Lappeenranta University of Technology.
Department of Information Technology.
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Master's Thesis:

Spectral Analysis of Buffers in Communication Systems.

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Abstract.

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Extension of the modified matrix geometric technique to more general queuing models is the main scope of the presented work. A queuing system, consisting of several subsystems with finite capacities and state independent phase-type distributed service times is studied in the thesis. The structure of the underlying finite Markov chain with level independent block matrices of a quasi birth death structure and several boundary states is discussed and presented. A representation of its steady state probability vector by matrix geometric terms, obtained by means of applying the modified matrix geometric solution, is stated as the main result of the thesis.

Tiivistelmä.

Lappeenrannan Teknillinen Yliopisto
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Denis Dyakov

Spectral Analysis of Buffers in Communication Systems

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Avainsanat: *QBD* prosessi, matriisi-geometrinen analyysi, vakaita analyysin olotiloja, numeraaliset metodit.

Muokatun matriisi-geometrian tekniikan kehitus yleimmäksi jonoksi on esitelty tässä työssä. Jonotus systeemi koostuu useista jonoista joilla on rajatut kapasiteetit. Tässä työssä on myös tutkittu PH-tyypin jakaautumista kun ne jaetaan. Rakenne joka vastaa lopullista Markovin ketjua jossa on itsenäisiä matriiseja joilla on QBD rakenne. Myös eräitä rajallisia olotiloja on käsiteltu tässä työssä. Sen esitteleminen matriisi-geometrisessä muodossa, muokkamalla matriisi-geometristä ratkaisua on tämän opinnäytetyön tulos.

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Acknowledgements

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1 Introduction.

1.1 History of The Subject.

In recent years, the extensive application of queuing systems of different kind has been widely witnessed. The search of numerical solutions for queuing models applied in different branches of science has practical importance and remains a hot topic for research.

While the complexity of queuing system was growing, the need for new approaches for their analysis became stronger, and a new branch of science, namely, the Queuing Theory, had appeared. Erlang was the first one, who suggested to describe the queuing process with Markov chains, and from this time the following developing of the theory was going on in this direction. This allowed to apply the Probability theory to the queuing systems, what, in turn, gave an opportunity to receive analytical solutions of almost all problems, somehow connected to the Queuing theory. The fact that the main consumers of the obtained results were the telephone services providers, resulted in reducing of the interest in the future development of the theory. But with the appearance of computer networks, the problem of performance analysis of computer systems has been posed, and the already developed techniques were very inconvenient for any applications to the problems of the kind. The need for algorithmic approach was clearly drawn, and the Queuing theory continued its development. Despite all the disadvantages of the algorithmic approach, it is oriented to certain real-life applications, what sometimes, may be much more useful than the analytical formulas, which can not be effectively implemented.

The researches in the mentioned direction resulted in the development of several efficient methods for the steady-state analysis of certain types of queuing systems [1,2,4,6,12,17,21,22,24]. The most popular one is the Quasi Birth Death process, because it can be used to obtain solutions for several applied queuing models such as[1] $M/PH/1/\infty$, $PH/M/n/\infty$, $PH/PH/1/\infty$, $MAP/PH/n/\infty$, $\sum_{i=1}^{n} MAP_i/PH/1/\infty$, $\sum_{i=1}^{n} MAP_i/PH/1/m$ ([22]). Owing to its wide applicability, the QBD process has gained a lot of research attention, and before moving ahead to discussion, let's now

[1]using Kendall notation

begin with the brief review of numerical methods available in literature so far that were developed for steady state analysis of *QBD* processes.

1.2 Existing Techniques.

Basically, *QBD* process is a two dimensional Markov chain (phase and level), where state changes are allowed only between ones, having adjacent levels. *QBD* process was first introduced by Wallace [23] in the late sixties, and from this time plenty of different techniques were proposed.

The most detailed discussion of *QBD* processes is done by Neuts [22]. In his book, Neuts studies infinite-state *QBD* processes using matrix geometric approach. His methodology is based on the fact that subvectors of the steady-state probability vector are related to one another in a matrix geometric form. The key element of this method is the iterative calculation of *rate matrix* R, by which the geometric relation is defined. However, the original matrix geometric method has some disadvantages mainly in terms of computational time. A generalization of this approach to multiple boundaries is done by Hajek [12]. The techniques, proposed by Latouche and Ramaswami [16,17] and Naumov [21] are the improved versions of the classical matrix geometric method.

Ram Chakka developed an exact computational method called *spectral expansion* for *QBD* process [5,7]. Instead of using the geometric relation between the state probability vectors, a special expression of the state probabilities vectors is introduced. The expression is defined by eigenvalues and eigenvectors of the characteristic matrix polynomial constructed from the process' parameters.

In [24] the authors presented an efficient *folding method*, that can be applied for finite *QBD* processes. The odd-even permutation achieved inside the transition matrix and the use of the principle of finite Markov chain reduction are the key elements of this approach.

Nail Akar approach the solution of *QBD* process from a novel side [1,2]. His method basically relies on the theory of invariant subspace and on the computation of matrix sign function with iterative procedure. The rate matrix R is obtained from a calculated invariant subspace of an adequately constructed matrix. The method is believed to be fast and stable.

Bini and Mini stated [4] a fast, quadratically convergent and numerically stable algorithm called *cyclic reduction algorithm* for *QBD* problems. Regarding to the time complexity, this algorithm is considered to be equivalent with Naumov's algorithm.

Recently, a novel method has been proposed in [8]. The method is named ETAQA (*Efficient Technique for the Analysis of QBD processes by Aggregation*). In this technique, the state space of a *QBD* chain is divided into several equivalent classes by a certain specific partitioning rule. Instead of computing the probability distribution of all states in the chain, only the aggregate probability distribution of the states in each class is evaluated. The authors show that those aggregate probabilities contain sufficient information to compute performance measures of interest such as the mean queue length or any higher moments.

The presented techniques do not form a complete list of ones, but these are considered to be the most known and widely applied to analysis of the queuing systems, which arise from modelling of telecommunication and computer networks.

1.3 The Current Work.

However, the presented methodologies are used only for the queuing systems, which can be represented as a single process. But, evidently, there is a huge class of systems, which can be modelled as a set of processes, where the state changes are allowed not only within one process between the states having adjacent levels, but also between the boundary states of different processes. There are only few of works, dealing with this problem, e.g. [15], and only special cases are covered there.

Being motivated by such observation, the current research work focuses on developing a numerical method for steady-state solution of queuing systems, which consist of several subsystems, represented as *QBD* processes. The fundamental material of the proposed method is the modified matrix geometric solution for finite

QBD processes, proposed by V.Naumov [21]. Matrix-analytic methods provide flexible models for the analysis of systems with complex behavior. This was the main reason for choosing it for the following extension and development.

The main idea of the suggested technique, is to build a Markovian-based model for complex queuing systems, apply a modified matrix geometric method to each subsystem, and using the boundary equations, obtain the general solution.

However, obviously, the modified matrix geometric method can not be directly applied to the posed problem, because of several reasons like, for example, it is not clear yet, how the structure of the boundary states looks like, what is the relation between the states of different processes etc. All the posed questions will be discussed in the thesis, and a representation of the steady state vector by the matrix geometric terms will be derived.

1.4 Organization.

The presented work is organized as follows. The section 2 is devoted to a brief overview of Markov chains, introducing the basic notations and definitions used throughout the paper. The modified matrix-geometric method for a single *QBD* process is introduced in the section 3. The main concepts of the method are briefly described in order to be prepared for the following improvement of the technique. The section 4.1 is devoted to the detailed description of a multiple processes queuing system, namely, the structure of the boundary and the inner states is discussed, with the purpose to introduce the generator matrix in 4.1.3. A general matrix geometric representation of the steady state probability vector of the queuing system is introduced and discussed in the section 4.2, and the steady state vectors' exact representation is given in the section 4.3. The implementation of the proposed technique is briefly discussed in 5. And finally, the results are summarized in the conclusion.

2 Background.

2.1 Basic Notation.

The following designations are used throughout the thesis:

* upper case Greek letters, except Φ, to to indicate sets (e.g. Θ);
* uppercase Roman letters to indicate matrices (e.g. $A, B, \ldots$);
* lower case Roman or Greek letters, topped with arrow, to indicate vectors (e.g. $\vec{\rho}, \vec{a}, \vec{b}$);
* single subscripts to indicate the levels number (e.g. $\vec{k}_k$);
* double subscripts to indicate a vectors element, a matrix element or the phase of a single process (e.g. $A_{ij}, \vec{\pi}_{ik}$);
* subscripts and superscripts in round brackets to indicate the correspondent process number (e.g. $A^{(i)}, \vec{\pi}_{(i)}$);
* $\vec{e}$ to indicate a column vector of all ones of the appropriate dimension;
* $\mathbf{0}$ to indicate a vector or a matrix of all zeros of the appropriate dimensions;
* Q to indicate the generator matrix of the system;
* $Q^{(i)}$ to indicate the generator matrix of the i-th process;
* $A^{\#}$ to indicate the general group inverse of A;
* $A^{\perp}$ to indicate any inverse of A, depending on the context;
* QBD stands for Quasi Birth Death process.

2.2 Markov Processes.

Let $\{X_n,\ n \geq 0\}$ be a sequence of random variables, which has a finite $\{\Theta_1, \ldots,\ \Theta_m\}$ or countable $\{\Theta_1,\ \Theta_2, \ldots\}$ set of possible states. Making a bijection between the all states and their numbers, we'll define a set Υ of all numbers of states, which will be used for the convenience. A system of random variables $\{X_n,\ n \geq 0\}$ is called a *Markov chain* if for all $n \geq 0$ and $i_0,\ i_1, \ldots, i_{t-1},\ j\ \in \Upsilon$

$$P\left(X_t = j \mid X_0 = i_0, \ldots,\ X_{t-1} = i_{t-1}\right) = P\left(X_t = j \mid X_{t-1} = i_{t-1}\right). \tag{2.1}$$

In another words, given the present, the future is conditionally independent of the past.

