One-Dimensional Model for the Nonlinear Resistive Electric Field Control in Medium-Voltage Rotating Electrical Machines

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One-Dimensional Model for the Nonlinear Resistive Electric Field Control in Medium-Voltage Rotating Electrical Machines

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ABSTRACT

This study provides an insight in the analytical estimation of the electric field along the surface of a stress grading system (SGS) in the end-winding region of an electrical machine. An analytical approach is derived for the field grading materials (FGMs) having power-law electric-field-dependent conductivity. Despite some limitations, the introduced model can be used for the preliminary analysis of SGS with specific geometric and dielectric parameters both during sinus and pulsed voltage condition. The analytical considerations are verified with finite element simulations for 15 markedly different nonlinear FGMs, which were previously reported in the applications related to the medium voltage rotating electrical machines.

Index Terms — rotating machines, nonlinear field dependent resistive field control, analytical models, stress grading tape, converter fed motors, end windings, nonlinear field grading material, electric field distribution, stress grading system

1 INTRODUCTION

In medium voltage rotating electrical machines, the electrical field grading technique is used to control the spatial distribution of the electric field along the main insulation. Partial discharge (PD) activity at the surface of insulation arises because of the local increase of electric field strength above the air ionization threshold. The chemical and thermal stresses under PDs may result in fast degradation of a machine’s insulation system [1].

The conventional stress grading system (SGS) of a medium voltage rotating electrical machine comprises conducting armor tape (CAT) and stress grading tape (SGT) as it is depicted in Figure 1. CAT prevents the PD activity on the surface of the main insulation inside machine stator stack region. CAT ends with SGT, which ensures a smooth transition of the electric potential from the phase electrodes potential to the CAT potential. The geometry and thermal limitations in the end-winding region of a machine have promoted the use of SGTs with electric field dependent electric properties [2].

The design of an SGS nowadays mostly relies on finite element analysis (FEA). The FEA simulations may be time consuming as the simulation time increases with the rise of the insulating material nonlinearity. Therefore, a simple analytical model is highly desirable at least for preliminary analysis of SGS with specific geometry, dielectric properties of system components, and signal shape of supply voltage. The analytical solutions for the electric potential on the surface of an SGS has been provided only for materials with field-independent and extremely nonlinear dielectric properties [3–5], which are not suitable for practical SGTs. Several optimization algorithms were implemented aiming at reducing the total calculation time of the FEA-based analysis of machines SGS [6].

Figure 1. Schematic view of the SGS of a medium voltage rotating electrical machine in a radially symmetrical coordinate system. The phase electrode is covered by insulation with conductivity \( \sigma_i \) and relative permittivity \( \varepsilon_i \). CAT with conductivity \( \sigma_{CAT} \) and relative permittivity \( \varepsilon_{CAT} \) comes out of the slot region and ends at distance \( l_{CAT} \) from the stator stack. SGT with conductivity \( \sigma_{SGT} \) and relative permittivity \( \varepsilon_{SGT} \) finalizes CAT and propagates to the distance \( l_{SGT} \) from CAT’s end. The electric potential of phase electrode is a function of time \( V_p(t) \). The parameters \( r_{SGT} \), \( r_{CAT} \), \( r_i \) and \( r_{cond} \) refer to the radius of SGT, CAT, insulation, and phase conductor, respectively.
This study contributes to the theory of the nonlinear resistive electric field control in electrical machines. The distribution of electric potential along SGS is estimated analytically in the end-winding region for AC and pulse rise condition for materials having power-law field dependent conductivity. The paper is organized as follows. Section 2 provides the analytical formulation for the electric potential and electric field strength in the machine’s end winding region. The available data on field grading materials (FGMs) with field-dependent electrical conductivity for medium voltage rotating electrical machines are provided in Section 3. Section 4 contains the comparison of FEA and model simulated results. The limitations of the introduced model are discussed in Section 5. Section 6 concludes the paper.

