

LAPPEENRANTA UNIVERSITY OF TECHNOLOGY

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VASICEK INTEREST RATE MODEL:

**PARAMETER ESTIMATION, EVOLUTION OF THE SHORT-TERM INTEREST RATE
AND TERM STRUCTURE**

Bachelor's thesis

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1. INTRODUCTION

Interest rates and especially their dynamics provide probably the most computationally challenging part of modern financial theory. The introduction of derivatives for fixed-income securities has complicated the analysis even further. This is a great foundation to develop new and innovative solutions to model the term structure, which is of huge importance both from a financial and monetary point of view, yet the older and simpler models haven't been forgotten. The Vasicek model described in this paper represents one of the first and most well known interest rate models and is still in active use. More importantly it is a good starting point for understanding the complex world of interest rate modelling.

In this study we are not going to address its ability to model the dynamics of the term structure to great extent since this is done in many previous studies and one can draw his own conclusions based on them. Our objective is to show how to estimate and use the single factor model but we will also show how this can be extended to the multifactor framework. We are also going to explain the mathematic terminology behind the model. This paper will follow the parameter estimation process explained and recommended by James and Webber (2004).

We begin with a brief theoretical background section, which is going to discuss the concept of term structure in general and the crucial part it plays in economics. Then we will look at the very basic theory of interest rate models and gradually focus our attention to the Vasicek model.

In the third section we will take a look and explain the methodology behind parameter estimation and the data used in this study. This is also the part, which includes most of the mathematical equations. In the fourth chapter it is time to estimate the actual parameters using data from the European fixed-income market. We are going to estimate the parameters with an easy to use Excel-program instead of Matlab-code or Eviews methods.

The fifth chapter will explain how we have developed and how we can use an Excel-based simple program in estimating the process for the short-term interest rate. In this estimation we will need the parameters estimated in the fourth section. In the sixth section an Excel-program is developed for the estimation of the actual term structure using again the parameters from the fourth section. Lastly we will conclude the results and most important parts of this paper.

2. THEORETICAL BACKGROUND

2.1 Term structure

2.1.1 Description

The term structure of interest rates (also known as the yield curve) illustrates the yields of securities, which only differ with respect to their time to maturity. It is constructed from the yields of benchmark fixed-income default-free zero-coupon bonds. Securities issued by the government are often used as the benchmark since they are considered risk free and therefore are homogenous in every aspect other than their yields and maturities. The term structure is graphed as though each coupon payment was a zero-coupon bond that matures on the coupon payment date. This is necessary since zero-coupon bonds with a maturity of over one year are rare. (Lanne, 1994, p. 1-2)

2.1.2 Importance of the term structure

The yield curve is of great importance both in financial and monetary economics and acts as a link between them. From a financial point of view the term structure can be used for hedging portfolios against risk and pricing derivatives such as bond options. Interest rates are also used for time discounting and hence are the backbone of pricing all market instruments and beyond from stocks to corporate investment decisions. (Benniga & Wiener, 1998, p. 1)

In risk management the term structure plays a key role as well. When calculating value at risk for a fixed-income portfolio one can simulate different paths for the term structure with an interest rate model and then generate a distribution of the values for the underlying portfolio in each scenario. (Bolder, 2001, p. 3)

From a macroeconomic point of view the term structure can be interpreted as an estimation of future short-term interest rates, inflation and economic activity. These forecasts are used by companies in their investment decisions, consumers in their saving decisions and economists in their policy decisions. Piazzesi (2003, p. 3) reminds that the central bank can adjust the interest rate in the short end of the yield curve but what matters to the real economy are the long-term yields. In order to conduct efficient

monetary policy a central bank can use an interest rate model to understand how movements in the short rates translate into movements in the long rates. According to Seppälä & Viertiö (1996, p. 33) empirical studies suggest that the term structure can predict consumption growth even better than auto regressions or highly complex econometric models. Bernard & Gerlach (1996, p. 6) show in their study that the same goes for predicting recessions for up to two years ahead from the term spread between the short and long rates. These are quite compelling arguments in favour of why we should care about term structure modelling.

2.1.3 Shape of the term structure

The yield curve can of course be basically of any shape, but there are some basic cases of how it should look in different situations. There are four main patterns that can characterize the term structure: normal, flat, inverted and humped. The different shapes represent different expectations about the future.

Under normal market conditions the investor's don't expect any major changes in the economy and think the economy will continue to grow at the normal prevailing pace. During these conditions, investors will expect higher yields for instruments with longer maturities. This is a very reasonable assumption since short-term instruments are generally thought to hold less risk because time is a significant source of uncertainty.

Flat yield curve can be interpreted so that the market environment is sending mixed signals about the future development of interest rates and hence investors aren't quite sure, how to read the market. Some indicators may suggest that short rates will rise and others that long rates will fall.