Define the *transition probabilities* as $p_{ij} = P\left(X_t = j \mid X_{t-1} = j\right)$, where $i,\ j\ \in \Upsilon$. The matrix

$$P = (p_{ij}) = \begin{pmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

is called a *transition probabilities matrix* or a *Markov matrix.* The following relation is valid for any Markov chain

$$\sum_{j \in \Upsilon} p_{ij} = 1, \quad i \in \Upsilon.$$

Any matrix with nonnegative elements that suits this relation is referred to as *stochastic matrix.*

Markov process with continuous time and discrete set of states can be defined, according to (2.1), as a process $\{\eta(t),\ t \geq 0\}$, for which

$$\begin{aligned} &P\left(\eta(t_{n+1}) = i_{n+1} \mid \eta(t_1) = i_1, \ldots,\ \eta(t_n) = i_n\right) = \\ &\quad = P\left(\eta(t_{n+1}) = i_{n+1} \mid \eta(t_n) = i_n\right) \end{aligned} \tag{2.2}$$

is valid for any moments of time $0 \leq t_1\ < \ldots < t_{n+1}$, and $i_1, \ldots, i_{n+1} \in \Upsilon$. Denote also $p_i(t) = P\left(\eta(t) = i\right)$ the probability of that at the moment t, the process $\eta(t)$ is in state i, and $p_{ij}(t) = P\left(\eta(t+s) = j | \eta(s) = i\right)$ – the probability of changing the state i to j in time t.

Define the *transition rates* as

$$
\begin{aligned}
q_{ij} &= \lim_{t \downarrow 0} \frac{p_{ij}(t)}{t}, \quad i,j \in \Upsilon,\ i \neq j, \\
q_{ii} &= \lim_{t \downarrow 0} \frac{p_{ii}(t) - 1}{t}, \quad i \in \Upsilon.
\end{aligned} \tag{2.3}
$$

It is proved that

$$-\infty \leq q_{ii} \leq 0,\ 0 \leq q_{ij} < \infty,\ i,j \in \Upsilon,\ i \neq j,$$

and

$$\sum_{j \in \Upsilon} q_{ij} \leq 0,\ i \in \Upsilon.$$

The matrix $Q = (q_{ij})$ is referred to as an *infinitesimal matrix.* A Markov process is said to be *conservative*, if

$$\sum_{j \in \Upsilon} q_{ij} = 0,\ i \in \Upsilon.$$

We will deal only with conservative processes, for which $-\infty < q_{ii}$.

2.3 QBD Process.

In the presented work we consider only a special class of Markov processes, called, the *Quasi Birth Death* (*QBD*) processes.

Consider a queueing system that can be modelled by a two-dimensional Markov process, meaning that at any time of observation the system can be described by two integer variables j and i. The former one can be either unbounded (infinite case) or bounded (finite) case and is referred to as a level of the system. The latter one is bounded and is referred to as a phase. If the possible changes of j are only : j, $j+1$ and $j-1$, the corresponding process is known as a *QBD* process.

Denote, m – the number of levels in a process plus one, that is, the length of queue, and n – the number of phases. Then the transition rates of a *QBD* process can be given by the following matrices:

B – purely phase transitions: from state (j,i) to state (j,k), where $j = 0,\ldots,m$, $k,i = 1,\ldots,n$, $k \neq i$;

A – one step upward transitions: from state (j,i) to state $(j+1,k)$, where $j = 0,\ldots,m-1$, $k,i = 1,\ldots,n$;

C – one step backward transitions: from state (j,i) to state $(j-1,k)$, where $j = 1,\ldots,m$, $k,i = 1,\ldots,n$.

If the matrices A, B and C are the same for all levels, except, maybe, the zeroth and the last ones, then such *QBD* process is referred to as *level-independent* or *homogeneous.*

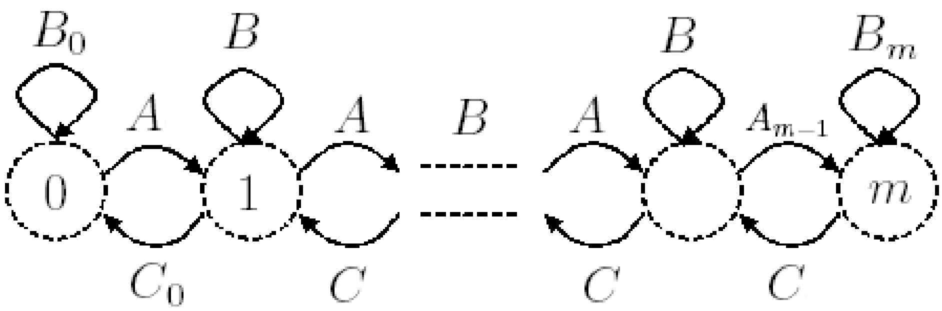

Fig. 1: Finite Homogeneous *QBD* process

The infinitesimal generator matrix for finite homogeneous process has the following structure:

$$Q = \begin{pmatrix} B_0 & A_0 & & & \mathbf{0} \\ C_0 & B & \ddots & & \\ & C & \ddots & A & \\ & & \ddots & B & A_{m-1} \\ \mathbf{0} & & & C_m & B_m \end{pmatrix}. \tag{2.4}$$

As it was stated above, the diagonal elements of Q are negative, and all the rest are nonnegative. The row sum of Q is equal to zero.

The problem for a single *QBD* process can now be stated as to find such vector $\vec{x}$ that:

$$\begin{cases} \vec{x}Q &= \mathbf{0} \\ \vec{x}\vec{e} &= 1 \end{cases},$$

where $\vec{e}$ is a column vector of all ones, and $\vec{x}$ represents a vector of steady state probabilities. Define the partition of the vector $\vec{x}$ as

$$\vec{x} = (\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_m), \tag{2.5}$$

where $\vec{x}_i,\ i = 0, \ldots, m$ represent the probabilities of being in the i^{th} level. Each entry of $\vec{x}_i$ represent a probability of being in a certain phase of the level i.

According to (2.4), the boundary conditions look as:

$$\begin{cases} \vec{x}_0 B_0 + \vec{x}_1 C_0 = \mathbf{0} \\ \vec{x}_{m-1} A_{m-1} + \vec{x}_m B_m = \mathbf{0} \end{cases}. \tag{2.6}$$

2.4 Phase-Type Distribution.

Material from [8] is used in the current section. Any continuous distribution, which can be obtained as the distribution of time until absorbtion in a continuous time Markov chain with a single absorbing state is said to be of phase-type (PH). A PH distribution represents random variables that are measured by the time v that the underlying Markov chain spends in its transient portion till absorption. From this perspective, a row vector $\vec{\tau}$ of size n is associated with the underlying Markov chain of a PH distribution and represents the initial probability vector for each of its transient states. The infinitesimal generator of the underlying Markov chain of a PH distribution is

$$U = \begin{pmatrix} 0 & \mathbf{0} \\ \vec{t} & T \end{pmatrix},$$

where the zero row represent the absorbing state of the chain. The matrix T is square of the size n and represents the transitions among the transient states, and $\vec{t}$ is a column vector of size n, which represents the transitions from the transient states to the absorbing state. Matrix T and vector $\vec{t}$ relate to each other as $\vec{t} = -T\vec{e}$. The PH distribution is fully represented by the vector $\vec{\tau}$ and the matrix T, $(\vec{\tau}, T)$. The basic characteristics of PH distribution are:

* the cumulative distribution function: $F(x) = 1 - \vec{\tau} e^{Tx} \vec{e}$,

* the density function: $f(x) = \vec{\tau} e^{Tx} \left(-T\vec{e}\right)$,

* the n^{th} moment: $m_n = (-1)^n \, n! \vec{\tau} T^{-n} \vec{e}$.

The term e^{Tx} is defined as

$$e^{Tx} = \sum_{n \geq 0} \frac{1}{n!} (Tx)^n$$

Since the structure of the underlying Markov chain is arbitrary, a PH distribution covers a wide range of characteristics including high variability. The PH random variables are independently identically distributed, because the initial probability vector is the same, every time the Markov chain starts in its transient portion in the correspondent renewal process.

Considering a finite buffer single server queue, where the interarrivals times have the phase-type distribution $(\vec{\tau}, T)$ of dimension n and the mean λ^{-1}, and service times, which are also of the phase distribution $\left(\vec{\beta}, S\right)$ of dimension m with the mean μ^{-1}, and defining

$$\vec{t^0} = -T\vec{e}, \quad \vec{s}^0 = -S\vec{e},$$

the block matrices in (2.4) can be presented as:

$$B_0 = T, \quad C_0 = T \otimes \vec{s}^0, \quad A_0 = \left(\vec{t^0}\vec{\tau}\right) \otimes \vec{\beta},$$

$$A = \left(\vec{t^0}\vec{\tau}\right) \otimes S, \quad C = T \otimes \left(\vec{s}^0\vec{\beta}\right), \quad B = \left(\vec{t^0}\vec{\tau}\right) \otimes \left(\vec{s}^0\vec{\beta}\right) + T \otimes S,$$

$$B_m = B + A, \quad A_{m-1} = A, \quad C_m = C;$$

where $\otimes$ denotes the Kronecker product. In this case, the queuing model is said to be of the $PH/PH/1/K$ type, where K is the queue capacity.

2.5 Matrix Computations.

The modified matrix geometric technique includes such operations as solving a quadratic matrix equations, and determining the general group inverse. The current section provides a brief review of these topics. Also, the definition of reducible and irreducible matrices is given here.

For any square matrix A of the order n and rank r consider the following equations [18]:

$$
\begin{aligned}
AXA &= A, && (2.1)\\
XAX &= X, && (2.2)\\
AX - XA &= \mathbf{0}. && (2.3)
\end{aligned}
$$

If some matrix X satisfies the equations (2.1, 2.2, 2.3), then it is referred to as *general group inverse* and denoted as $A^{\#}$.

As one of the methods for obtaining the general group inverse, the general determinant representation is given here.