2 1-DIMENSIONAL MODEL OF END-WINDING REGION

The distribution of the electric potential \( V(x, t) \) along the surface of SGS in the end winding region of a machine can be fairly well modelled with 1-dimensional equivalent circuit in Figure 2. The details may be found in [3, 4]. The current along the FGM \( I_{\text{F}}(x, t) \) can be described as follows [3]:

\[
I_{\text{F}}(x, t) = \left( \frac{\partial S_{\text{F}}}{\partial x} \right) \frac{V(x, t)}{\bar{C}} + \left( \frac{\partial^2 C_{\text{F}}}{\partial t \partial x} \right) \frac{V(x, t)}{\bar{C}}
\]

where \( \bar{C} \) is the relative permittivity of the FGM (CAT or SGT).

\[
S_{\text{F}} = \pi \times \sigma_{\text{F}} \left( \rho_{\text{F}}^2 - r_i^2 \right)
\]

\[
C_{\text{F}} = \pi \times \varepsilon_{\text{F}} \left( \rho_{\text{F}}^2 - r_i^2 \right)
\]

The Kirchhoff condition \( I_0 \) \( (x, t) + I_1 \) \( (x, t) = \frac{\partial}{\partial t} \bar{C} \theta(x, t) \) results in the following partial differential Equation:

\[
S_{\text{F}} \frac{\partial V(x, t)}{\partial t} - C_{\text{F}} \frac{\partial^2 V(x, t)}{\partial t^2} = \frac{\partial S_{\text{F}}}{\partial x} \frac{\partial V(x, t)}{\partial x} + \frac{\partial C_{\text{F}}}{\partial x} \frac{\partial V(x, t)}{\partial x}
\]

which is the basis for the further considerations of this paper.

2.1 CAPACITIVE-RESISTIVE FIELD CONTROL WITH LINEAR FIELD GRADING MATERIAL

The conventional CAT is manufactured with both field independent conductivity \( \sigma_{\text{CAT}} \) and relative permittivity \( \varepsilon_{\text{CAT}} \). Setting parameters of CAT in Equations (2), (3), (7) and writing \( V(x, t) = V_i(x) \times f(t) \) results in the second order differential Equation:

\[
V_i(x) = k_{\text{ind}}(t) \frac{\partial V_i(x)}{\partial x}
\]

where

\[
k_{\text{ind}}(t) = \frac{S_{\text{CAT}} \times f(t) + C_{\text{CAT}} \times \partial f(t)/\partial t}{S_i \times f(t)/(\rho - r_{\text{cond}}) + C_i \times \partial f(t)/\partial t}
\]

Setting \( V_i \) \( |x = 0, \rho = V_{\text{ph}}(t) \) and \( \partial V_i / \partial x \) \( |x = \rho, t = 0 \) = 0 (no current at the end of SGS) as boundary conditions, the distribution of the electric potential along the SGS with linear FGM may be described as follows:

\[
V(x, t) = V_{\text{ph}}(t) \left( 1 - e^{-\alpha \sqrt{\frac{V_{\text{ph}}(t)}}{E}} \right)
\]

The solution with sinuousoidal supply with angular frequency \( \omega \) can be obtained in the frequency domain, i.e. \( f(t) = e^{j \omega t} \). The solution during a voltage pulse with the pulse rise time \( \tau_p \) is considered at time \( 0 \leq t \leq \tau_p \) with \( f(t) = \tau_p \).