Inverted yield curve is the opposite of a normal situation. In such market conditions, bonds with maturity dates further into the future are expected to yield less than bonds with shorter maturities. The market expects the interest rates to fall in the future, which means that the yields of long-term bonds will decline. Investors are still however interested in long-term bonds as well since an inverting yield curve can be an indication

of economic slowdown, which will lower future yields even more so it's better to lock your money into long-term investments at the present prevailing yields.

2.2 Interest rate models

2.2.1 Description

Interest rate models can be used to model the dynamics of the yield curve, which is vital in pricing and hedging of fixed-income securities and also of great importance from a macro economical point of view as we have stated earlier. Traditionally these models specify a stochastic process for the term structure dynamics in a continuous-time setting. A stochastic process means that the outcome depends both from a determined and a random component and is thus given the form

$$dr(t) = A_0 dt + A_1 dW(t) \quad (1)$$

Here the first term is deterministic and called the drift-term. The second term describes the randomness of the process. We will get more acquainted with this type of differential function later on in the paper. (Bolder, 2001, p. 7)

Models can be roughly divided into equilibrium models and no-arbitrage models. Only equilibrium models are described here and from those only the Vasicek model used will be covered in greater detail. There are single or multifactor versions available of most models and the factors used vary but the first factor is usually the instantaneous interest rate. The Vasicek model used in this paper is one of the first term structure models to appear on the market in 1977. It has stood the tests of time well and inspired a whole bunch of following models. It has been used extensively in valuing bond options, futures and other fixed-income derivatives that require the estimation of the term structure.

2.2.2 Previous empirical research

The empirical studies of the interest rate models have offered mixed results about their performance. Most studies have concentrated on the single factor models. This was especially the case until the late nineties, when the multifactor models were developed and tested in growing numbers as computer technology enabled the calculation of more complex systems. It became evident that the multifactor models were able to explain a

greater variety of term structure movements. This isn't a surprise since the dynamics are depicted by three things: changes in the curvature, in the slope and in the level of the yield curve. The first factor is nevertheless dominating in explaining the variance of the spot rates and thus the shape of the yield curve. In a study by Heitmann & Trautmann (1995, p. 27), who used German bond market data, concluded that the first factor explained, depending from the sample, 73-84 % of the variance of the spot rates. Despite this, in the same study they also found that two factors were needed to capture the volatility smile effect present in some of the samples. The use of a single factor model yielded significantly different (and incorrect) option prices for these situations. This goes to show that in some cases the choice of a model can have a huge impact on how the instruments in the market are priced by dealers.

Another improvement came in the early nineties as Chen & Scott (1993) and Pearson & Sun (1994) opted for a different approach to multifactor models. They didn't set the factors beforehand but instead let the estimation method detect the parameters from the observed bond yields.

2.2.3 Equilibrium models of the yield curve

These models usually specify a process for the short-term interest rate. This process is based on assumptions about economic variables. Because of this, the models don't fit the real market data at any point in time. The no-arbitrage models however are specified so that they fit the current yield curve. (Copeland et al, 2005, p. 265)

The most important equilibrium models are the Vasicek and the Cox-Ingersoll-Ross (from now on CIR) models. These offer benefits since they reduce to differential functions, which are numerically quite simple and easy to solve with computers. They are also very helpful in finding the important factors. Many different models have been developed in this group, some of them single and some of them multifactor models. The Vasicek and the CIR model are interesting and widely tested empirically since they offer closed form solutions of their conditional and steady state density functions. (Käppi, 1997, p. 37)

Several other models with a mean-reverting process like in Vasicek and CIR have also been developed. Usually the difference is only in modelling the second term describing uncertainty of future developments. In Vasicek the coefficient of the Wiener-process (explained in the next section) is simply σ and thus independent of r . In CIR it is $\sigma \sqrt{r}$. Courtadon (1982) models it as σr , Chan et al. (1992) as σr^α and Duffie & Kan (1993) as $\sqrt{\sigma_0 + \sigma_1 r}$. Ait-Sahalia (1996) finds that the linearity of the drift term $\alpha (\gamma - r)dt$ is actually the main source of misspecification and not the second term as thought before. To overcome the problem of linearity it is possible to add jumps to the process, which means unfortunately that closed form solutions of the bond prices become unattainable. (Copeland et al, 2005, p. 268-269)

Duffie (2000) states several negative aspects of these equilibrium models. For one he explains that the models cannot simultaneously allow for negative correlations between state variables and guarantee that interest rates would be positive. They can't also capture the nonlinearities in the data in a satisfying manor. Ahn, Dittmar and Gallant (2002) created a model to overcome these drawbacks. It specifies yields as quadratic functions of the state variables. In the Black-Karasinski (1991) model the state variable is identified as $\log r$ to avoid the problem of negative interest rates (Leippold & Wiener, 2000, p. 2). (Copeland et al, 2005, p. 269)