Denote the minor of A, containing $w_1, \ldots, w_t$ rows and $v_1, \ldots, v_t$ columns by

$$A\begin{pmatrix} w_1, \ldots, w_t \\ v_1, \ldots, v_t \end{pmatrix},$$

and the algebraic complement, corresponding to the element a_{ij} as

$$A_{ij}\begin{pmatrix} w_1, \ldots, w_{p-1}, i, w_{p+1}, \ldots, w_t \\ v_1, \ldots, v_{q-1}, j, v_{q+1}, \ldots, v_t \end{pmatrix} = (-1)^{p+q}\, A\begin{pmatrix} w_1, \ldots, w_{p-1}, w_{p+1}, \ldots, w_t \\ v_1, \ldots, v_{q-1}, v_{q+1}, \ldots, v_t \end{pmatrix}.$$

Then as follows from [18], the general group inverse of A, $A^{\#} = \left(a_{ij}^{\#}\right)$, exists if

$$u = \sum_{1 \le w_1 < \ldots < w_r \le n} A\begin{pmatrix} w_1, \ldots, w_r \\ w_1, \ldots, v_r \end{pmatrix} \neq 0,$$

and $A^{\#}$ has the following representation

$$a_{ij}^{\#} = \frac{1}{u^2} \sum_{1 \le w_1 < \ldots < w_r \le n} \; \sum_{1 \le v_1 < \ldots < v_r \le n} A\begin{pmatrix} v_1, \ldots, j, \ldots, v_r \\ w_1, \ldots, i, \ldots, w_r \end{pmatrix} A_{ij}\begin{pmatrix} w_1, \ldots, i, \ldots, w_r \\ v_1, \ldots, j, \ldots, v_r \end{pmatrix}.$$

There are several methods for solving the quadratic matrix equations. The Logarithmic Reduction algorithm is presented here, as it is believed to be the fastest one. Suppose, it is required to solve a left quadratic matrix equation

$$A + RB + R^2C = \mathbf{0}.$$

Note that sometimes it is needed to solve the right task, that is,

$$A + BR + CR^2 = \mathbf{0},$$

but it can be transformed to the left tack by means of transposing.
The following procedure, known as logarithmic reduction technique, gives the solution matrix R [15].

$$\begin{array}{l} S = B \\ V = A \\ T = C \\ W = B \\ do \\ \qquad \begin{array}{rcl} X &=& -S^{-1}V \\ Y &=& -S^{-1}T \\ Z &=& VY \\ W &=& W + Z \\ S &=& S + Z + TX \\ V &=& VX \\ T &=& TY \end{array} \\ while \quad (\|Z\|_\infty \geq \varepsilon) \\ R = -AW^{-1} \end{array}.$$

Fig. 2: Logarithmic Reduction algorithm.

A square $n \times n$ matrix $A = (a_{ij})$ is called *reducible* if the indices $1, 2, \ldots, n$ can be divided into two disjoint nonempty sets $i_1, i_2, \ldots, i_k$ and $j_1, j_2, \ldots, j_l$ with $k + l = n$, such that $a_{i_p j_q} = 0$, for $p = 1, 2, \ldots, k$, $q = 1, 2, \ldots, l$. A square matrix, which is not reducible, is said to be *irreducible.*

2.6 Matrix Geometric Solution.

This section is devoted to a brief overview of the matrix geometric solution technique for *QBD* processes.

The key element of the matrix geometric solution for the generator (2.4) is the assumption that the elements of the stationary state probability vector $\vec{\pi}_i$ follow the relation

$$\vec{\pi}_i = \vec{\pi}_1 R^{i-1}, \quad \forall i \geq 1,$$

where R is the solution of the matrix equation

$$A + RB + R^2 C = \mathbf{0}.$$

The above equation is obtained from the flow balance equations of the repeating portion of the process. The elements $\vec{\pi}_0$ and $\vec{\pi}_1$ are obtained from the boundary equations by substituting the matrix geometric relation. The whole vector is normalized using the normalizing condition.

3 Modified Matrix Geometric Solution.

The modified matrix-geometric solution was proposed by V. Naumov, and is known to be the only one, which can handle the case, when $\vec{\pi}A\vec{e} = \vec{\pi}C\vec{e}$. As it is a modification of a matrix-geometric technique, the method has the same accuracy and time complexity and it will be used as a basis for solving the tasks for more complex systems, consisting of several QBD processes.

The proofs of the theorems are omitted here, as they can easily be found from [21], and the goal of the section is to introduce the modified matrix geometric solution for a single-process system with the point to extend it later to a more general case.

3.1 General Notes.

Let's consider a finite QBD process with $m+1$ levels and a generator matrix

$$Q = \begin{pmatrix} B_0 & A & & & \\ C_0 & B & \ddots & & \\ & C & \ddots & A & \\ & & \ddots & B & A_{m-1} \\ & & & C & B_m \end{pmatrix},$$

where A, B, C are all $n \times n$ matrices. Suppose that $H(z)$ is defined as

$$H(Z) = A + BZ + CZ^2$$

and assume that

a) the generator matrix $H(1)$ is irreducible,
b) $\det(H(Z)) \neq 0$ for some $Z = \tilde{Z}$.

Let F and G be the minimal nonnegative solutions of matrix quadratic equations

$$A + BF + CF^2 = \mathbf{0}, \quad AG^2 + BG + C = \mathbf{0}, \tag{3.1}$$

and define matrix Φ as

$$\Phi = AG + B + CF. \tag{3.2}$$

Denote $\vec{\pi}$ the stationary probability vector of $H(1)$.

Let V and W be the minimal nonnegative solutions of the equations

$$V = -\left(B + AVC\right)^{-1}, \quad W = -\left(B + VWA\right)^{-1}. \tag{3.3}$$

As shown in [21], the solutions of (3.3) always exist.
It is proved in [14] that for any solution $\vec{x}_0, \ldots, \vec{x}_m$ of linear system, which represents a difference equation for a QBD process

$$\vec{x}_{k-1}A + \vec{x}_k B + \vec{x}_{k+1}C = \mathbf{0}, \quad 0 < i < m, \tag{3.4}$$

there exist vectors

$$\vec{f} = \vec{x}_0\left(B + CWA\right) + \vec{x}_1 C, \quad \vec{g} = \vec{x}_m\left(B + AVC\right) + \vec{x}_{m-1}A, \tag{3.5}$$

such that $\vec{x}_0, \ldots, \vec{x}_m$ satisfy a linear system

$$\vec{x}_k \Phi = \vec{f} F^k + \vec{g}\, G^{m-k}, \quad 0 \le k \le m. \tag{3.6}$$

The problem is to find appropriate vectors $\vec{f}$ and $\vec{g}$, when vectors $\vec{x}_0, \ldots, \vec{x}_m$ are unknown, and to find solution $\vec{x}_0, \ldots, \vec{x}_m$ of (3.6), which is the solution of (3.4).

3.2 Nonsingular matrix Φ

If matrix Φ is nonsingular, then vectors $\vec{x}_0, \ldots, \vec{x}_m$ satisfy linear system (3.4) if and only if for some vectors $\vec{f}$ and $\vec{g}$ they could be expressed in a modified matrix-geometric form as

$$\vec{x}_k = \left(\vec{f} F^k + \vec{g}\, G^{m-k}\right)\Phi^{-1}, \quad 0 \le k \le m. \tag{3.7}$$

This result was obtained in [21]. It is also proved there, that vectors $\vec{f}$ and $\vec{g}$ that can be used in (3.7) are unique for any solution $\vec{x}_0, \ldots, \vec{x}_m$ of linear system (3.4) and are given by (3.5).

In practice, the vectors $\vec{f}$ and $\vec{g}$ are computed from the boundary conditions (2.6) and the normalizing condition by substituting all the $\vec{x}_k$ in the form of (3.7). The resulting system will have a unique solution, and then all the $\vec{x}_k$ should be obtained directly from (3.7).

3.3 Singular matrix Φ

Here the matrix Φ is assumed to be singular and irreducible. It is possible [21] if only and only if $\vec{\pi}A\vec{e} = \vec{\pi}C\vec{e}$. In this case matrices F and G are stochastic, matrix Φ is a generator matrix and $\vec{\pi}$ is its stationary probability vector. Condition a implies that $\vec{\pi}A\vec{e} = \vec{\pi}C\vec{e} \neq 0$.
Let's denote $\Phi^{\#}$ generalized group inverse of Φ. Since Φ is irreducible,

$$\Phi^{\#}\Phi = \Phi\Phi^{\#} = I - \vec{e}\vec{\pi}.$$

In the depicted case, vectors $\vec{f}$ and $\vec{g}$ in (3.5) satisfy the equality

$$\vec{f}\vec{e} + \vec{g}\vec{e} = 0. \tag{3.8}$$

As it is proved in [21], the following proposition is valid.
Proposition. Let matrix Φ be singular and irreducible, then vectors $\vec{x}_0, \ldots, \vec{x}_m$ satisfy linear system (3.4) if and only if for some vectors $\vec{f}$ and $\vec{g}$ satisfying (3.8) they could be expressed as

$$\vec{x}_k = \vec{y}_k + (\vec{\pi}C\vec{e})^{-1}\, h_k\vec{\pi}, \tag{3.9}$$

with vectors $\vec{y}_k$ defined by

$$\vec{y}_k = \left(\vec{f}F^k + \vec{g}\,G^{m-k}\right)\Phi^{\#}, \quad 0 \leq k \leq m, \tag{3.10}$$

and constants h_k satisfying equalities

$$h_k - h_{k-1} = \vec{f}\vec{e} - \vec{y}_kC\vec{e} + \vec{y}_{k-1}A\vec{e}, \quad 0 < k <\leq m. \tag{3.11}$$

Vectors $\vec{f}$, $\vec{g}$ and constants h_k are unique for any solution of linear system (3.4).

The constants h_k can also be expressed as

$$h_k = h_0 + k\left(\vec{f}\vec{e}\right) + \sum_{j=0}^{k-1} \vec{f}F^j\vec{\phi} - \sum_{j=m-k}^{m-1} \vec{g}G^j\vec{\gamma}, \quad 0 \le k \le m, \tag{3.12}$$

where the column vectors $\vec{\phi}$ and $\vec{\gamma}$ are defined by

$$\vec{\phi} = \Phi^{\#}A\vec{e} - F\Phi^{\#}C\vec{e}, \quad \vec{\gamma} = \Phi^{\#}C\vec{e} - G\Phi^{\#}A\vec{e}. \tag{3.13}$$

Just as in nonsingular case, the vectors $\vec{x}_k$ are obtained from the boundary conditions (2.6) along with the normalizing condition by means of substituting (3.9) and solving the resulting system in terms of $\vec{f}, \vec{g}$ and h_0.

4 Multiple Processes Queuing System.

Following the main scope of the paper, let's turn to more complex queuing systems, which can not be described as a single QBD process, but it is possible to represent them as a set of different processes, which have the quasi birth death structure. The main goal of the section is to obtain the steady state probability vector of the whole system by means of applying the modified matrix geometric solution to each of the QBD process within the system.

Fig. 3: Finite Homogeneous QBD process

The figure (3) represents the same as the figure 1, namely, one QBD process of the length $m + 1$. That is, we'll plot only the boundary states of each process in order to simplify a picture of a whole system. Also, since we assume all the processes to be level-independent, it is not necessary to draw the repeating parts of them.

