radial deformation parameter. We do not consider here initial stresses the angular coordinate remains the same \( 0 \leq \Phi \leq 2\pi \).

From (36) it is possible to receive the gradient deformation \( F \), which has the form

\[
F = \begin{bmatrix}
  \frac{r}{r_R} & 0 & 0 \\
  0 & \frac{r}{R} & 0 \\
  0 & 0 & \lambda_z
\end{bmatrix},
\]  

(37)

where \( \frac{r}{r_R} \) denotes differentiation of \( r \) with respect to \( R \). From (37) it immediately follows that

\[
F^{-1} = \begin{bmatrix}
  \frac{1}{r_R} & 0 & 0 \\
  0 & \frac{R}{r} & 0 \\
  0 & 0 & \frac{1}{\lambda_z}
\end{bmatrix},
\]  

(38)

and Cauchy–Green tensor \( C \) (11) and its invariants as

\[
I_1 = \text{tr} C, \\
I_2 = \frac{1}{2} (\text{tr} C^2 + \text{tr}^2 C), \\
I_3 = \text{det} C.
\]  

(39)

Materials under consideration are incompressible and therefore, the incompressibility constraint can be written in the form \( \text{det} F = 1 \) or

\[
J = \lambda_1 \lambda_2 \lambda_3 = 1,
\]  

(40)
where stretches
\[ \lambda_1 = \lambda = \frac{r}{R}, \]
\[ \lambda_2 = \frac{1}{\lambda_1 \lambda_3}, \]
\[ \lambda_3 = \lambda_z. \]

Then, the invariants \( I_1, I_2 \) can be rewritten as functions of two stretches \( \lambda_1 \) and \( \lambda_3 \)
\[ I_1 = \lambda_1^2 + \lambda_3^2 + \lambda_1^{-2} \lambda_3^{-2}, \]
\[ I_2 = \lambda_1^{-2} + \lambda_3^{-2} + \lambda_1^2 \lambda_3^2, \]
also using (40) we can get

\[ r = \sqrt{\frac{R^2 - A^2}{\lambda_z} + a^2}, \quad (41) \]
where \( a \) is the inner radius of cylinder in current configuration, \( a \leq r \leq b \). From (41) some relations can be derived.

\[ dr = d \left[ \sqrt{\frac{R^2 - A^2}{\lambda_z} + a^2} \right] = \frac{dR}{\lambda_z \lambda}, \]
\[ d\lambda = (1 - \lambda z \lambda^2) \frac{dR}{\lambda_z \lambda R}, \quad (42) \]
\[ \frac{dr}{R} = \frac{d\lambda}{(1 - \lambda z \lambda^2)}. \]

All constitutive relations in continuum mechanics of any samples can be defined by potential energy function per unit reference volume. Here we will use the Helmholtz’s potential energy density function used in the number of works, for example, in [9, 62, 52], as follows:

\[ \Psi = \frac{c}{2} (I_1 - 3) + k_1 k_2 \left( e^{k_2 (\alpha I_1 + (1-3\alpha)I_4 - 1)} - 1 \right), \quad (43) \]
where \( k_1, k_2, c \) are elastic material parameters. \( k_1 > 0 \) and \( c > 0 \) are stress-like material
parameters, where $c$ is the Neo-Hookean parameter, $k_2$ is a dimensionless parameter, $\kappa$ is the dispersion of family collagen fibers. The dispersion factor $\kappa$ influences importantly on the anisotropic hyperelastic response varying from 0 to $\frac{1}{3}$. In case of $\kappa = \frac{1}{3}$ (fully dispersed fibers) it corresponds to isotropic model and does not depend on the fiber direction [9], i.e. angle $\gamma$ in Figure 17. All these parameters except $\kappa$ and $I_4$ can be determined from mechanical tests of the tissue $\kappa$ is determined from histological data of the tissue [9], description of $I_4$ will be given below.