2.2.4 Vasicek interest rate model

The Vasicek model uses a mean-reverting stochastic process to model the evolution of the short-term interest rate. Mean reversion is one of the key innovations of the model and this feature of interest rates can also be justified with economic arguments. High interest rates tend to cause the economy to slow down and borrowers require less funds. This causes the rates to decline to the equilibrium long-term mean. In the opposite situation when the rates are low, funds are of high demand on the part of the borrowers so rates tend to increase again towards the long-term mean. (Zeytun & Gupta, 2007, p. 2)

The model assumes that the current short interest rate is known for sure and the subsequent values of it follow this stochastic differential equation

$$dr_t = \alpha(\gamma - r_t)dt + \sigma dz_t \quad (2)$$

which is interpreted as,

$$\int dr_t = \int \alpha(\gamma - r_t)dt + \int \sigma dz_t$$

(Vasicek, 1977, p. 9)

$r(t)$ is a continuous function of time (no jumps) and follows a Markovian process, which means that the system has no memory. That is, future developments of the short rate are independent of past movements. Since this is a continuous Markovian process it's called a diffusion process. The model assumes the market to be efficient. (Svoboda, 2003, p. 5)

In the formula γ is the long-term mean of the short-term interest rate r (the shortest possible - usually understood as instantaneous). In some advanced models the long-term mean isn't constant but a function of time. α is the speed of adjustment with which r closes on the long-term mean γ . If $r > \gamma$ then the coefficient $\alpha > 0$ makes the drift-term negative and thus the rate will be pulled back down towards γ . The opposite happens when $r < \gamma$. (Zeytun & Gupta, 2007, p. 2) Because there is only one factor, all interest rates are ultimately dependent on the shortest interest rate r . All interest rate models are additive and that is why the $t+\Delta t$ interest rate is of type: $r_t + \Delta r$. This characteristic will be used in section 5, where we develop a program for the evolution of the short rate.

The second term aims to capture the instantaneous volatility caused by possibly infinite number of unpredictable factors. The symbol σ is the volatility and z_t is a Wiener-process, which means that the value z_T at time T conditional on its value z_t is normally distributed with a mean z_t and variance $T-t$, where T is larger than t (James & Webber,

2002, p. 54). The market price of risk is assumed to be a constant and usually negative or zero to guarantee a positive premium for bond prices (Vasicek, 1977, p. 7 & 9). In order to construct the term structure (a function of r_t) we only need the long-term mean, the speed of adjustment and the standard deviation. This procedure and a program that uses it, is described more thoroughly in section 6.

The yield curves start at the current level of instantaneous interest rate $r(t)$ and approach the following infinite yield

$$R_\infty = \gamma - \frac{\sigma^2}{2\alpha^2} \quad (3)$$

The model can produce upward or downward sloping or humped yield curves. If

$$r(t) \leq R_\infty - \frac{\sigma^2}{4\alpha^2} \quad (4)$$

the yield curve is monotonically increasing, that is, upward sloping. If

$$R_\infty + \frac{\sigma^2}{2\alpha^2} \geq r(t) \geq R_\infty - \frac{\sigma^2}{4\alpha^2} \quad (5)$$

the yield curve is humped. The last scenario is when

$$r(t) \geq R_\infty + \frac{\sigma^2}{2\alpha^2} \quad (6)$$

and this produces a monotonically decreasing curve. (Vasicek, 1977, p. 10)

The model can be extended into a multifactor model simply by taking other factors into the process and handling them in the same way as the short-term interest rate is handled (mean-reverting process). This was done by Langetieg in 1980. Actually the famous one-factor model is simply a special case. In the single factor model the short rate R_t equals the state variable but in the multifactor model the short rate is the sum of independent state variables y_{it}

$$R_t = \sum_{i=1}^k y_{it} \quad (7)$$

where, k is the number of the factors. (Piazzesi, 2003, p. 32)

The discount bond price has a well-known closed form solution, which is, when generalized into the multifactor form, the following

$$P_t = \prod_{i=1}^k A_{it} e^{-B_{it} r_{it}} \quad (8)$$

where,

$$A_{it} = \exp\left(\frac{(B_{it} - t)(\alpha_i(\alpha_i \gamma - q_i \sigma_i) - \sigma_i^2 / 2) - \frac{\sigma_i^2 B_{it}^2}{4\alpha_i}}{\alpha_i^2}\right) \quad (9)$$

and,

$$B_{it} = \frac{1 - \exp(-\alpha_i t)}{\alpha_i} \quad (10)$$

In the equation A_{it} we can see the market risk premium q for the first time. This is set to zero in this paper since we want to model the interest rate process and the term structure in a risk-neutral world where bond and fixed income derivatives are priced. It's good to remind that the real world development of $r(t)$ does not matter in the pricing process. (Hull, 2003, p. 537)

2.2.5 Drawbacks of the single factor Vasicek model

We must however acknowledge that the single factor model used here, although empirically very tractable, is also subject to some criticism. The dependence on a single factor greatly limits the possible shapes of the yield curve and often leads into situations, where the theoretical yield curve does not correspond to the market yield curve. In fact this is the case nearly always. Another drawback is the undesired property that the yields of all maturities are perfectly correlated. This is an unrealistic assumption about the behaviour of yields. (Käppi, 1997, p. 35)