2.2 NONLINEAR RESISTIVE FIELD CONTROL

The conductivity of SGT in the nonlinear resistive control may be defined with a power-law relation, i.e. with low state conductivity \( \sigma_0 \) threshold field \( E_0 \) and nonlinearity coefficient \( \alpha \) as follows:

\[
\sigma_{\text{SGT}}(E) = \sigma_0 \times \left| 1 + \frac{E}{E_0} \right|^\alpha
\]

Setting \( V(x,t) = V_i(x) \times f(t) \) and neglecting both capacitive effects and field independent conductivity term \( \sigma_0 \) in SGT, partial differential Equation (7) results in:
Thus, Equation (12) turns to:

\[ V_i(x) = \frac{\sigma_0 \pi |r_{\text{int}}-r_{\text{amb}}|}{E_0} \left( \frac{f(t)}{C_i} + \frac{\partial V_i(x)}{\partial x} \right), \]

(13)

where

\[ k_{\text{mn}}(t) = \frac{\sigma_0 \pi |r_{\text{int}}-r_{\text{amb}}|}{E_0} \left( \frac{f(t)}{C_i} + \frac{\partial V_i(x)}{\partial x} \right). \]

(14)

Applying actual SGT enables the transition of the electric potential of the phase electrode \( V_{ph} \) to the potential at the end of CAT \( V_h \), and the cancelation of the electric field at the end of SGT, i.e., \( V_i(\xi) = V_{ph}(t) \), \( V_i(\xi) = V_h \), \( \xi = \infty, \xi = 0 \), and \( \partial V_i / \partial x \bigg |_{\xi=\infty,0} = 0 \). Thus, the analytical solution of Equation (13) for \( 0 \leq V(x) \leq V_{ph} \) is of the form:

\[ x = C + \sqrt{\frac{\alpha + 2}{\alpha} \left[ V(x) - V_{ph}(t) \right]} \left( \frac{(\alpha + 2) |V(x) - V_{ph}(t)|^2}{k_{\text{mn}}(t) |\alpha + 1|} \right)^{-\frac{1}{\alpha + 1}}, \]

(15)

\[ \frac{\partial V(x)}{\partial x} = \sqrt{\frac{\alpha + 2}{\alpha} \left[ V(x) - V_{ph}(t) \right]} \left( \frac{(\alpha + 2) |V(x) - V_{ph}(t)|^2}{k_{\text{mn}}(t) |\alpha + 1|} \right)^{-\frac{1}{\alpha + 1}}, \]

(16)

where \( C \) is estimated as follows:

\[ C = -L_{\text{CAT}} + \sqrt{\frac{(\alpha + 2)}{\alpha} \left[ V_{ph}(t) - V_{ph}(t) \right]} \left( \frac{(\alpha + 2) |V_{ph}(t) - V_{ph}(t)|^2}{k_{\text{mn}}(t) |\alpha + 1|} \right)^{-\frac{1}{\alpha + 1}}. \]

(17)

To verify the presented analytical considerations, some practical nonlinear resistive field grading materials for medium voltage electrical machines are discussed further.

### 3 Existing Commercial Nonlinear Resistive Field Grading Materials for Electrical Machines

Existing nonlinear resistive field grading materials are combinations of one or several fillers placed in an insulating matrix. The usual filler materials used are silicon carbide (SiC) and zinc oxide (ZnO). The filler concentration is typically above the percolation threshold, which enables continuous conducting paths in the composite. The nonlinear resistive properties of field grading materials may originate from two distinct mechanisms. The nonlinearity in SiC-based composites arises from particle contacts, whereas the nonlinear resistive behavior of ZnO-varistor-ceramics-based FGMs is the internal property of the ZnO filler itself. A detailed discussion on the topic is out of the scope of this study and may be found e.g. in [7]. The available measured data on the resistive nonlinear properties of FGMs for medium voltage rotating electrical machines are depicted in Figure 3 [1, 7–17]. The data was reproduced from respective references at temperatures close to 80 °C for comparative purposes. Table 1 shows the dielectric properties of materials in Figure 3, which are estimated with Equation (11). The measured relative permittivity \( \varepsilon_{SGT} \) of nonlinear resistive FGMs is in the range from 6 to 20 [7, 8, 13–16].

![Figure 3](image-url)

**Figure 3.** The reported measured data on the electrical conductivity of the practical resistive nonlinear FGMs used for SGTs.