The function (43) allows to modulate the object reinforced by two families of fibers and defines the form of stress tensors. In other words the response on any type of loading which body are subjected. Also this function can be extended on the cases with viscoelastic behavior. But as it was mentioned in 1.3 we restrict our study only to the mechanical elasticity response and we will use model which is reinforced by two similar fibers families. Now we give descriptions of all variables which were used in (43). $I_1$ is already given, it is the first principle invariant of $C$. $I_4$ is a so-called pseudoinvariant of two tensors $A$ and $C$. To describe $I_4$ and receive tensor $A$ we introduce unit vector field $a_0 (X)$ which gives the direction of fiber family in each point $X \in \Omega_0$ and can be defined by two angels $X$ and $\Theta$.

![Figure 16](image)

**Figure 16.** Characterization of angels $X$ and $\Theta$ for definition of fiber orientation in Cartesian coordinate system [8].

In vector form $a_0$ has the view

$$a_0 = \begin{bmatrix}
\sin (\Theta) \cos (X) \\
\sin (\Theta) \sin (X) \\
\cos (X)
\end{bmatrix}.$$
$a_0(X)$, like $\kappa$, is determined from histological data of the tissue [9]. It can be easy seen that $a_0$ describes the basic geometry of one family fibers in referent configuration.

**Figure 17.** Thin-wall approximation of the adventitial layer with two embedded families of fibers. The mean orientations and the dispersion of the collagen fibres are characterized by $\kappa$ and $\gamma$, respectively.

Now we define the structural tensor of order two

$$A_0 = a_0 \otimes a_0.$$ 

Using the tensors $A_0$ and $C$ it is possible to characterize the stretch of fiber in the form

$$I_4 = C : A_0,$$  \hspace{1cm} (44)

or in terms of independent stretches it can be expressed as

$$I_4 = \lambda_1^2 \cos(\gamma)^2 + \lambda_2^2 \sin(\gamma)^2$$

According to [7, 9, 63, 64] response can be described in actual configuration

$$T = 2J^{-1}F \frac{\partial \Psi}{\partial C} F^T,$$

but this formula is true for all materials without any restrictions, for material (43), such it
is incompressible.

\[
\mathbf{T} = -p \mathbf{I} + 2 \mathbf{F} \frac{\partial \Psi}{\partial \mathbf{C}} \mathbf{F}^T,
\]

(45)

where \( p \) is a Lagrange multiplier and reveals hydrostatic pressure, \( \mathbf{T} \) is Cauchy stress tensor. In our case of incompressible material which is reinforced by two families of fibres (43) we have

\[
\mathbf{T} = -p \mathbf{I} + 2 \psi_1 \mathbf{B} + 2 \psi_2 \left( I_1 \mathbf{B} - \mathbf{B}^2 \right) + 2 \psi_4 \mathbf{a} \otimes \mathbf{a}.
\]

(46)

Here \( \mathbf{B} \) denotes the left Cauchy–Green tensor, \( \mathbf{a} = \mathbf{F} \mathbf{a}_0 \) and \( \psi_i = \frac{\partial \Psi}{\partial I_i} \). In matrix form we received

\[
\mathbf{T} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & \sigma_{32} & \sigma_{33} \end{bmatrix}.
\]

Based on expressions of \( \mathbf{T} \), the equilibrium equations for deformation of a incompressible inhomogeneous hollow cylinder can be derived. The set of elastostatics equilibrium equations for incompressible solid in vector form

\[
\text{div} \mathbf{T} = 0.
\]

(47)

This leads to the equilibrium expression as follows

\[
r \sigma_{11, r} + \sigma_{11} - \sigma_{22} = 0,
\]

(48)

where \( r \sigma_{11, r} \) denotes derivative respect to \( r \). If \( P \) denotes pressure of the cylinder is applied to inner surface then \(-P = \sigma_{11} (a)\) and integrates of (48) leads to

\[
P = \int_{a}^{b} (\sigma_{22} - \sigma_{11}) \frac{dr}{r} = \int_{a}^{b} \lambda W_\lambda \frac{dr}{r},
\]