The most problematic point in the Vasicek model is that it can produce negative interest rates. This isn't a big problem with real interest rates since they often can be negative but nominal ones are unlikely to be negative. In the long run it is useful to point out that negativity isn't a large problem usually since the distribution of the short rate is

stationary and Gaussian with expected rate and variance (when $\alpha > 0$) of γ and $\frac{\sigma^2}{2\alpha}$ respectively. (Xie, 2006, p. 1; Brigo et al, 2007, p. 29) These values can be derived from

$$Er(T) = \gamma + (r(t) - \gamma)e^{-\alpha(T-t)} \quad (11)$$

When $T \rightarrow \infty$ the second term approaches 0.

And,

$$\text{var } r(T) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(T-t)}) \quad (12)$$

When $T \rightarrow \infty$ the coefficient $(1 - e^{-2\alpha(T-t)})$ approaches 1. (Vasicek, 1977, p. 9)

Another drawback is also that it hasn't passed many empirical studies. For example Murto (1992, p. 33-34) rejects it in his study about interest rate models using data from the Finnish money market because of fitting errors. Käppi also finds that the single factor Vasicek model fits the data poorly but multifactor versions do better (Käppi, 1997, p. 61). Despite the drawbacks, the Vasicek model is very useful and widely used in practice. (Benninga & Wiener, 1998, p. 6)

Like with all interest rate models the data is a major source of misspecification. The sample should cover well over ten years in order to catch the effects of mean reversion. This is a serious problem since some datasets simply don't have such a long history. For example the LIBOR swap data popularly used in estimations is too short due to the fact that the market hasn't existed that long. Regime changes, like the creation of the European Central Bank and the harmonization of short-term interest rates across European countries, have also ruined many national samples. (Kim & Orphanides, 2005, p. 5-6)

3. ESTIMATION METHODOLOGY AND DATA

3.1 Maximum likelihood estimation of the parameters

In order to estimate the parameters α , γ and σ of the model we use a maximum likelihood method as suggested in James & Webber (2004, p. 506-507). The maximum likelihood method finds the parameter values so that the actual outcome has the maximum probability. An advantage of the maximum likelihood method over for example generalized methods of moments (GMM) is that it provides an exact maximum likelihood estimator. On the downside it assumes that the variance of the stochastic variable is constant between the discrete observations. This seems like oversimplifying the problem but in practice it does not seem to affect the estimation too much. (Episcopos, 2000, p. 4)

First we need to define the likelihood function. Then we have to find the parameter values that maximize the value of this likelihood function. Those are then the maximum likelihood estimates. One can think of them as a parameter set $\theta = \{\alpha, \gamma, \sigma\}$.

The likelihood function is

$$L = \prod_{i=1}^{N-1} \left(2\pi \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha\Delta t_i}) \right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} v^2(r_{ii}, r_{ii+1}, t_i) \right) \quad (13)$$

where,

$$v(r_{ii}, r_{ii+1}, \Delta t) = \frac{r_{ii+1} - (\gamma + (r_{ii} - \gamma)e^{-\alpha\Delta t_i})}{\sqrt{\text{var}_{ii}}} \quad (14)$$

and,

$$\text{var}_{ii} = \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha\Delta t_i}) \quad (15)$$

We can however simplify the formula slightly since our observations are at equal time increments, so $\Delta t_i = \Delta t$. L becomes

$$L = \left(2\pi \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha\Delta t}) \right)^{\frac{N-1}{2}} \exp\left(-\frac{1}{2} \sum_{i=1}^{N-1} v^2(r_{it}, r_{it+1}, \Delta t) \right) \quad (16)$$

It is easier to maximize the log-likelihood function instead of the likelihood function itself.

We take natural logarithms from both sides to get

$$\ln L = -\frac{N-1}{2} \ln 2\pi - \frac{N-1}{2} \ln\left(\frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha\Delta t}) \right) - \frac{1}{2} \sum_{i=1}^{N-1} v^2(r_{it}, r_{it+1}, \Delta t) \quad (17)$$

The maximum likelihood parameter set $\hat{\theta}$ can then be found using the following argument

$$\hat{\theta} = \arg \max_{\theta} \ln L(\theta) \quad (18)$$

3.2 Data

In equilibrium term structure models the state variable is extracted from observed bond yields or interest rates. In the next chapter we are going to estimate the parameters that determine the behaviour of the state variable and therefore need some interest rate data. In multifactor specification we would need more than one time series, for example a series of a short interest rate and a series of a longer interest rate.

In this paper we are going to use one week Euro Interbank Offered Rate (Euribor) collected from the Datastream database. The sample period is from January 18, 1999 to November 18, 2008 and contains 119 monthly observations. The ECB was founded in January 1999 so we have data right from the beginning of the Euribor-era. We have chosen the one week rate since it is likely to give a more accurate estimation of the parameters because it is closer to the unobserved short rate than longer interest rates.