**Table 1.** Nonlinear resistive properties of FGMs estimated with Equation (11) from the data in Figure 3.

<table>
<thead>
<tr>
<th>Field grading material</th>
<th>Ref.</th>
<th>Low state conductivity ( \sigma_0 ) ([S/m])</th>
<th>Threshold field ( E_b ) ([kV/m])</th>
<th>Nonlinearity coefficient ( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>FGM1 [8]</td>
<td>2.07 \times 10^6</td>
<td>120</td>
<td>3.24</td>
<td></td>
</tr>
<tr>
<td>FGM2 [7]</td>
<td>2.67 \times 10^6</td>
<td>100</td>
<td>4.16</td>
<td></td>
</tr>
<tr>
<td>FGM3 [7]</td>
<td>2.38 \times 10^6</td>
<td>100</td>
<td>3.43</td>
<td></td>
</tr>
<tr>
<td>FGM4 [9]</td>
<td>1 \times 10^6</td>
<td>90</td>
<td>3.8</td>
<td></td>
</tr>
<tr>
<td>FGM5 [10]</td>
<td>4.12 \times 10^6</td>
<td>150</td>
<td>5.1</td>
<td></td>
</tr>
<tr>
<td>FGM6 [10]</td>
<td>7.05 \times 10^6</td>
<td>200</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>FGM7 [10]</td>
<td>4.96 \times 10^6</td>
<td>200</td>
<td>8.16</td>
<td></td>
</tr>
<tr>
<td>FGM8 [11]</td>
<td>8.73 \times 10^6</td>
<td>100</td>
<td>3.66</td>
<td></td>
</tr>
<tr>
<td>FGM9 [12]</td>
<td>6.07 \times 10^7</td>
<td>130</td>
<td>3.76</td>
<td></td>
</tr>
<tr>
<td>FGM10 [1]</td>
<td>1.54 \times 10^7</td>
<td>70</td>
<td>2.40</td>
<td></td>
</tr>
<tr>
<td>FGM11 [13]</td>
<td>2.89 \times 10^6</td>
<td>140</td>
<td>10.60</td>
<td></td>
</tr>
<tr>
<td>FGM12 [14]</td>
<td>7.48 \times 10^6</td>
<td>225</td>
<td>6.50</td>
<td></td>
</tr>
<tr>
<td>FGM13 [15]</td>
<td>9.94 \times 10^6</td>
<td>250</td>
<td>3.84</td>
<td></td>
</tr>
<tr>
<td>FGM14 [16]</td>
<td>4.01 \times 10^6</td>
<td>125</td>
<td>3.65</td>
<td></td>
</tr>
<tr>
<td>FGM15 [17]</td>
<td>6.83 \times 10^6</td>
<td>100</td>
<td>3.17</td>
<td></td>
</tr>
</tbody>
</table>
The data in Table 1 shows markedly different nonlinear resistive properties of the materials studied. The low state conductivity $\sigma_0$ varies in the range $10^{-11}$ – $10^{-7}$ [S/m] depending on the material. The materials demonstrate moderate nonlinearity coefficient $\alpha = 2.4 – 10.6$. Recently, stress grading composite materials with $\alpha$ up to 45 were reported in [18, 19]. The calculated threshold field $E_{th}$ is in the range from 70 to 250 kV/m. The data in Table 1 is used further in FEA simulations to evaluate the introduced analytical model.

4 FINITE ELEMENT SIMULATION

The finite element analysis of the actual SGS is based on 2D COMSOL Multiphysics with AC/DC module. The relevant dimensions of the system and dielectric properties of the materials are reproduced from [20]. The sketch of the model geometry is depicted in Figure 3. The frequency-stationary analysis is used for pure sinusoidal supply [21]. The behavior of the system under voltage pulses with magnitude $V_{peak}$ and pulse rise time $t_p$ is calculated for the time period $0 \leq t \leq t_p$ with transient time domain study. Parameters used in the simulation are depicted in Table 2. The effect of the Joule generated loss on the parameters of the system are not considered, since it is out of the scope of this study.