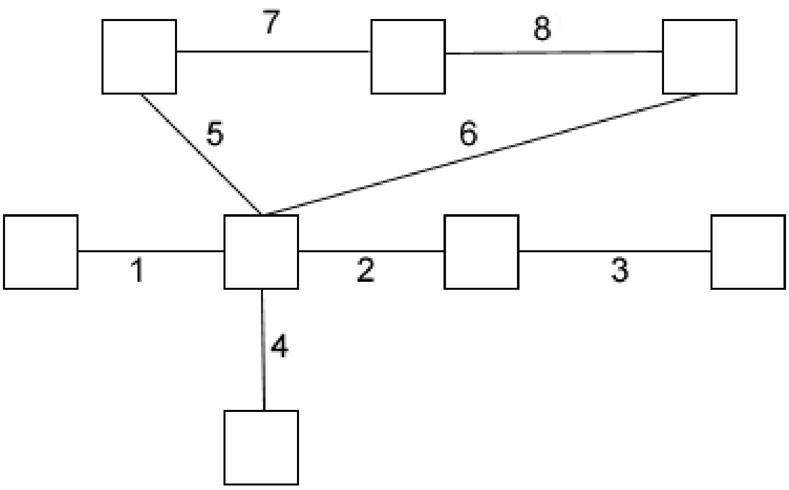

Fig. 4: Example of a queuing system.

The figure (4) presents an example of a queuing system that consists of eight QBD processes, which, in turn, are somehow related to each other, but have different matrices A, B, C and different dimensions, that is the number of phases and levels of different processes may be not the same. The boxes on the figure are not marked with the levels numbers, because the intersections of the adjacent QBD processes can hardly be assigned to one of them.

Assume also that the transitions in the queuing system are allowed only between the boundary states of different QBD processes and within each process as described in the section 2.3. That is, the queuing system can be regarded as a network, in which each node can be represented as a queue. Suppose also, that there is a single arrival stream for the whole system and a single absorbing state.

The high level idea is to split the state space of the whole system into two subsets, namely, the boundary states and the inner states. As it was described in the section 3, the matrix geometric terms can be computed from the boundary equations, and then, all the other components can be obtained from the matrix geometric relation. Such approach can result in the significant benefit in computational cost, because the linear system of boundary equations has the order, equal to the dimension of the boundary state space, which is essentially less, than the one of the systems state space. Following this idea, the next several sections are devoted to the definition of the boundary state space and to derivation of the boundary equations, as they are the key element of the technique. Also, certain modifications are introduced to the technique, described in the section 3, so that it would be possible to apply it to a multi-processes system. The modifications concern mainly handling the regular and the singular cases of the QBD processes using the common representation of the steady state vector. This is done due to the fact, that it is not specified in advance, how many and which exactly processes will be singular or regular, and it is unlikely to separate these case, as was done to the single-process system, because we intend to obtain the representation of a steady state vector for the whole system. And as the processes within the system are somehow related to each other, they should be treated in the same manner.

4.1 Queuing Model.

In our study we consider queuing system with the following properties. The arrival stream is a superposition of r independent arrival processes. The service consists of

m subsystems, $m \geq r$, with state independent phase-type distributed service times. Each of them is represented as a finite homogeneous *QBD* process with an irreducible infinitesimal generator $Q^{(i)},\ i = 1..m$. The customers can arrive to any of r subsystem, and if it is idle, the customer occupies it for a random service period or is redirected to another subsystem. If the subsystem is busy, the arriving customer joins the waiting line. If no waiting positions are available, the arriving customer is either lost and has no further impact on the system, either is served by another subsystem.

Each *QBD* process, that is, subsystem, is defined by its length $m^{(i)}$, the number of phases $n^{(i)}$, the inner transition rate matrices $A^{(i)},\ B^{(i)},\ C^{(i)}$ and the boundary matrices $A_0^{(i)}$, $B_0^{(i)}$, $C_0^{(i)}$, $A_{m^{(i)}-1}^{(i)}$, $B_{m^{(i)}}^{(i)}$, $C_{m^{(i)}}^{(i)}$, $i = 1..m$.

The state space Ψ of the system can be represented as a triple

$$\Psi = \{(i, k, l)\ \mid i = 1, \ldots, m,\ k = 0..m^{(i)},\ l = 0, \ldots, n^{(i)}\}.$$

That is, each state is described by the serving subsystem, the length of queue in this subsystem and the phase, of the just served customer. The states are ordered lexicographically.

Let's divide Ψ into two subsets
$\Lambda^{(i)} = \{(j, l)\ \mid j = 0, m^{(i)},\ l = 1, \ldots, n^{(i)}\}$, $i = 1, \ldots, m$.
$\Theta^{(i)} = \{(k, l)\ \mid k = 1, \ldots, m^{(i)} - 1,\ l = 1, \ldots, n^{(i)}\}$, $i = 1, \ldots, m$,

then $\bigcup\limits_{i=1}^{m} \Lambda^{(i)}$ represent the boundary states, and $\bigcup\limits_{i=1}^{m} \Theta^{(i)}$ stands for the inner states of the system. Evidently, $\Psi = \bigcup\limits_{i=1}^{m} \Psi^{(i)}$, where $\Psi^{(i)} = \Lambda^{(i)} \bigcup \Theta^{(i)}$, and $\Lambda^{(i)} \bigcap \Theta^{(i)} = \varnothing$. $\Psi^{(i)}$ is the state space for each subsystem. Using such representation, the whole system can be described by its boundary states $\Lambda^{(i)}$ and the inner states $\Theta^{(i)}$.

Let $\vec{p}$ denote the steady state probability vector of the queuing system. Define the vectors $\vec{p}_k^{(i)},\ k = 0, \ldots, m^{(i)},\ i = 1, \ldots, m$, as components of the partitioned steady state vector $\vec{p}$. Note that $\vec{p}_0^{(i)}$ and $\vec{p}_{m^{(i)}}^{(i)}$ stand for the probabilities of being in the states of $\Lambda^{(i)}$, and the others stand for $\Theta^{(i)}$.

Define also p_{00} as the probability of that the whole system is in idle state, $p_{00} = 1 - \sum_{i=1}^{m} \sum_{k=0}^{m^{(i)}} \vec{x}_k^{(i)} \vec{e}^{(i)}$. Then the steady state probability vector $\vec{p}$ can be written as $\vec{p} = \left(p_{00}, \vec{p}_k^{(i)} \right)$, where $\vec{p}_k^{(i)}$ are ordered lexicographically too. The vectors $\vec{p}_0^{(i)}$ in such representation are of the size $(1 \times n^{(i)})$, as they do not include the idle states probabilities of the subsystems. Such changing of variables can be regarded as the nonlinear transform to the lower dimensional state space, namely, one idle state of the system is introduced instead of m states.

To construct the generator matrix of a queuing system, we need to define all its components. As it was mentioned, we divide the system into the boundary and the inner states and introduced the systems idle state. Now let's turn to the detailed description of these notations.

4.1.1 The Boundary States.

Let's discuss a bit about the matrices $B_0^{(i)}$, $A_0^{(i)}$, $C_0^{(i)}$, $B_{m^{(i)}}^{(i)}$, $A_{m^{(i)}-1}^{(i)}$, $C_{m^{(i)}}^{(i)}$. It was stated, that the arrival stream of the system is supposed to be a superposition of r independent arrivals, while the system consists of $m \geq r$ subsystems. The arrivals can be represented as the transitions from the idle state to the zeroth states, and these transitions in our representation are stored in the first row of $B_0^{(i)}$. Subsequently, the first column of $B_0^{(i)}$ contains the transition rates to the absorbing state. Since only r of m matrices $B_0^{(i)}$ have a nonzero first row, it is good to present another notation for B_0 in order to avoid the case, when a generator matrix of the system will not be irreducible. To achieve this, let's split the matrices $B_0^{(i)}$, $A_0^{(i)}$, $C_0^{(i)}$ in the following way

$$B_0^{(i)} = \begin{pmatrix} b_0^{(i)} & \vec{b}_1^{(i)} \\ \vec{b}_2^{(i)} & \underline{B_0}^{(i)} \end{pmatrix}, \qquad A_0^{(i)} = \begin{pmatrix} \mathbf{0} \\ A^{(i)} \end{pmatrix}, \qquad C_0^{(i)} = \begin{pmatrix} \mathbf{0} & \underline{C_0}^{(i)} \end{pmatrix}, \tag{4.1}$$

where zeros are the zero vectors of the appropriate size. The vectors $\vec{b}_1^{(i)}$ and $\vec{b}_2^{(i)}$ represent the first row and the first column of $B_0^{(i)}$ starting from the second element. The scalar $b_0^{(i)}$ is the first element of the matrix $B_0^{(i)}$. The square matrix $\underline{B_0}^{(i)}$

represents the transitions within the zeroth level and has the order of $n^{(i)}$. As it was mentioned, $\vec{b}_1^{(i)}$ and $\vec{b}_2^{(i)}$ describe the arrival process and the transitions to the absorbing state of each subsystem. As it was stated above, we consider a single idle for the system, so a set of the vectors $\vec{b}_1^{(i)}$ and $\vec{b}_2^{(i)}$ can be regarded as the description of the arrival process of the whole system. In the rest of the paper $\underline{B_0}^{(i)}$ will be denoted as $B_0{}^{(i)}$, and $\underline{C_0}^{(i)}$ will be denoted as $C_0{}^{(i)}$.

Such representation is used to construct an irreducible generator matrix of the whole system. The probability p_{00} was introduced with the same purpose. The point is, that in such representation we will deal with an idle state of the whole system, but not with ones of each subsystem. This will allow us to make the formulas for the steady state probabilities a bit more simple, because of the fact that there won't be any necessity to deal with the vectors of different length within each process.