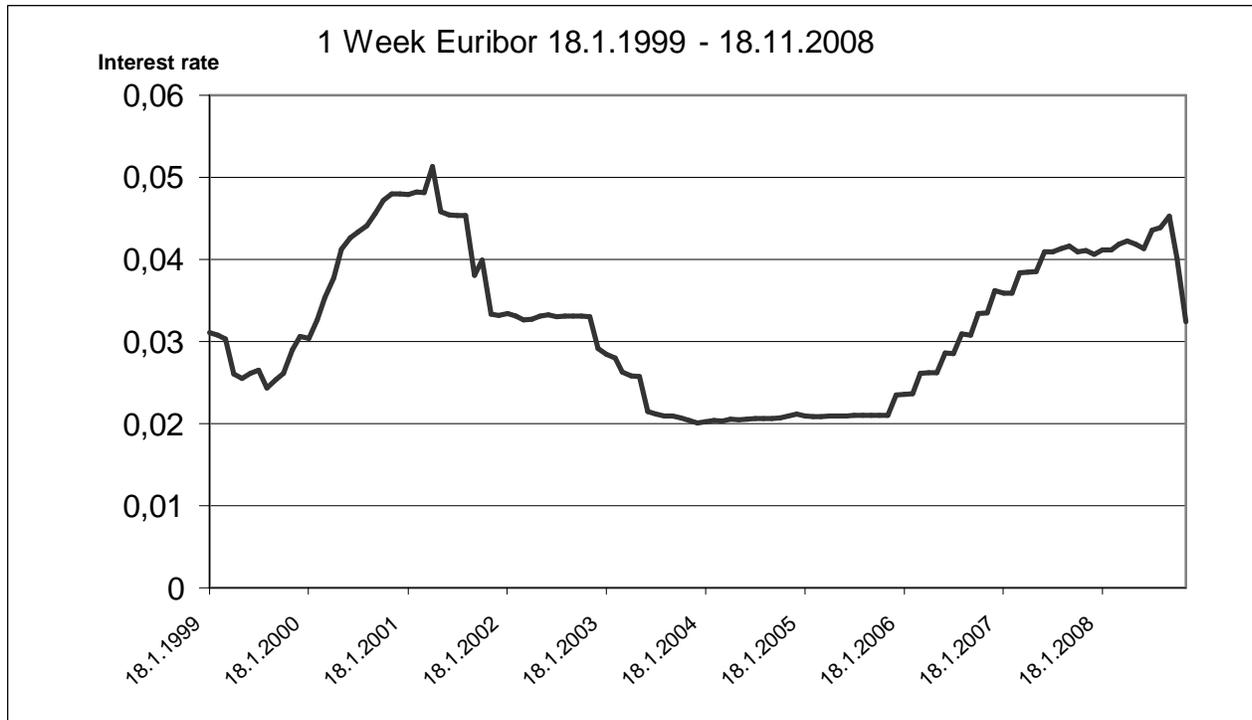


Figure 3.1. Development of the 1 week Euribor-rate from 18.1.1999 to 18.11.2008

From the graph we can clearly see how the ECB first raised the interest rate as the economy was strong in 2000 and then tried to stimulate the economy after the .com bubble burst in 2001-2002. The same pattern can be seen with the credit crunch taking its toll on the economy and how the ECB is in the process of lowering its offered rate. One needs to remember that the Euribor isn't the same as the main policy rate but these two are strongly linked to each other through ECBs open-market operations, where it lends money to European banks.

4. PARAMETER ESTIMATION WITH EXCEL

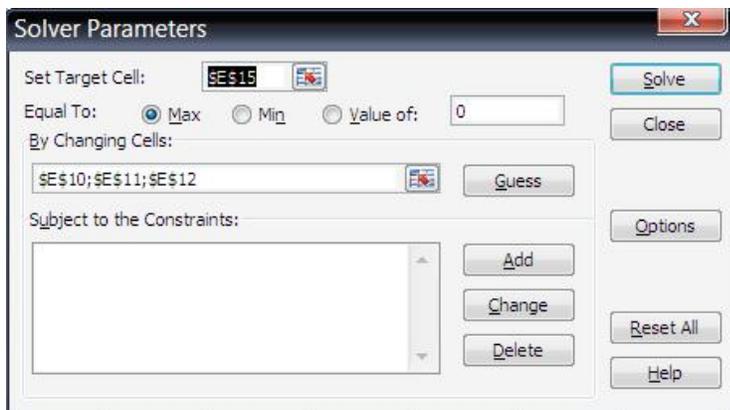
There are several Matlab-codes available in the internet for the estimation of the parameters but their use requires some previous experience with the program in order to use them correctly and some of them seem to be faulty. That is why we develop here an Excel-based easy to use solution for estimating the parameters. The application is based around Excel's solver add-in.