The maximum electric field $E_{max}$ on the surface and the SGT’s length along the axial direction are crucial parameters that need to be considered during the design of an actual SGS. The enhancement of the electric field above the air ionization threshold facilitates the PD activity on the surface of the insulation. Insufficient length of the SGT along the axial direction may result in PDs at the end of SGT. The analytical solution neglects the constant conductivity component $\sigma_0$ and the capacitive current term in Equation (7), which dominates at low field strengths, i.e. at the region of the SGT surface with potential close to the potential of the phase electrode. The length of the SGT along the axial direction is analyzed with penetration length $\Delta$, which is calculated as follows:

$$\Delta = x \left( \frac{V_{th} - \left( \frac{V_{th} - V}{e} \right)}{\sigma_0} \right) \cdot x(t).$$  

The penetration length estimates the distance from the end of CAT where the potential along the SGT changes by the fraction $(1 - e)/e$. Figure 4 depicts the comparison of the analytical and FEA estimated results for FGM7 as an example. The comparison between FEA and the model estimated penetration length $\Delta$ with respect to the maximum electric field strength along SGT $E_{max}$ are depicted in Figure 5a and Figure 5b for pure sinusoidal and pulse supplies, respectively.

The space charge limiting field (SCLF) $E_{SCLF}$ is frequently used to evaluate the approximate maximum electric field strength in a dielectric with field dependent conductivity [22–24]. Study [22] estimates $E_{SCLF}$ at the point where the resistive and capacitive components of the current density are equal; $E_{SCLF}$ can be calculated as follows in this very case:

$$E_{SCLF} = \sqrt{\frac{\sigma_0 \cdot \left( f(\phi) \right)^n \cdot E_{th}^2}{\sigma_0 \cdot \left( f(\phi) \right)^n \cdot E_{th}^2}}.$$  

Table 3 depicts the relative error in the maximum electric field

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase conductor radius $r_{cond}$ [m]</td>
<td>$10 \times 10^{-3}$</td>
</tr>
<tr>
<td>Radius of the main insulation $r_l$ [m]</td>
<td>$12.4 \times 10^{-3}$</td>
</tr>
<tr>
<td>Radius of the conducting armor tape $r_{CAT}$ [m]</td>
<td>$12.9 \times 10^{-3}$</td>
</tr>
<tr>
<td>Radius of the stress grading tape $r_{SGT}$ [m]</td>
<td>$13.1 \times 10^{-3}$</td>
</tr>
<tr>
<td>Length of the CAT along axial direction $l_{CAT}$ [m]</td>
<td>$55 \times 10^{-3}$</td>
</tr>
<tr>
<td>Electric conductivity of the insulation $\sigma_l$ [S/m]</td>
<td>$2.01 \times 10^{-15}$</td>
</tr>
<tr>
<td>Relative dielectric permittivity of the insulation $\varepsilon_l$</td>
<td>4</td>
</tr>
<tr>
<td>Electric conductivity of the CAT $\sigma_{CAT}$ [S/m]</td>
<td>$10^{-5}$</td>
</tr>
<tr>
<td>Electric conductivity of the air $\sigma_{air}$ [S/m]</td>
<td>$10^{-18}$</td>
</tr>
<tr>
<td>Relative dielectric permittivity of the CAT $\varepsilon_{CAT}$</td>
<td>1</td>
</tr>
<tr>
<td>Relative dielectric permittivity of the air $\varepsilon_{air}$</td>
<td>20</td>
</tr>
<tr>
<td>Pulse rise time $t_r$ [s]</td>
<td>$1.2 \times 10^{-5}$</td>
</tr>
<tr>
<td>Peak value of the pulse $V_{peak}$ [V]</td>
<td>11500</td>
</tr>
<tr>
<td>Electrical frequency of the supply voltage at sine supply $f$ [Hz]</td>
<td>50</td>
</tr>
<tr>
<td>Magnitude of the phase voltage at sine supply $V_{rms}$ [V]</td>
<td>5500/2</td>
</tr>
</tbody>
</table>