Now let's turn to the discussion about the boundary states themselves. In order to make it more clear, at first let's consider a simple example. Assume that a queuing system consists only of two *QBD* processes, for example the first and the second ones from the figure 4. It this case there will be three boundary states. It is clear that they will be described by the following matrices

$$\left(B_0^{(1)}\right), \qquad \begin{pmatrix} B_{m^{(1)}}^{(1)} & P_{(12)} \\ P_{(21)} & B_0^{(2)} \end{pmatrix}, \qquad \left(B_{m^{(2)}}^{(2)}\right), \tag{4.2}$$

where $P_{(12)}$ and $P_{(21)}$ are of the orders $n^{(1)} \times n^{(2)}$ and $n^{(2)} \times n^{(1)}$ correspondingly. Therefore, the second boundary matrix in (4.2) is square of the order $\left(n^{(1)} + n^{(2)}\right)$. It is clear that if the whole system consists of a sequence of *QBD* processes, its infinitesimal generator can be represented in the same manner as in (4.2), namely, as a direct sum of the generator matrices of each *QBD* process, that is $Q = Q^{(1)} \oplus \ldots \oplus Q^{(m)}$ [15]. But in a more general case, the transitions between the *QBD* processes are possible not only between the last and the zeroth levels, but between any states in $\bigcup_{i=1}^{m} \Lambda^{(i)}$. To handle such situation, let's introduce the transition rate matrices, which carry the information about the transitions between the processes within a system:

M_{ij} – from state $(0,k)$ of the i^{th} process to the state $(0,h)$ of the j^{th}, where $k = 0,\dots,n^{(i)}$, $h = 0,\dots,n^{(j)}$, $i,j = 1,\dots,m$,$i \neq j$;

L_{ij} – from state $(0,k)$ of the i^{th} process to the state $(m^{(j)},h)$ of the j^{th}, where $k = 0,\dots,n^{(i)}$, $h = 0,\dots,n^{(j)}$, $i,j = 1,\dots,m$, $i \neq j$;

N_{ij} – from state $(m^{(i)},k)$ of the i^{th} process to the state $(0,h)$ of the j^{th}, where $k = 0,\dots,n^{(i)}$, $h = 0,\dots,n^{(j)}$, $i,j = 1,\dots,m$, $i \neq j$;

K_{ij} – from state $(m^{(i)},k)$ of the i^{th} process to the state $(m^{(j)},h)$ of the j^{th}, where $k = 0,\dots,n^{(i)}$, $h = 0,\dots,n^{(j)}$, $i,j = 1,\dots,m$, $i \neq j$.

It is clear, that the infinitesimal generator of a *QBD* process within a system, contains the matrices M_{ij}, L_{ij}, N_{ij}, K_{ij} as certain blocks of $B_0^{(i)}$ and $B_{m^{(i)}}^{(i)}$. But it is still more convenient to separate the processes from the transitions between them and introduce the rules for constructing, if needed, the boundary matrices.

In such representation, the boundary levels of each process are described by the matrices $B_0^{(i)}$, $C_0^{(i)}$, $A_{m^{(i)}-1}^{(i)}$, $B_{m^{(i)}}^{(i)}$, $i = 1..m$. The arriving process is additionally specified by $b_0^{(i)}$, $\vec{b}_1^{(i)}$, $\vec{b}_2^{(i)}$ .To describe the whole system let's add to the list the matrices M_{ij}, L_{ij}, N_{ij}, K_{ij}. Using these eight matrices, two vectors and one scalar, it is possible to specify the transitions, that is the boundary states, in any queuing system with the described properties.

4.1.2 The Inner States.

Each of the *QBD* processes in a queuing system may have its own characteristics, that is, the matrices $A^{(i)}$, $B^{(i)}$, $C^{(i)}$ and the lengths $m^{(i)}$, $i = 1..m$, are not the same for different processes. The point is that they represent a repeating portion of the length $m^{(i)} - 2$ in each process, and the transitions to the other processes from the inner states are not allowed. That is, the matrices $A^{(i)}$, $B^{(i)}$, $C^{(i)}$ depict the transition rates within $\Theta^{(i)}$.

As the modified matrix geometric solution will be used for obtaining the steady state probability vector, the same assumptions as in the single-process case should be done.

Suppose that $H^{(i)}(z)$ is defined as

$$H^{(i)}(z) = A^{(i)} + B^{(i)}z + C^{(i)}z^2$$

and assume that for all $i = 1..m$

a) the generator matrix $H^{(i)}(1)$ is irreducible,

b) $\det\big(H^{(i)}(z)\big) \neq 0$ for some $z = \tilde{z}$.

Let $F_{(i)}$ and $G_{(i)}$, $i = 1, \ldots, m$ be the minimal nonnegative solutions of matrix quadratic equations

$$A^{(i)} + B^{(i)}F_{(i)} + C^{(i)}F_{(i)}^2 = \mathbf{0}, \quad A^{(i)}G_{(i)}^2 + B^{(i)}G_{(i)} + C^{(i)} = \mathbf{0}, \tag{4.3}$$

and define matrix $\Phi_{(i)}$ as

$$\Phi = A^{(i)}G_{(i)} + B^{(i)} + C^{(i)}F_{(i)}. \tag{4.4}$$

Denote $\vec{\pi}^{(i)}$, $i = 1, \ldots, m$ the stationary probability vector of $H^{(i)}(1)$.

Just as in the case of a single process, the following is valid for each $i = 1, \ldots, m$:

1. There exist minimal nonnegative solutions $V_{(i)}$, $W_{(i)}$ of the matrix equations:

$$\begin{aligned} V_{(i)} &= -\left(B^{(i)} + A^{(i)}V_{(i)}C^{(i)}\right)^{(-1)} \\ W_{(i)} &= -\left(B^{(i)} + C^{(i)}W_{(i)}A^{(i)}\right)^{(-1)} \end{aligned}. \tag{4.5}$$

2. The matrices $F_{(i)} = W_{(i)}A^{(i)}$ and $G_{(i)} = V_{(i)}C^{(i)}$ are the minimal nonnegative solutions of the matrix quadratic equations (4.3).

3. $\mathrm{tr}\big(F_{(i)}\big) = 1$ if and only if $\vec{\pi}^{(i)}A^{(i)}\vec{e}^{(i)} \geq \vec{\pi}^{(i)}C^{(i)}\vec{e}^{(i)}$. In this case $F_{(i)}$ is stochastic.

4. $\mathrm{tr}\big(G_{(i)}\big) = 1$ if and only if $\vec{\pi}^{(i)}A^{(i)}\vec{e}^{(i)} \leq \vec{\pi}^{(i)}C^{(i)}\vec{e}^{(i)}$. In this case $G_{(i)}$ is stochastic.
5. The irreducible matrix $\Phi_{(i)}$ is singular if and only if $\vec{\pi}^{(i)}A^{(i)}\vec{e}^{(i)} = \vec{\pi}^{(i)}C^{(i)}\vec{e}^{(i)}$. In this case $\vec{\pi}^{(i)}$ is a stationary probability vector of $\Phi_{(i)}$.

The exact expression for the steady state probability vector of each process is given in the section 4.2.1.

4.1.3 Generator Matrix.

After all the required designations were introduced, the generator matrix of a whole system can be presented. In general, it can be viewed as the consequent description of the transition rates between the states of $\Lambda^{(i)}$ and $\Theta^{(i)}$, ordered lexicographically.

As we consider each subsystem to be of a *QBD* structure, the transition rates within each process, that is, between the states in $\Theta^{(i)}$ can be gathered into the following matrix

$$T_{\text{in}}^{(i)} = \begin{pmatrix} & 0 & \\ A^{(i)} & & \\ B^{(i)} & \ddots & \\ C^{(i)} & \ddots & A^{(i)} \\ & \ddots & B^{(i)} \\ & & C^{(i)} \\ & 0 & \end{pmatrix}, \tag{4.6}$$

of the size $\left(1 + \sum_{i=1}^{m} n^{(i)} \left(m^{(i)} + 1\right)\right) \times n^{(i)} \left(m^{(i)} - 1\right)$, and the first nonzero row starts at the line next to $\left(1 + \sum_{j=1}^{i-1} n^{(j)} \left(m^{(j)} + 1\right)\right)$.

The transition rates within the boundary states, that is, in $\Lambda^{(i)}$, can also be described in a similar manner, but, as we consider a single idle state, one column for it should be added, which represents the transition rates to the systems absorbing state. We examine only the transitions into the i^{th} level in each $\Lambda^{(i)}$, because all the nonzero transitions from one level can be regarded as the transitions to another level, so the structure of columns of the generator matrix is examined, that is the length of the repeating portion in each process is determined as the number of repeating columns, but not rows. The matrices $A_0^{(i)}$ and $C_{m^{(i)}}^{(i)}$ have the structure (4.1), also only due to this fact.

$$P_M = \begin{pmatrix} \sum_{i=1}^{m} b_0^{(i)} \\ \vdots \\ 0 \\ \vec{b}_2^{(i-2)} \\ 0 \\ \\ \vec{b}_2^{(i)} \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vec{b}_2^{(i+1)} \\ 0 \\ 0 \\ \vec{b}_2^{(i+2)} \\ \vdots \end{pmatrix}, T_{1\mathrm{b}}^{(i)} = \begin{pmatrix} \vec{b}_1^{(i)} \\ \vdots \\ N_{i-2,i} \\ M_{i-1,i} \\ 0 \\ N_{i-1,i} \\ B_0^{(i)} \\ C_0^{(i)} \\ 0 \\ \vdots \\ 0 \\ M_{i+1,i} \\ 0 \\ N_{i+1,i} \\ M_{i+2,i} \\ \vdots \end{pmatrix}, T_{2\mathrm{b}}^{(i)} = \begin{pmatrix} 0 \\ \vdots \\ K_{i-2,i} \\ L_{i-1,i} \\ 0 \\ K_{i-1,i} \\ 0 \\ \vdots \\ 0 \\ A_{m^{(i)}-1}^{(i)} \\ B_{m^{(i)}}^{(i)} \\ L_{i+1,i} \\ 0 \\ K_{i+1,i} \\ L_{i+2,i} \\ \vdots \end{pmatrix}, \tag{4.7}$$

where P_M is of the size $\left(1 + \sum_{i=1}^{m} n^{(i)} \left(m^{(i)} + 1\right)\right) \times 1$, and the matrices $T_{1\mathrm{b}}^{(i)}$ and $T_{2\mathrm{b}}^{(i)}$ are $\left(1 + \sum_{i=1}^{m} n^{(i)} \left(m^{(i)} + 1\right)\right) \times n^{(i)}$. The M_{ji} and L_{ji} stand on the lines staring from the same one, as the first $B_0^{(j)}$, and N_{ji} with K_{ji} stand on the lines staring from the same one, as the $B_{m^(j)}^{(j)}$.

The matrix $T_{1\mathrm{b}}^{(i)}$ and $T_{2\mathrm{b}}^{(i)}$ contain the transition rates between the states of $\Lambda^{(i)}$ and $\Lambda^{(k)},\ i, k = 1, \ldots, m,\ i \neq k$.