We begin by inputting the data set and determining the time increment between the observations and the number of observations. In this case the time increment is one month (1/12) and the number of observations is 119. The next step is to input some initial values for the parameters in order to prevent dividing with zero in the formulas. The application then calculates first the $v^2(r_{it}, r_{it+1}, \Delta t)$ for each individual observation and then sums them up. It then proceeds to calculate the value of the log-likelihood function as described in section 3.1.

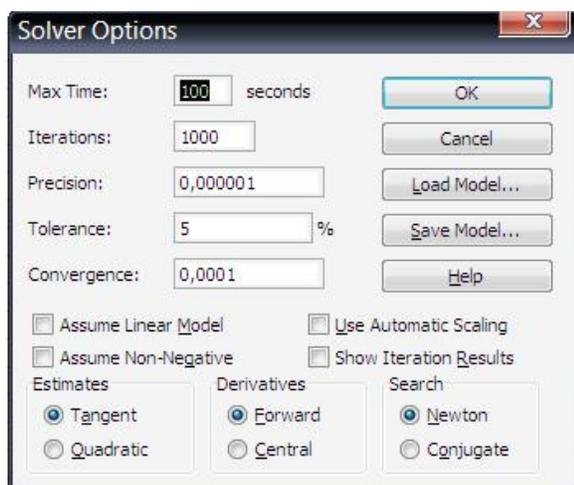
Next we use the solver to maximize the value of the log-likelihood function by changing the parameter values until the maximum is found. Excel does this by iterating using the Newton-Rhapson optimization method. From the options we can change the number of iterations Excel does, since with a lot of iterations and observations the calculations become quite time consuming. Below are screenshots of the program and of the solver settings.

| | A | B | C | D | E | F | G |
|----|--|---------------|----------|----------|----------|---|---|
| 1 | Vasicek parameter estimation | | | | | | |
| 2 | | | | | | | |
| 3 | Notation is the same as in the paper. | | | | | | |
| 4 | | | | | | | |
| 5 | Time increment between observations Δt | | | | 0,083333 | | |
| 6 | Number of observations N | | | | 119 | | |
| 7 | Pi | | | | 3,141593 | | |
| 8 | | | | | | | |
| 9 | Initial values of: | | | | | | |
| 10 | Mean reversion parameter α | | | | 0,0200 | | |
| 11 | Equilibrium rate γ | | | | 0,0300 | | |
| 12 | Volatility σ | | | | 0,0100 | | |
| 13 | | | | | | | |
| 14 | Sum of v^2 | | | | 48,0413 | | |
| 15 | Value of log-likelihood function ln L | | | | 557,6625 | | |
| 16 | | | | | | | |
| 17 | We use the solver to maximize the value of ln L by changing the parameter values | | | | | | |
| 18 | | | | | | | |
| 19 | Sample statistics: | | | | | | |
| 20 | Mean | | | | 0,03194 | | |
| 21 | Volatility | | | | 0,00912 | | |
| 22 | | | | | | | |
| 23 | Data table | | | | | | |
| 24 | Date | interest rate | v | v^2 | | | |
| 25 | 18.1.1999 | 0,03109 | -0,09645 | 0,009302 | | | |
| 26 | 18.2.1999 | 0,03081 | -0,16941 | 0,028701 | | | |
| 27 | 18.3.1999 | 0,03032 | -1,48022 | 2,19105 | | | |
| 28 | 18.4.1999 | 0,02605 | -0,1791 | 0,032076 | | | |
| 29 | 18.5.1999 | 0,02554 | 0,212378 | 0,045105 | | | |

Picture 4.1 Screenshot of the parameter estimation program



Picture 4.2 Screenshot of the Solver



Picture 4.3 Screenshot of the solver options used

When we run the solver the parameter estimates are the following

Target Cell (Max)

| Cell | Name | Original Value | Final Value |
|-------------|---------------------------------------|-----------------------|--------------------|
| \$E\$15 | Value of log-likelihood function In L | 557,6625 | 576.2173 |

Adjustable Cells

| Cell | Name | Original Value | Final Value |
|-------------|--------------------------|-----------------------|--------------------|
| \$E\$10 | Mean reversion parameter | 0,02 | 0,2475 |
| \$E\$11 | Equilibrium rate | 0,03 | 0,0325 |
| \$E\$12 | Volatility | 0,01 | 0,0064 |

Picture 4.4 Parameter estimates

The given estimates seem very reasonable. The pullback term of 0,2475 is in line compared with results in other studies, where the values have ranged from around 0,1 to 1. The long term equilibrium rate of 3,25% and a volatility of 0,64% seem reliable judging by the sample statistics. The mean of the observations is 3,19% and volatility 0,91%. This gives us confidence that the parameter estimates are fairly reliable despite the fact that our likelihood method does not provide us with standard deviations of the estimates.