Figure 4. Comparison between analytically and FEA calculated data along the surface of the SGS from the stator slot exit with FGM7 used for SGT. a) Electric potential distribution; b) Electric field strength distribution. The data for the pulse voltage signal is provided at time $t = t_p$, i.e. at the time where the maximum magnitude of the electric field strength occurs in SGT.
5 DISCUSSION

The analytical model with field independent dielectric properties of the FGM overestimates the transition rate of the electric potential along CAT as it can be seen in Figure 4a. The electric potential along CAT in an actual stress grading structure has a slower increase rate because of the electric charge accumulation phenomenon that arises at the boundary of the semiconducting CAT and the main insulation. This phenomenon is neglected in the equivalent circuit model. The analytically calculated potential distribution along SGT region in Figure 4a follows close to FEA-estimated data at the region where SGT starts. The analytical formulation neglects the capacitive current term, which is present in the actual SGT. The capacitive current affects the slope of the potential transition at the region where the voltage along the SGT is close to the voltage of the phase electrode. This region is characterized with the relatively small values of the electric field strength, i.e., the capacitive current term may be of comparable size with the field-dependent-electric conductivity-caused current term. The analytical solution is available only up to the potential of the phase electrode $V_{ph}(t)$. Thus, the voltage overshoot phenomenon at the end of the SGT cannot be modelled. The data in Figure 4b shows fair good agreement between FEA and analytically calculated electric field distribution along the SGS. The maximum electrical field strength $E_{\text{max}}$ occurs at the end of CAT in a correctly designed SGS. The presented analytical solution for this region is close to the FEA calculated data.

Figures 5a and 5b show similar trends for FEA and analytically estimated data on the penetration length $\Delta m$ and the maximum electric field along SGT $E_{\text{max}}$. The difference in $\Delta m$ is mainly caused by the capacitive current term that may be in the range of resistive current term at the relatively low values of the electric field strength. The data of $\eta_1$ in Table 3 shows that the FEA and model calculated values of $E_{\text{max}}$ are close to each other, except the data for FGM5, FGM6 and FGM13. These materials are discussed further. The relative error $\eta_2$ for other FGMs in Table 3 is the largest for FGM 9. The data of $\eta_2$ in Table 3 clearly shows that the concept of SCLM $E_{\text{SGT,CL}}$ underestimates the maximum electric field strength in the actual SGT with nonlinear conductivity. The inaccuracy of the presented model is significantly lower in comparison to the conventionally used SCLF approach [8, 24, 25].

The difference between the model and FEA estimated data for the FGM5, FGM6 and FGM13 is caused by the assumptions in the analytical model that neglect the field independent conductivity term $\sigma_0$ and the capacitive current term $\varepsilon_0 \varepsilon_{SGT,CL} \partial f(t)/\partial t$ in Equation (11). Figure 6 depicts the resistive current density component $J_{\text{res}}$ and capacitive current density component $J_{\text{cap}}$ in the discussed FGMs under AC supply and pulse voltage supply ($t = t_p$), which are calculated as follows:

$$J_{\text{res}} = \sigma_0 \left( \frac{1}{E_0} \cdot \varepsilon_0 \cdot f(t) \right) \cdot E \cdot \partial f(t) / \partial t, \quad (20)$$

$$J_{\text{cap}} = \varepsilon_0 \varepsilon_{SGT,CL} \cdot E \cdot \partial f(t) / \partial t. \quad (21)$$