(49)

where

\[
W_\lambda = \frac{\partial \Psi}{\partial \lambda}, W_\lambda_z = \frac{\partial \Psi}{\partial \lambda_z},
\]

\[
\sigma_{22} - \sigma_{11} = \lambda W_\lambda,
\]

\[
\sigma_{33} - \sigma_{11} = \lambda_z W_\lambda_z.
\]

Also we have adopted the notation \( \lambda_a = \frac{a}{A}, \lambda_b = \frac{b}{B} \) which also gives

\[
\frac{R^2}{A^2} = \frac{\lambda_a^2 \lambda_z - 1}{\lambda^2 \lambda_z - 1}.
\]

(50)
and using the expressions in (42) it is possible to rewrite the expression (49) in the following form

\[ P = \int_{a}^{b} \lambda W_\lambda \frac{d\lambda}{\lambda R} = \int_{\lambda_a}^{\lambda_b} \frac{W_\lambda d\lambda}{\lambda^2 \lambda_z - 1}. \] (51)

The second equation for equilibrium of the solid will be derived from loading conditions for the top and bottom of the cylinder surfaces

\[ N = 2\pi \int_{a}^{b} \sigma_{33} r dr, \] (52)

The (52) can be expressed as

\[ \frac{N}{\pi A^2} = (\lambda_a^2 \lambda_z - 1) \int_{\lambda_a}^{\lambda_b} \frac{\lambda d\lambda}{(\lambda^2 \lambda_z - 1)^2} + P \lambda_a. \] (53)

The expressions (51), (53) with

\[ B^2 - A^2 = \lambda_z (\lambda_a^2 \lambda_z - \lambda_b^2 \lambda_z) \]

give us the closed system of equations respect to variables \( \lambda_z, \lambda_a \) and \( \lambda_b \).

3.2.2 Solution method

Solutions for (51), (53) were done in Maple, to build the load diagrams we will use own Newton’s iteration method instead of implemented in Maple, cause it showed better convergence. For this we rewrite equations in the next form

\[ F_1 (\lambda_a, \lambda_z) = \int_{\lambda_a}^{\lambda_b} W_\lambda \frac{d\lambda}{\lambda^2 \lambda_z - 1} - P (\lambda_a, \lambda_z), \] (54)

\[ F_2 (\lambda_a, \lambda_z) = (\lambda_a^2 \lambda_z - 1) \int_{\lambda_a}^{\lambda_b} \frac{\lambda d\lambda}{(\lambda^2 \lambda_z - 1)^2} + P (\lambda_a, \lambda_z) \lambda_a^2 \frac{N (\lambda_a, \lambda_z)}{\pi A^2}. \] (55)
Use function $F_1$ and $F_2$ we can create the iteration system

$$
\begin{align}
F_1 (\lambda_a^i, \lambda_z^i) + \frac{\partial F_1 (\lambda_a^i, \lambda_z^i)}{\partial \lambda_a} (\lambda_a^i - \lambda_a^j) + \frac{\partial F_1 (\lambda_a^i, \lambda_z^i)}{\partial \lambda_z} (\lambda_z^i - \lambda_z^j) &= 0, \\
F_2 (\lambda_a^i, \lambda_z^i) + \frac{\partial F_2 (\lambda_a^i, \lambda_z^i)}{\partial \lambda_a} (\lambda_a^i - \lambda_a^j) + \frac{\partial F_2 (\lambda_a^i, \lambda_z^i)}{\partial \lambda_z} (\lambda_z^i - \lambda_z^j) &= 0.
\end{align}
$$

(56)

Also the inflation of a thin-walled tube characterized by the proposed anisotropic constitutive model (43) with inner and without axial pressure can be investigated numerically in the second way. This way is described in works [9, 52] and it will be used to compare results which will be obtained from (56). The deformation in the second way described the following system of equations

$$
\begin{align}
\lambda_z \frac{\partial \Psi}{\partial \lambda_z} - \frac{\lambda_z \left( \lambda R - \frac{H}{2\lambda_z} \right)^2}{2HR} P &= 0, \\
\lambda \frac{\partial \Psi}{\partial \lambda} - \left( \frac{\lambda^2 R}{H} - \frac{1}{2} \right) P &= 0,
\end{align}
$$

(57)

where $R$ is mean radius and $H$ is the wall thickness. The method of the two-dimensional solution is simpler than system of equations (51), (53) or system (56). But to compare them we need to take very thin tube to receive the comparable solutions. Because it is well-know that the two-dimensional theory can describe with the good accuracy the deformation only thin-walled construction.