5. SIMULATION OF THE SHORT-TERM INTEREST RATE PROCESS WITH EXCEL

In order to estimate the development of the short term (instantaneous) interest rate we need to use the Euler-discretisation of the stochastic differential equation (2) and the additive feature of our model. The Euler-discretisation is

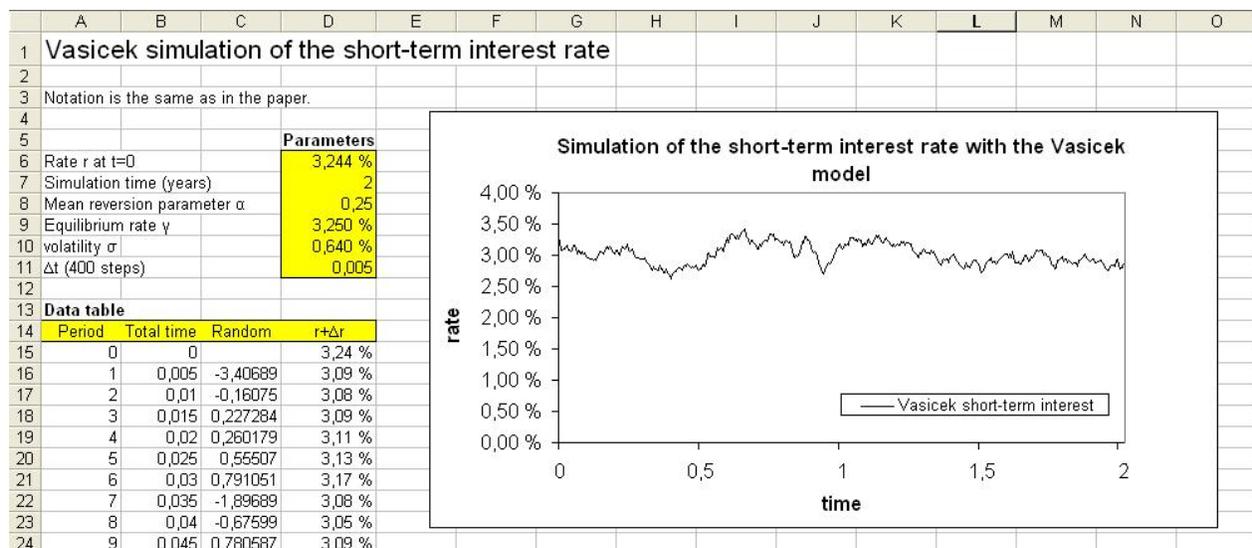
$$r_{t+\Delta t} = r_t + \alpha(\gamma - r)dt + \sigma \varepsilon_{t+\Delta t} \sqrt{\Delta t} \quad (19)$$

where,

$$\varepsilon_{t+\Delta t} \square N(0,1), t = 0, \dots, T - \Delta t \quad (20)$$

(Benniga & Wiener, 1998, p. 6)

Below is a screenshot of the program in order to clarify its functioning.



Picture 5.1 Screenshot of the Excel-program

We begin with some initial value of r at t=0 (cell D6). We then specify the estimation time horizon in cell D7. Next we need the parameters α , γ , and σ (D8-D10). In cell D11 we specify the desired amount of time steps. The time step needs to be quite small in order to achieve accuracy with the Euler-discretisation. Underneath the input values is the data table needed to construct the diagram. The period column indicates the time step (in this case the table has 400 rows). Total time is simply the total time elapsed in years

from $t=0$. The random row has random values drawn from a standardized normal distribution with a mean of 0 and variance of 1.

Now we can calculate the interest rate in the next time step simply by adding $r(=dr)$ to the r of the previous time step and then draw a diagram of the possible evolution of the short-term interest rate during the desired time period.

When one plays around with the parameters a bit it is easy to see that the mean reversion parameter has to be quite large in order to have a large effect on the process.

In most cases the noise-term $\sigma\varepsilon\sqrt{\Delta t}$ has such a great effect on the development of the interest rate that it overpowers the first drift-term $\alpha(\gamma - r)dt$. This can be seen easily with pressing F9 in the Excel-sheet in order to generate a new set of random values ε . These different graphs generated with different random values are called sample paths. Although the path is somewhat random it seems to be mostly due to the volatility, since once the volatility parameter σ is smaller the changes become more and more subtle and the interest rate seems to converge better towards the long-term equilibrium because of the drift-term. A large mean reversion parameter also speeds up the convergence. The time-horizon should however be kept rather short since the estimates become increasingly uncertain as the simulation time is increased. This is easy to understand since we are dealing here with the future.

6. SIMULATION OF THE TERM STRUCTURE WITH EXCEL

Our next program estimates the current term structure from the parameters of the model. We calculate the discount factors of bonds of different maturities and derive from those the corresponding yields to construct the term structure.