The introduced matrices present the block columns of the generator matrix, which now can be written as a block row:

$$Q = \begin{pmatrix} P_M & T_{1b}^{(1)} & T_{in}^{(1)} & T_{2b}^{(1)} & T_{1b}^{(2)} & T_{in}^{(2)} & T_{2b}^{(2)} & \dots & T_{1b}^{(m)} & T_{in}^{(m)} & T_{2b}^{(m)} \end{pmatrix}. \quad (4.8)$$

It is easy to notice, that Q is a square matrix of the size $\left(1 + \sum_{i=1}^{m} n^{(i)} \left(m^{(i)} + 1\right)\right)$.

The first row of Q represents the arrival process, and the first column stands for the transitions to the absorbing state.

The matrices $T_{1b}^{(i)}$ and $T_{2b}^{(i)}$ represent the boundary states of the whole system and can be considered as a one column $\left(T_{2b}^{(i)}\ T_{1b}^{(i)}\right)$. They are separated here, because of the two main reasons, namely, to depict the boundary states for $i = 1$ and $i = m$, where we have only $T_{1b}^{(1)}$ and $T_{2b}^{(m)}$ respectively without any difficulties. And the second reason, is that throughout the paper, the structure of the whole queuing system was examined according to the generators columns, but not rows or diagonals, although such approaches are also possible, but they lead to certain difficulties with constructing of the generator matrix.

In terms of the generator matrix, the separation of $B_0^{(i)}$ into four parts can be regarded as summarizing the corresponding rows and columns of Q and putting them to the first row and column respectively. Such procedure can spoil nothing, but the benefit is that, the m rows and columns were replaced with a single row and column. One can notice, that the generator matrix (4.8) now does not have a block tridiagonal structure, but it even didn't have one, because any transitions between the processes were allowed, and the corresponding transition rate matrices were separated from B_0 and B_m.

The generator matrix Q should have a zero row sum, that is, $Q\vec{e} = \mathbf{0}$, therefore, according to (4.7) and (4.6),the following rules can be obtained:

$$\begin{cases} C_0^{(i)}\vec{e} + B^{(i)}\vec{e} + A^{(i)}\vec{e} = \mathbf{0}, \\ C^{(i)}\vec{e} + B^{(i)}\vec{e} + A^{(i)}\vec{e} = \mathbf{0}, \\ C^{(i)}\vec{e} + B^{(i)}\vec{e} + A^{(i)}_{m^{(i)}-1}\vec{e} = \mathbf{0}, \\ B_0^{(i)}\vec{e} + A_0^{(i)}\vec{e} + \sum_{j=1, j\neq i}^{m} (M_{ij}\vec{e} + L_{ij}\vec{e}) = -\vec{b}_2^{(i)}, \\ B^{(i)}_{m^{(i)}}\vec{e} + C^{(i)}_{m^{(i)}}\vec{e} + \sum_{j=1, j\neq i}^{m} (N_{ij}\vec{e} + K_{ij}\vec{e}) = \mathbf{0}, \end{cases} \quad i = 1..m, \tag{4.9}$$

where $\vec{e}$ is a column vector of all ones and $\mathbf{0}$ is a column zero vector of the appropriate size.

4.2 The Steady State Distribution.

In this section we propose a procedure to calculate the steady state distribution vector $\vec{p}$ of the queuing system. It is determined by the generator matrix Q in (4.8) as unique nonnegative solution of the linear system

$$\vec{p}Q = \mathbf{0}, \tag{4.10}$$

subject to the normalizing condition

$$\vec{p}\vec{e} = 1. \tag{4.11}$$

Using (4.8) the system (4.10) can be rewritten for $i = 1, \ldots, m$ as

$$\vec{p}T_{\text{in}}^{(i)} = \mathbf{0} \tag{4.12}$$

$$\vec{p}P_M = \mathbf{0} \tag{4.13}$$

$$\vec{p}T_{\text{1b}}^{(i)} = \mathbf{0} \tag{4.14}$$

$$\vec{p}T_{\text{2b}}^{(i)} = \mathbf{0} \tag{4.15}$$

Our goal is to show that the components of $\vec{p}$, just as in the case of a single *QBD* process (3.6), can be represented by linear combinations of matrix geometric terms.

According to our representation of the state space Ψ of the queuing system, the linear system (4.12 – 4.15) can be also divided into two linear systems, namely, one, which contains the steady state probabilities of $\Theta^{(i)}$ and the other for $\Lambda^{(i)}$ for $i = 1, \ldots, m$. The former one does not need to be solved directly, because the probabilities of being in the inner states of the system are expressed by matrix geometric terms. The latter one represents the boundary equations and is used for determining the matrix geometric terms. Therefore, the size of a system to be solved is reduced from the size of Q to $\left(1 + 2\sum_{i=1}^{m} n^{(i)}\right)$ that is, to the size of the boundary states. So, the system to be solved is formed by the equations (4.13 – 4.15) and is defined by the matrices $P_M,\ T_{1\mathrm{b}}^{(i)},\ T_{2\mathrm{b}}^{(i)}$.

4.2.1 Matrix Geometric Form.

Suppose that the queuing system under consideration consists of d processes, for which the matrix $\Phi^{(i)}$, defined as in (4.4) is regular, and q processes, for which $\Phi^{(i)}$ is singular, with $d+q = m$. As we intend to find the steady state distribution of the whole system, we should not totally separate the singular and regular cases, namely, it would be good to obtain a representation of the steady state probabilities by the matrix geometric terms, which could be applied to both cases. That is, let's rewrite the expressions (3.7) and (3.9), taking into account, that now a considered queuing model consists of several processes.

The matrices $F_{(i)},\ G_{(i)},\ \Phi_{(i)}$ are as they are defined in (4.3) and (4.4) respectively. We'll use the same technique as for a single process case, described in section 3. The goal of the current section is to combine the results, obtained in 3.2 and 3.3 into one expression for the elements of the steady state probability vector, so that there won't be any necessity to separate the processes into the singular and regular ones. This can be done, because, as it can be seen from (3.7) and (3.9), the regular and singular cases differ from each other by a linear term.

The matrices $\Phi_{(i)}$ are assumed to be irreducible. Let's introduce the following designations:

$$
\begin{aligned}
q_{(i)} &= \left(\vec{\pi}^{(i)} C^{(i)} \vec{e}^{(i)}\right)^{-1}, \\
y_{(i)} &= \left[\begin{array}{ll} 0 & \text{if} \quad \det \Phi_{(i)} \neq 0, \\ 1 & \text{if} \quad \det \Phi_{(i)} = 0. \end{array}\right., \\
\Phi_{(i)}^{\perp} &= \left[\begin{array}{ll} \Phi_{(i)}^{-1} & \text{if} \quad \det \Phi_{(i)} \neq 0, \\ \Phi_{(i)}^{\#} & \text{if} \quad \det \Phi_{(i)} = 0 \end{array}\right.,
\end{aligned}
\tag{4.16}
$$

$$
\vec{\phi}_{(i)} = \Phi_{(i)}^{\#} A^{(i)} \vec{e}_{(i)} - F_{(i)} \Phi_{(i)}^{\#} C^{(i)} \vec{e}^{(i)}, \quad \vec{\gamma}_{(i)} = \Phi_{(i)}^{\#} C^{(i)} \vec{e}^{(i)} - G_{(i)} \Phi_{(i)}^{\#} A^{(i)} \vec{e}_{(i)}.
$$

Let's formulate the results as a combination of two theorems, proposed and proved in [21] for a single QBD process case.

Proposition. Let the matrices $\Phi_{(i)}$ be irreducible. Then the components of the steady state probability vector $\vec{p}$ satisfy the linear system 4.12 if and only if for some vectors $\vec{f}^{(i)}$ and $\vec{g}^{(i)}$, such that, if $\Phi_{(i)}$ is singular, then $\vec{f}^{(i)} \vec{e}^{(i)} + \vec{g}^{(i)} \vec{e}^{(i)} = 0$, and constants $h_0^{(i)}$ they could be expressed as

$$
\vec{p}_k^{(i)} = \left(\vec{f}^{(i)} F_{(i)}^k + \vec{g}^{(i)} G_{(i)}^{m^{(i)}-k}\right) \Phi_{(i)}^{\perp} + y_{(i)} q_{(i)} h_k^{(i)} \vec{\pi}^{(i)}, \quad \left[\begin{array}{l} k = 0, \ldots, m^{(i)}; \\ i = 1, \ldots, m. \end{array}\right., \tag{4.17}
$$

where

$$
h_k^{(i)} = h_0^{(i)} + k\left(\vec{f}^{(i)} \vec{e}^{(i)}\right) + \sum_{j=0}^{k-1} \vec{f} F_{(i)}^j \vec{\phi}_{(i)} - \sum_{m^{(i)}-k}^{m^{(i)}-1} \vec{g} G_{(i)}^j \vec{\gamma}_{(i)}. \tag{4.18}
$$

4.3 Matrix Geometric Solution.

4.3.1 General Representation.

After the general representation of the components of the steady state probability vector has been defined, we can turn to the solution procedure itself.

The solution of the linear system (4.10) will be obtained in two main steps, namely, determining the matrix geometric terms from the boundary conditions (4.14, 4.15, 4.13), and then calculating all the other components of the steady state probability vector using (4.17).

The system (4.14, 4.15, 4.13) can be rewritten in the following way. Assume

$$T = \left(P_M,\ T^{(1)}_{1\mathrm{b}},\ T^{(1)}_{2\mathrm{b}},\ T^{(2)}_{1\mathrm{b}},\ T^{(2)}_{2\mathrm{b}}, \ldots, T^{(m)}_{1\mathrm{b}},\ T^{(m)}_{2\mathrm{b}}\right).$$

Then the linear system of the boundary equations can be presented as

$$\vec{p}T = \mathbf{0}. \tag{4.19}$$

Denote H_{ij} as the elements of the partition of columns of T according to the partition of $\vec{p}$, then T will have the following structure:

$$T = \begin{pmatrix} H_{0,0} \\ H_{0,1} \\ \vdots \\ H_{m^{(1)},1} \\ H_{0,2} \\ \vdots \\ H_{m^{(m)},m} \end{pmatrix}.$$

In such notation the matrix H_{00} presents the arrival process, and $H_{0,i}$, $H_{m^{(i)},i}$ present the transitions from the boundary states of the i^{th} process to the absorbing state and to the other boundary states.