The resistive current component is the same for AC supply and at time $t = t_p$ during the pulse voltage supply. The FEA and model calculated maximum electric field strengths at SGS with FGM5
Figure 6. The comparison of the resistive and capacitive ($\omega_{\text{cap}} = 20$) current density components in SGT for FGM5, FGM6, FGM7, and FGM13 at AC supply (frequency domain) and at the pulse supply ($t = \tau_0$).

during AC supply ($f = 50$ Hz) are 821 and 819 kV/m, as it can be seen in Figure 5a. The data in Figure 6 for FGM5 shows $J_{\text{res}} / J_{\text{cap}} \approx 5.4$; the resistive current term dominates over the capacitive current in the SGT. Thus, the model and FEA calculated results are close to each other. The magnitude of the capacitive current exceeds the resistive current term in FGM5 at least at order during the pulse voltage supply, as it can be seen in Figure 6. Therefore, the distribution of the electric field strength in SGT is governed mostly by capacitive effects and the model results become inadequate. Similarly, the inaccuracy between FEA and the model-calculated results for FGM6 arises from the dominant length $\lambda$ in an actual SGS with specific geometry, dielectric properties of FGMs under both AC and pulse voltage supplies. The assumptions in the analytical formulation may result in a significant error in case when the capacitive current density or the field independent term of resistive current density have a dominant share in comparison to nonlinear field dependent current density. Such a situation may be predicted with careful analysis of the current components during the specific operation mode.

APPENDIX

This Appendix gives the derivation of Equations (15)–(17). Setting $y = V(x), p(y) = y' = \partial V(x) / \partial x$, and $y'' = p(y) \partial p(y) / \partial y$ Equation (13) converts to:

$$y = k_m \cdot (\alpha + 1) \left( \frac{\partial p}{\partial y} \right).$$

Separating variables and finding indefinite integral transform Equation (22) as follows:

$$\frac{\partial V(x)}{\partial x} = \frac{\sqrt{2} \left( x \right)}{\alpha} \left( \frac{(\alpha + 2) \left( x \right) V(x)}{k_m(t) (\alpha + 1)} \right)^{\frac{1}{2}}.$$

The boundary conditions at the end of SGS $V_1 |_{x = \xi, \eta} = 0$, and $\partial V_1 / \partial x |_{x = \xi, \eta} = 0$ results in $C_1 = 0$. The indefinite integral of Equation (23) results in the following expression for $V(x)$:

$$x + C_2 = \sqrt{2} \left( x \right) \left( \frac{(\alpha + 2) \left( x \right) V(x)}{k_m(t) (\alpha + 1)} \right)^{\frac{1}{2}}.$$

where $C_2$ is a subject of boundary conditions. Considering $V(x) = V(l) + V_{\text{ph}}(l)$ (see Figure 2) and $V_1 |_{x = \xi_\text{CAT}, \eta} = V_1 - V_{\text{ph}}(l)$ Equation (24) results in:

$$x + C_2 = \sqrt{2} \left( x \right) \left( \frac{(\alpha + 2) \left( x \right) V(x) - V_{\text{ph}}(l)}{k_m(t) (\alpha + 1)} \right)^{\frac{1}{2}}.$$

where

$$C_2 = -\xi_\text{CAT} + \sqrt{2} \left( \frac{(\alpha + 2) \left( \xi_\text{CAT} - V_{\text{ph}}(l) \right)}{\alpha} \left( \frac{(\alpha + 2) \left( \xi_\text{CAT} - V_{\text{ph}}(l) \right)}{k_m(t)(\alpha + 1)} \right)^{\frac{1}{2}}.$$

The derivative of Equation (25) with respect to $\partial V(x)$ estimates $\partial V(x) / \partial x$ as follows:

$$\frac{\partial V(x)}{\partial x} = \sqrt{2} \left( x \right) \left( \frac{(\alpha + 2) \left( \xi_\text{CAT} - V_{\text{ph}}(l) \right)}{k_m(t)(\alpha + 1)} \right)^{\frac{1}{2}}.$$

REFERENCES