To validate derived solution we consider hollow cylinder. The geometrical and physical data were taken from [9]. $c = 7.64$ kPa, $k_1 = 996.6$ kPa, $k_2 = 524.6$, parameters of anisotropic structure, i. e. $\kappa$ and $\gamma$, which characterizes angle between vector $a_0$ and vector $e_\phi$ (one of basic vectors in referent configuration), are varied during numerical experiments. Here in figures it is possible to compare the obtained results, it reveals that there is no differences with the results received in [9] with approach (57). The first figure shows the deformation of $\lambda$ under the inner pressure (in the graphics the $P$ is given in kPa).
The second figure shows the deformation of $\lambda_z$ or other words the change of length of cylinders under the inner pressure. It is obviously that without uniaxial load the result should be monotonically decreasing functions in all cases. Which will be obtained as a result of calculations.
The results are confirmed with the result obtained from two-dimensional theory proposed in the work [9]. Such as the results of two-dimensional theory were checked in the work using finite element methods we can conclude that this part of thesis work has done correctly and that all mathematical transformations were made correctly.
3.2.3 Deformation of solid cylinder

Here we will describe the deformation of solid body such it was describe in 2.4, but in the case of solid cylinder. Received analytical solution is also possible to adopt for that aim (in the pictures and tables it is written as Maple). However, it is restricted in case $a \neq 0$. In the case then it needs to consider the solid cylinder it is often to take little shift $\varepsilon$ that $\varepsilon > 0$, $a = \varepsilon$ and it is considered the deformation of cylinder $\varepsilon \leq r \leq b$, but such as $\varepsilon$ is so small, it is assumed that the cylinder is solid [65].

3.2.4 Neo-Hookean material

Firstly, it was considered the deformation of bar from Neo-Hookean material, such one of the simplest one. It was taken the bar with $L = 1$ m, $R = 0.1$ m, the physical features was given with constant $\mu = 900$ kPa. In the case of analytical solution inner radius was $a = 0.001$ m.
Table 12. Neo-Hookean material with $\mu = 900$ kPa, $d = 0$ (for ANSYS) and $d = 10^{-7}$ (for ANCF)

<table>
<thead>
<tr>
<th>Maple</th>
<th>ANSYS</th>
<th>ANCF</th>
<th>Total force (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00118</td>
<td>0.00115296</td>
<td>0.001198</td>
<td>100</td>
</tr>
<tr>
<td>0.00235</td>
<td>0.0023059</td>
<td>0.002399</td>
<td>200</td>
</tr>
<tr>
<td>0.003552</td>
<td>0.00345883</td>
<td>0.003603</td>
<td>300</td>
</tr>
<tr>
<td>0.00475</td>
<td>0.00461174</td>
<td>0.00481</td>
<td>400</td>
</tr>
<tr>
<td>0.005924</td>
<td>0.00576462</td>
<td>0.006019</td>
<td>500</td>
</tr>
<tr>
<td>0.007129</td>
<td>0.00691749</td>
<td>0.007232</td>
<td>600</td>
</tr>
<tr>
<td>0.008306</td>
<td>0.00807032</td>
<td>0.008448</td>
<td>700</td>
</tr>
</tbody>
</table>

### 3.2.5 Mooney-Rivlin material

Here it will be demonstrated the deformation of circular beam from five-constant Money-Rivlin material. It was taken the bar with the same geometrical features $L = 1$ m, $R = 0.1$ m, the physical ones was given by constants in Pa from [58]

$$c_{10} = -7.7 \cdot 10^5, c_{01} = 9.1 \cdot 10^5, c_{20} = -2.7 \cdot 10^5, c_{11} = 1.03 \cdot 10^6, c_{02} = -5.9 \cdot 10^5,$$

To remind the potential energy of deformation in that case is given with the formula

$$\Psi = c_{10}(T_1 - 3) + c_{01}(T_2 - 3) + c_{20}(T_1 - 3)^2 + c_{11}(T_1 - 3)(T_2 - 3) + c_{02}(T_2 - 3)^2 + \Psi_{vol}.$$
As it is possible to notice ANCF element gives the better approximation than ANSYS in terms of analytical solution, they are almost matching.