Then the boundary conditions can now be represented as

$$\vec{p}T = p_{00}\, H_{0,0} + \sum_{i=1}^{m} \sum_{k=0}^{m^{(i)}} \vec{x}^{(i)}_k H_{k,i} = 0. \tag{4.20}$$

Note that the system (4.20) contains only the probabilities of being in the boundary states of the processes, or more precisely, all the other components of the steady state probability vector have the zero coefficients, as the correspondent $H_{k,i}$ are the zero matrices.

Substituting (4.17) into (4.20) yields the linear system

$$p_{00}\, H_{0,0} + \sum_{i=1}^{m} \left[\vec{f}^{(i)} \sum_{k=0}^{m^{(i)}} F_{(i)}^{k} \Phi_{(i)}^{\perp} H_{k,i} + \right.$$
$$\left. + \vec{g}^{(i)} \sum_{k=0}^{m^{(i)}} G_{(i)}^{m^{(i)}-k} \Phi_{(i)}^{\perp} H_{k,i} + y_{(i)} q_{(i)} \sum_{k=0}^{m^{(i)}} h_k^{(i)} \vec{\pi}^{(i)} H_{k,i} \right] = \mathbf{0} \ , \tag{4.21}$$

where h_k are given by (4.18). The system (4.21) has a new vector of unknowns

$$\vec{\psi} = \left(p_{00}, h_0^{(1)}, \ldots, h_0^{(m)}, \vec{f}^{(1)}, \ldots, \vec{f}^{(m)}, \vec{g}^{(1)}, \ldots, \vec{g}^{(m)} \right),$$

If $0 \leq q \leq m$ is the number of processes, for which the matrix $\Phi_{(i)}$ is singular, then $(m-q)$ of $h_0^{(i)}$ will have zero coefficients, according to the values of the corresponding $y_{(i)}$. Also, exactly q equations of $y_{(i)} \vec{f}^{(i)} \vec{e}^{(i)} + y_{(i)} \vec{g}^{(i)} \vec{e}^{(i)} = 0$, $i = 1, \ldots, m$ should be added to the system (4.21). After that, the vector of unknowns will be

$$\vec{\psi} = \left(p_{00}, h_0^{(1)}, \ldots, h_0^{(q)}, \vec{f}^{(1)}, \ldots, \vec{f}^{(m)}, \vec{g}^{(1)}, \ldots, \vec{g}^{(m)} \right), \tag{4.22}$$

of the length $1 + q + 2 \sum_{i=1}^{m} n^{(i)}$.

Due to the regularity of the system (4.10), (4.11), the solution $\vec{\psi}$ of (4.21) is unique up to a constant multiplier that is determined from the normalizing condition (4.11), which, in turn, can be represented by matrix geometric terms as

$$p_{00} + \sum_{i=1}^{m} \left(\vec{f}^{(i)} \vec{\alpha}_{(i)} + h_0^{(i)} \varkappa_{(i)} + \vec{g}^{(i)} \vec{\beta}_{(i)} \right) = 1, \tag{4.23}$$

where

$$\vec{\alpha}_{(i)} = \sum_{k=0}^{m^{(i)}} \left(F_{(i)}^{k} \right) \Phi_{(i)}^{\perp} \vec{e}^{(i)} + y_{(i)} \left(q_{(i)} \frac{m^{(i)} \left(m^{(i)} + 1 \right)}{2} \vec{e}^{(i)} + q_{(i)} \sum_{k=0}^{m^{(i)}} \sum_{j=0}^{k-1} \left(F_{(i)}^{j} \right) \vec{\phi}_{(i)} \right),$$

$$\vec{\beta}_{(i)} = \sum_{k=0}^{m^{(i)}} \left(G_{(i)}^{m^{(i)}-k} \right) \Phi_{(i)}^{\perp} \vec{e}^{(i)} - y_{(i)} \left(q_{(i)} \sum_{k=0}^{m^{(i)}} \sum_{j=m^{(i)}-k}^{m^{(i)}} \left(G_{(i)}^{j} \right) \vec{\gamma}_{(i)} \right),$$

$$\varkappa_{(i)} = y_{(i)} q_{(i)} \left(m^{(i)} + 1 \right),$$

and $y_{(i)}$, $q_{(i)}$, $\vec{\gamma}_{(i)}$, $\vec{\phi}_{(i)}$ are defined in (4.16).

So, the equations (4.21) and (4.23) allow to obtain the vector $\vec{\psi}$, defined as in (4.22), which, in turn, contains all the required parameters to use the expression (4.17) in order to obtain the whole steady state probability vector of the queuing system.

Summarizing altogether, the procedure for determining the steady state vector $\vec{p}$ can be separated into four steps:

1. Define the boundary and the inner states of the system.
2. Given the transition rate matrices, check if the conditions (4.9) and the assumptions, made in the section 4.1.2 are suited.
3. Solve the linear system (4.21), subject to the (4.23).
4. Having calculated $\vec{\psi}$, substitute its elements into (4.17) to obtain the steady state vector $\vec{p}$.

The main benefit of the technique, is that the order of a linear system to be solved was reduced from $\left(1+\sum_{i=1}^{m} n^{(i)}\left(m^{(i)}+1\right)\right)$, as in (4.10), to $\left(1+q+2\sum_{i=1}^{m} n^{(i)}\right)$, as in (4.21), and at the same time, the computational complexity increased insignificantly. Notice that, it is not required to construct the whole generator matrix in order to obtain the solution, because only the boundary states are considered directly, and probabilities of being in the inner states are obtained by means of matrix geometric terms. Having obtained the steady state probability vector $\vec{p}$ of the system the additional measures of interest, such as the average queue length and any higher moments can be easily computed.

4.3.2 In Terms of Transition Rate Matrices.

The section is devoted to representing the linear system (4.21) in terms of the transition rate matrices, introduced in sections 4.1.1 and 4.1.2. This is done, because the already given representation is not very convenient for a computer implementation, as, investigating the structure of the matrices $P_M, T_{1_b}^{(i)}, T_{2_b}^{(i)}$, one can notice, that the equation (4.20) contains only the probabilities of being in the zeroth, first, penultimate and the last levels of each process, and all the others have zero coefficients, so it would be good to present the boundary equations only in nonzero terms.

All, that is done in the current section is just substituting the matrices (4.7) into (4.21) and making some simplifications. All the designations remain the same.

The normalizing condition (4.23) does not contain the transition rate matrices, so it won't change.

Substituting (4.7) into the boundary equations (4.20) yields the linear system

$$p_{00}\sum_{i=1}^{m} b_0^{(i)} + \sum_{i=1}^{m} \vec{p}_0^{(i)} \vec{b}_2^{(i)} = 0, \tag{4.24}$$

$$p_{00}\vec{b}_1^{(i)} + \vec{p}_0^{(i)} B_0^{(i)} + \vec{p}_1^{(i)} C_0^{(i)} + \sum_{k=1,\ k\neq i}^{m} \left(\vec{p}_0^{(k)} M_{k,i} + \vec{p}_{m^{(k)}}^{(k)} N_{k,i} \right) = \mathbf{0}, \tag{4.25}$$

$$\vec{p}_{m^{(i)}-1}^{(i)} A_{m^{(i)}-1}^{(i)} + \vec{p}_{m^{(i)}}^{(i)} B_{m^{(i)}}^{(i)} + \sum_{k=1,k\neq i}^{m} \left(\vec{p}_0^{(k)} L_{k,i} + \vec{p}_{m^{(k)}}^{(k)} K_{k,i} \right) = \mathbf{0}, \tag{4.26}$$

for $i = 1, \ldots, m$. Evidently, the equations (4.24), (4.25) and (4.26) present the same as (4.13), (4.14), (4.15), being rewritten subject to (4.7).

Now, substituting (4.17) into (4.24), (4.25) and (4.26), and collecting the coefficients before $\vec{f}, \vec{g}, h_0$ we'll obtain

$$p_{00}\sum_{i=1}^{m} b_0^{(i)} + \sum_{i=1}^{m} \left(\vec{f}^{(i)} \vec{\delta}_{(i)} + h_0^{(i)} \varepsilon_{(i)} + \vec{g}^{(i)} \vec{\eta}_{(i)} \right) = 0, \tag{4.27}$$

with the coefficients, defined as

$$\vec{\delta}_{(i)} = \Phi_{(i)}^{\perp} \vec{b}_2^{(i)},$$

$$\varepsilon_{(i)} = y_{(i)} q^{(i)} \vec{\pi}_{(i)} \vec{b}_2^{(i)},$$

$$\vec{\eta}_{(i)} = G_{(i)}^{m^{(i)}} \Phi_{(i)}^{\perp} \vec{b}_2^{(i)}.$$

The equation (4.27) corresponds to the first row of the generator matrix 4.8. It contains only the transition rates from the zeroth states of the processes to the common absorbing state. If some another type of service discipline was considered, then all the transition rates to the absorbing state with the correspondent to their levels coefficients were included here.

The other two equations contain the transition rates to the zeroth and the last levels of the i^{th} process from all the other boundary states of the rest processes. Also, the arrival process is described in the first of the equations.
The next equation (4.25), in turn, will be

$$\begin{aligned} & p_{00}\vec{b}_1^{(i)} + \vec{f}^{(i)} D^1_{(i)} + \vec{g}^{(i)} D^2_{(i)} + h_0^{(i)} \vec{d}^3_{(i)} + \\ & + \sum_{k=1,\ k\neq i}^{m} \left[\vec{f}^{(k)} D^1_{(k)} + \vec{g}^{(k)} D^2_{(k)} + h_0^{(k)} \vec{d}^3_{(k)} \right] = \mathbf{0}, \end{aligned} \tag{4.28}$$

where

$$\begin{aligned} D^1_{(i)} &= \Phi^{\perp}_{(i)} B_0^{(i)} + \left(F_{(i)} \Phi^{\perp}_{(i)} + y_{(i)} q_{(i)} \vec{e}^{(i)} \vec{\pi}_{(i)} \right) C_0^{(i)} \\ D^2_{(i)} &= G_{(i)}^{m^{(i)}} \Phi^{\perp}_{(i)} B_0^{(i)} + G_{(i)}^{m^{(i)}-1} \Phi^{\perp}_{(i)} C_0^{(i)} \\ \vec{d}^3_{(i)} &= y_{(i)} q_{(i)} \vec{\pi}_{(i)} \left(C_0^{(i)} + B_0^{(i)} \right) \\ D^1_{(k)} &= \Phi^{\perp}_{(k)} M_{k,i} + \left(F_{(k)} \Phi^{\perp}_{(k)} + y_{(k)} q_{(k)} \left(m^{(k)} \vec{e}^{(k)} + \sum_{j=0}^{m^{(k)}-1} \left(F^j_{(k)} \right) \vec{\phi}_{(k)} \right) \vec{\pi}_{(k)} \right) N_{k,i} \\ D^2_{(k)} &= G_{(k)}^{m^{(k)}} \Phi^{\perp}_{(k)} M_{k,i} + \left(\Phi^{\perp}_{(k)} - y_{(k)} q_{(k)} \sum_{j=0}^{m^{(k)}-1} \left(G^j_{(k)} \right) \vec{\gamma}_{(k)} \vec{\pi}_{(k)} \right) N_{k,i} \\ \vec{d}^3_{(k)} &= y_{(k)} q_{(k)} \vec{\pi}_{(k)} \left(M_{k,i} + N_{k,i} \right). \end{aligned}$$