### Table 13. Mooney-Rivlin $d = 0$ (for ANSYS) and $d = 10^{-7}$ (for ANCF)

<table>
<thead>
<tr>
<th>Total force (N)</th>
<th>ANSYS</th>
<th>ANCF</th>
<th>Maple</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.00381913</td>
<td>0.00392</td>
<td>0.0039</td>
</tr>
<tr>
<td>200</td>
<td>0.00784526</td>
<td>0.00809</td>
<td>0.00805</td>
</tr>
<tr>
<td>300</td>
<td>0.0121068</td>
<td>0.012553</td>
<td>0.01249</td>
</tr>
<tr>
<td>400</td>
<td>0.0166418</td>
<td>0.017365</td>
<td>0.01729</td>
</tr>
<tr>
<td>500</td>
<td>0.0215012</td>
<td>0.022602</td>
<td>0.022509</td>
</tr>
<tr>
<td>600</td>
<td>0.0267538</td>
<td>0.028371</td>
<td>0.02825</td>
</tr>
<tr>
<td>700</td>
<td>0.0324954</td>
<td>0.034831</td>
<td>0.03469</td>
</tr>
<tr>
<td>800</td>
<td>0.0388665</td>
<td>0.042235</td>
<td>0.04207</td>
</tr>
<tr>
<td>900</td>
<td>0.0460609</td>
<td>0.051036</td>
<td>0.05085</td>
</tr>
<tr>
<td>1000</td>
<td>0.054362</td>
<td>0.062207</td>
<td>0.06199</td>
</tr>
<tr>
<td>1100</td>
<td>0.065224</td>
<td>0.078874</td>
<td>0.07863</td>
</tr>
<tr>
<td>1150</td>
<td>0.070678</td>
<td>0.09541</td>
<td>0.09508</td>
</tr>
<tr>
<td>1200</td>
<td>0.079174</td>
<td>–</td>
<td>0.11273</td>
</tr>
</tbody>
</table>

#### 3.2.6 Incompressible anisotropic material

Now we consider the deformation of anisotropic material which is provided Helmholtz energy (43). Where $c = 7.64$ kPa and $k_1 = 996.6$ kPa, $k_2 = 524.6$ and $\kappa = 0$ [9], the direction of fibers was taken $\gamma = 45$. 

![Graph](image.png)
As it can be seen in it is difficult to describe the behaviour such as in different situation under some loads gives various level of estimation of the deformation.

Table 14. anisotropic material with $\gamma = 45$, $d = 0$ (for ANSYS) and $d = 10^{-7}$ (for ANCF)

<table>
<thead>
<tr>
<th>Total force (N)</th>
<th>ANSYS</th>
<th>ANCF</th>
<th>Maple</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.0677853</td>
<td>0.043497</td>
<td>0.05686</td>
</tr>
<tr>
<td>200</td>
<td>0.134955</td>
<td>0.0951248</td>
<td>0.109</td>
</tr>
<tr>
<td>300</td>
<td>0.201023</td>
<td>0.1563219</td>
<td>0.157</td>
</tr>
<tr>
<td>400</td>
<td>0.264014</td>
<td>0.22829521</td>
<td>0.2</td>
</tr>
<tr>
<td>450</td>
<td>0.297001</td>
<td>0.26857046</td>
<td>0.221244</td>
</tr>
<tr>
<td>500</td>
<td>–</td>
<td>0.3116774</td>
<td>0.2404</td>
</tr>
</tbody>
</table>

Now we consider the deformation of the same bar from anisotropic material, but now it was changed the fibres direction, it is equal to $\gamma = 30$.

Unfortunately, then the load came to the meaning $N > 130$ the analytical solution stopped to converge, but as it is possible to see before that level it coincidences with Matlab solution for ANCF element.
Table 15. Anisotropic material with $\gamma = 30$, $d = 0$ (for ANSYS) and $d = 10^{-7}$ (for ANCF)

<table>
<thead>
<tr>
<th>Total force (N)</th>
<th>ANSYS</th>
<th>ANCF</th>
<th>Maple</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>0.0169747</td>
<td>0.017431</td>
<td>0.0176</td>
</tr>
<tr>
<td>40</td>
<td>0.0271552</td>
<td>0.028223</td>
<td>0.0285</td>
</tr>
<tr>
<td>50</td>
<td>0.0339292</td>
<td>0.03556</td>
<td>0.0359</td>
</tr>
<tr>
<td>60</td>
<td>0.0407169</td>
<td>0.043012</td>
<td>0.0431</td>
</tr>
<tr>
<td>75</td>
<td>0.0508779</td>
<td>0.05441</td>
<td>0.0548</td>
</tr>
<tr>
<td>100</td>
<td>0.0677874</td>
<td>0.074</td>
<td>0.07399</td>
</tr>
<tr>
<td>110</td>
<td>0.0745425</td>
<td>0.082047</td>
<td>0.08265</td>
</tr>
<tr>
<td>120</td>
<td>0.0812861</td>
<td>0.090216</td>
<td>0.09054</td>
</tr>
<tr>
<td>130</td>
<td>0.0880231</td>
<td>0.098508</td>
<td>0.09914</td>
</tr>
<tr>
<td>150</td>
<td>0.10147</td>
<td>0.115463</td>
<td>–</td>
</tr>
<tr>
<td>175</td>
<td>0.118211</td>
<td>0.137354</td>
<td>–</td>
</tr>
<tr>
<td>200</td>
<td>0.134917</td>
<td>0.16551</td>
<td>–</td>
</tr>
<tr>
<td>250</td>
<td>0.16803</td>
<td>0.215187</td>
<td>–</td>
</tr>
</tbody>
</table>
4 SUMMARY

In this work several tasks have been considered.

1. The chapter 2 was devoted to the deformation of rectangular and circular beams from hyperelastic materials. That research was performed only with finite element point of view. For that in Matlab it was written the code to describe the elongations of the hyperelastic beams under the extension loads, which were normally applied to the one of the smallest areas. The obtained results were checked with several commercial software, namely ABAQUS and ANSYS. Usage of commercial software was dictated by the fact that the results obtained from ANCF required some verification. As it was already said the numerical experiments have been done for beams from a number of hyperelastic materials, among them compressible one, which was represented with Blatz&Ko material, several incompressible isotropic models. They are Neo-Hookean material and Mooney-Rivlin two and five constant models, and anisotropic Holzapfel model [9]. In the last case, it was considered material which was reinforced with two similar family fibers. Also, it was considered several orientations of fibers. Difference between approaches was represented in the table and graphic forms. The obtained results revealed that ANCF element gives the good approximation of deformations of the beam from mentioned above materials, but the element gives values of deformation slightly higher than other software. However, there are no differences in the dynamic of the behavior even then it was considered non-linear one, like in five-constant Mooney-Rivlin case.

Furthermore, for the behavior of incompressible solids, it was shown the influence of compressibility parameter. Solutions revealed that the correct choice of this parameter has a huge impact on both the result, computational time and convergence of solutions. In some cases when the parameter was not small enough the results became significantly higher than expected.

In conclusion, it was demonstrated that it was possible to describe big deformation, non-linear effects and anisotropic behavior with ANCF element. In the future, the following complication of the system will be considered by including in the model viscoelastic properties of the material as the possible complication of the shape of the structure to become closer to the real description of artery and tendon behavior.