Finally, the last boundary condition can be written as:

$$\vec{f}^{(i)} U^1_{(i)} + \vec{g}^{(i)} U^2_{(i)} + h_0^{(i)} \vec{u}^3_{(i)} + \sum_{k=1,\ k\neq i}^{m} \left[\vec{f}^{(k)} U^1_{(k)} + \vec{g}^{(k)} U^2_{(k)} + h_0^{(k)} \vec{u}^3_{(k)} \right] = \mathbf{0}, \tag{4.29}$$

with the following coefficients

$$
\begin{aligned}
U^1_{(i)} &= \left(F^{m^{(i)}-1}_{(i)}\Phi^{\perp}_{(i)} + y_{(i)}q_{(i)}\left((m^{(i)}-1)\vec{e}^{(i)} + \sum_{j=0}^{m^{(i)}-2}\left(F^j_{(i)}\right)\vec{\phi}_{(i)}\right)\vec{\pi}_{(i)}\right)A^{(i)}_{m^{(i)}-1} + \\
&\quad + \left(F^{m^{(i)}}_{(i)}\Phi^{\perp}_{(i)} + y_{(i)}q_{(i)}\left(m^{(i)}\vec{e}^{(i)} + \sum_{j=0}^{m^{(i)}-1}\left(F^j_{(i)}\right)\vec{\phi}_{(i)}\right)\vec{\pi}_{(i)}\right)B^{(i)}_{m^{(i)}} \\
U^2_{(i)} &= \left(G_{(i)}\Phi^{\perp}_{(i)} - y_{(i)}q_{(i)}\sum_{j=1}^{m^{(i)}-1}\left(G^j_{(i)}\right)\vec{\gamma}_{(i)}\vec{\pi}_{(i)}\right)A^{(i)}_{m^{(i)}-1} + \\
&\quad + \left(\Phi^{\perp}_{(i)} - y_{(i)}q_{(i)}\sum_{j=0}^{m^{(i)}-1}\left(G^j_{(i)}\right)\vec{\gamma}_{(i)}\vec{\pi}_{(i)}\right)B^{(i)}_{m^{(i)}} \\
\vec{u}^3_{(i)} &= y_{(i)}q_{(i)}\vec{\pi}_{(i)}\left(A^{(i)}_{m^{(i)}-1} + B^{(i)}_{m^{(i)}}\right) \\
U^1_{(k)} &= \Phi^{\perp}_{(k)}L_{k,i} + \left(F^{m^{(k)}}_{(k)}\Phi^{\perp}_{(k)} + y_{(k)}q_{(k)}\left(m^{(k)}\vec{e}^{(k)} + \sum_{j=0}^{m^{(k)}-1}\left(F^j_{(k)}\right)\vec{\phi}_{(k)}\right)\vec{\pi}_{(k)}\right)K_{k,i} \\
U^2_{(k)} &= G^{m^{(k)}}_{(k)}\Phi^{\perp}_{(k)}L_{k,i} + \left(\Phi^{\perp}_{(k)} - y_{(k)}q_{(k)}\left(\sum_{j=0}^{m^{(k)}-1}\left(G^j_{(k)}\right)\vec{\gamma}_{(k)}\right)\vec{\pi}_{(k)}\right)K_{k,i} \\
\vec{u}^3_{(k)} &= y_{(k)}q_{(k)}\vec{\pi}_{(k)}\left(L_{k,i} + K_{k,i}\right).
\end{aligned}
$$

The system of equations (4.27), (4.28), (4.29) with the vector of unknowns, defined as in (4.22), together with the normalizing condition (4.23) and the certain amount of the equations $y_{(i)}\vec{f}^{(i)}\vec{e}^{(i)} + y_{(i)}\vec{g}^{(i)}\vec{e}^{(i)} = 0$ allows to obtain all the components of $\vec{\psi}$ and then to apply (4.17) in order to obtain the steady state probability vector $\vec{p}$. The equations (4.27), (4.28), (4.29) are much more convenient for implementation, as they do not require construction of the matrices P_M, $T^{(i)}_{1\mathrm{b}}$, $T^{(i)}_{2\mathrm{b}}$, T, H_{ki}, and can be applied directly to the given transition rate matrices.

5 Implementation.

The described technique was implemented in BoDyTool application. The current section is devoted to a brief description of the tools architecture and its input and output.

Since the proposed technique requires different kind of matrix manipulations, like solving matrix quadratic equations, linear system solving etc., the application was divided i in order to]

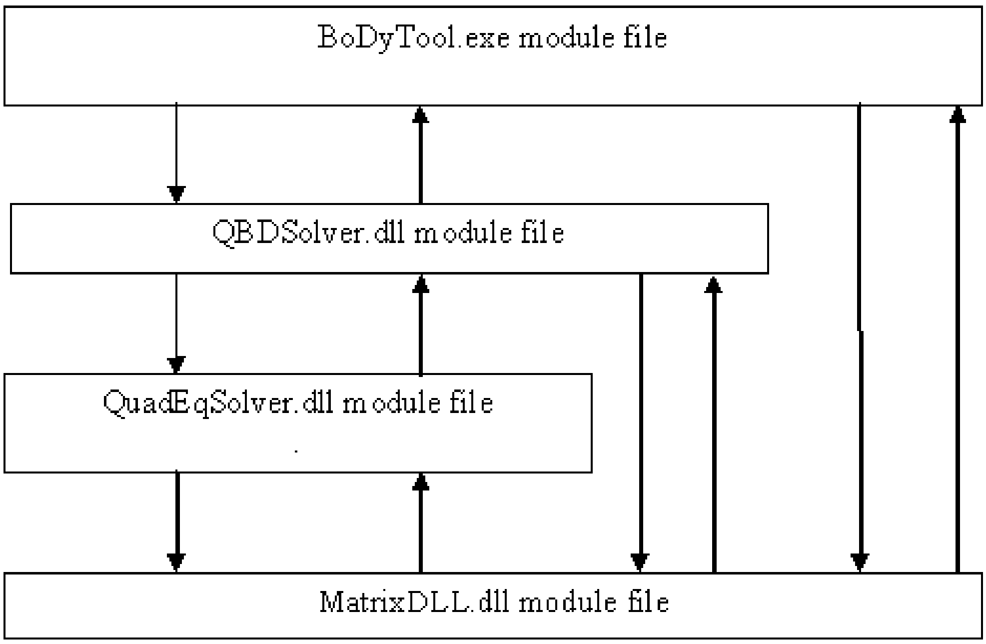

Fig. 5: The BoDyTool structure.

BoDyTool.exe implements a user interface. It provides a user with all of the objects, which can be viewed, for example any sorts of error windows, dialog boxes and so on. Now calculations are performed in the module. The main purpose of the module is to provide data input and output operations, and data checking.

MatrixDLL.dll is the module that implements all the basic matrix operations such as matrix assignments, additions, subtractions, multiplications, and inversions.

QuadEqSolver.dll implements an improved logarithmic reduction algorithm for solving the matrix quadratic equations.

QBDSolver.dll is the module that contains the developed technique. It implements the expressions, presented in the section 4.3.2.

5.1 Input and Output.

The BoDyTool input is stored in the set of ASCII files, which contain all the necessary information to describe the queuing system. Basically, there are two kind of files, namely, for description of the inner states of the system, and for description of the boundary states. Each process in the system is described by its own files.

The first type of input files contains the following:

* The matrices A, B and C, which describe forward, local and backward transitions;
* the length m of the process.

The files for the boundary states include:

* The matrices A_0, B_0, C_0, A_{m-1}, B_m and C_m, which specify the transition rates within the boundary states;
* The transition rate matrices, which represent the transitions between the boundary states of different QBD processes.

The output of the programm is also an ASCII text file, containing the computed steady state probabilities for each QBD process. The QBD processes are ordered lexicographically, and the corresponding steady state probability vector follows the processes number.

6 Conclusion and Future Work.

The queueing system, consisting of several subsystems with FIFO scheduling and state independent phase-type distributed service times was studied in the presented work. The system was assumed to be a superposition of several QBD processes.

The general structure of the underlying Markov chain with level independent block matrices and several boundary states was discussed and presented. A modified matrix geometric technique was improved in order to apply it to each process in the system, and then, using the boundary equations, to represent the steady state probability vector of the whole system by matrix geometric terms. Two possible representations were given, namely, the general one, and a more detailed that is more suitable for implementation.

The main advantage of the proposed technique is that by this time it is the only one, which can evaluate the steady state probability vector of the described model. All the other of the existing techniques can not be applied to the queuing models, where the states are described by some subsystems. The developed method has as its basics the modified matrix geometric method, which is considered to be among the best in time and storage complexity, so being applied to simple queuing systems, the performance characteristics are the same as for the modified matrix geometric solution. Low time complexity follows mainly from the significant reducing of the size of the linear system, which is to be solved. It is good to mention, that increasing the number of waiting positions in the subsystems increases only the computational cost, but the execution time of the method, because only the linear system for determining the matrix geometric terms does not include the inner states of the queuing model. With increasing of the number of phases in subsystems, the performance characteristics of the technique become worse, but the same can be noticed for any other methodology.

The main disadvantages of the technique result mainly from the computational complexity, because a lot of different calculations, like solving the matrix quadratic equations, evaluation of the general group inverse and others, should be performed.

The future work can go on in several directions. One of them, is that the developed technique can be improved in different ways in order to be suitable for applying to more complex queuing systems, for example, consisting of level-dependent quasi birth death processes, different type of arrival stream and service times distribution may be considered and so on. Also, another algorithm can be taken as the basics, e.g. the Folding algorithm, which may suit for certain problems better, than the matrix geometric technique.

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