2. In the chapter 3.2.1, the deformation of the cylinder from hyperelastic anisotropic material has been analytically derived. The object under consideration was described using so-called semi-inverse method and considered from the 3D point of
view. With the usage of the non-linear elasticity theory and this method, the equilibrium equations in three-dimensional form were obtained. To validate this way of solution, it was applied to the description of the deformation of thin-walled constructions. The obtained results were compared with solutions received from two-dimensional theory [9].

The solutions obtained from 3D formulation and ones based on two-dimensional theory did not show significant differences. There are several small variations in the predictions of the behavior of anisotropic cylinder, which are probably caused due to some geometrical approximations and distinguishes between the two model formulations. In fact, if we consider a three-dimensional geometrical object, it is reasonable to think that 2D implementation would give some differences. We assume that the 3D model is more proper to use because during its implementation we based on only one assumption about possible ways of deformation. That cylinder after deformation would become an object with the form which is very close to the cylindrical one.

There is a limitation of this three-dimensional solution. It can be applied only to the bodies of simple forms, like a sphere, a beam or a cylinder. But on the other hand, there are some benefits. Firstly, the proposed exact solution can be easily extended to the deformation of bodies from any types of materials, it needs only potential energy expressions. Secondly, just adding some parameters in the expression (36), it is possible to include the disinclinations and dislocations into the object. That helps to regulate the level of initial stress. Thirdly, the represented way allows complicating the construction of solid with adding new layers with different physical features, the example of that object can be found in [52]. Also, it is possible to describe the behavior of the solid cylinder, but only after taking one more assumption, it has been done in 3.2.

To sum up, the three-dimensional approach has some benefits, but in spite of them, it requires lots of knowledge in the framework of the non-linear elasticity theory, mathematical and programming skills. So, we assume that the main way of usage of this approach is to check the solutions which are obtained from other approaches. For example, some problems can arise during the calculation with finite element methods, to overcome them it is useful to have the exact solution.
3. In the 3.2, the deformation of the circular bar was considered. This part can be regarded as a logical extension of the previous two. On the one hand, we considered a continuous object whose behavior was modeled with finite elements as it was done in the chapter 2 and as it was shown models on the base of ANCF element did it with good accuracy. On the other hand a hollow cylinder deformation was considered for which an analytical solution was derived in the chapter 36. After taking one assumption about inner radius it was possible to adapt the analytical solution for the description of the solid one. This allowed us to give a more accurate estimation of the deformation of bodies which are modeled with ANCF elements. And it is interesting to note that the cases of isotropic material and one case of anisotropic solutions received with ANCF element the solutions were closed to analytical one. In that chapter, only ANSYS was used for checking from FEM points of view.
5 DISCUSSION

During this work it was considered several specific questions mentioned in motivation part. As well as some subquestion which arose during the work in each chapter. To reveal the answers we firstly identified those properties that were considered as basic and whose influence we wanted to study such as non-linearity, elasticity, incompressibility. We did it to simplify our model as much as possible to avoid working with lots of additional parameters, but in spite of limits of this model we tried to save the ability to represent tendons and artery behavior. To receive better prediction it should be implemented other features like viscoelastisity, existence of initial stresses, complexity of the forms etc. Another important thing to say it is a common problem for all biomechanics studies is lack of data about real physical parameters of models to deal with. The constants of the models, used in this study, were taken from the literature and can be found in reference part. However, this work was mostly devoted to the tendons behavior, the constants for tissues were taken from the experiments on the arteries deformation. And in spite of the simplifications that were made, the results obtained both analytically and with the help of finite element programming based on the ANCF element were demonstrated. They all were in good agreement with the results obtained by other scientists, and received from proven commercial software.

The further development of this work will be the inclusion into the model of dynamic behavior, viscoelastic properties and complication of the form of samples. All this will allow to more accurately simulate the deformation of tendons and other biological objects. It is also possible to adopt our solutions to the behavior of arteries etc., but a bit more work will have to be done because our solution is not ready to include hollows in the structures. That all will even allow us to compare the results obtained with computer modulation with the results of real experiments. Which help to reduce such thing, which was already mentioned, as lack of data about real physical parameters of models to deal with.
REFERENCES