The price at time t of a zero-coupon bond paying \$1 at time T can be calculated from the equation derived from the multifactor version presented already in section 3. This time however we expect the market price of risk to be zero for the reasons we stated earlier.

$$P(t, T) = A(t, T)e^{-B(t, T)r(t)} \quad (21)$$

where,

$$A(t, T) = \exp \left[(B(t, T) - T + t) \left(\frac{\alpha^2 \gamma - \frac{\sigma^2}{2}}{\alpha^2} \right) - \frac{\sigma^2 B(t, T)^2}{4\alpha} \right] \quad (22)$$

and,

$$B(t, T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha} \quad (23)$$

(Hull, 2003, p. 540)

The yields can then be calculated easily for all maturities using the price of the corresponding zero-coupon bonds

$$r(T) = \frac{\ln\left(\frac{1}{P(T)}\right)}{T} \quad (24)$$

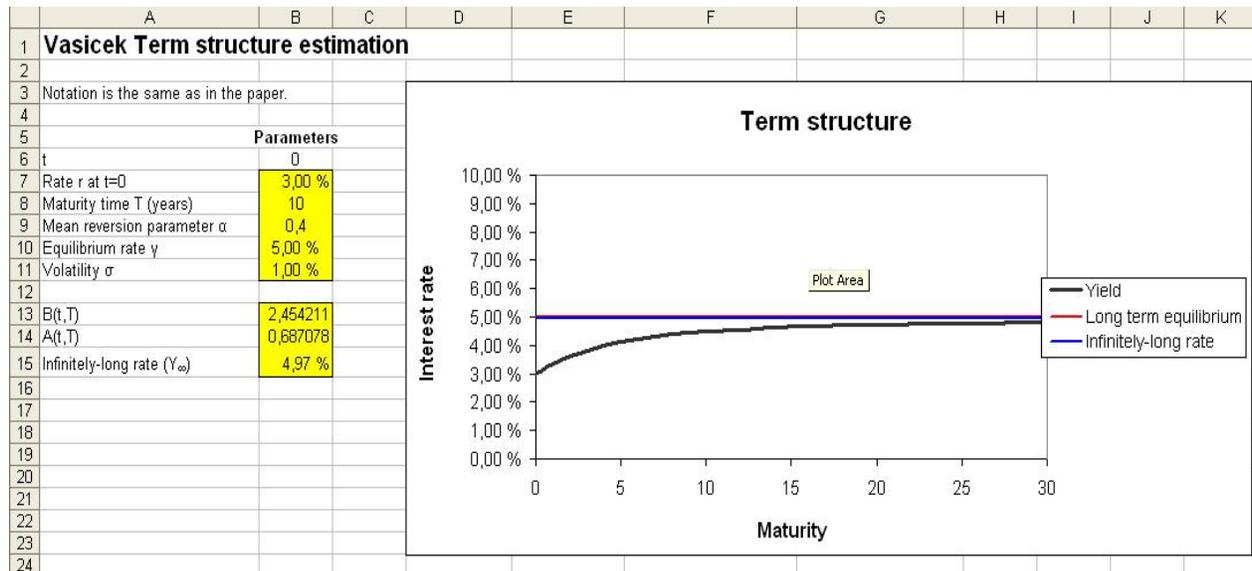
derived from,

$$P(T) = e^{-r(T)T} \quad (25)$$

where $r(T)$ represents the yield for a bond maturing at time T . While we are in the process of constructing the current yield curve, t equals 0. The above equations allow

however the pricing of bonds at time t in the future. This is notable since for example the pricing of derivative contracts requires this feature.

Below is a screenshot of the program:



Picture 6.1: A screenshot of the Excel-program for estimating the term structure.

We start with the parameter estimates α , γ , and σ (cells B9-B11). We must also specify the interest rate prevailing at time $t=0$ and the maturity of our zero-coupon bond. In this case $T=10$ years. The program then calculates the values of $A(t,T)$ and $B(t,T)$ and ultimately the price $P(t,T)$ according to the formulas above. In order to construct the yield curve we then proceed to develop a data table (see picture below) in which the prices (and ultimately yields) for bonds with various maturities are calculated, whilst keeping the other parameters constant.

| | A | B | C | D | E | F | G | H | I | J |
|----|---|-------------------|----------|----------|------------|--------|----------------------------|----------------------|---|---|
| 28 | | | | | | | | | | |
| 29 | | | | | | | | | | |
| 30 | | | | | | | | | | |
| 31 | | Data table | | | | | | | | |
| 32 | | T | A | B | Bond price | yield | Long-term equilibrium rate | Infinitely-long rate | | |
| 33 | | 0,001 | 1,000000 | 0,001000 | 0,999970 | 3,00 % | 5,00 % | 4,97 % | | |
| 34 | | 0,1 | 0,999901 | 0,098026 | 0,996965 | 3,04 % | 5,00 % | 4,97 % | | |
| 35 | | 0,5 | 0,997663 | 0,453173 | 0,984192 | 3,19 % | 5,00 % | 4,97 % | | |
| 36 | | 1 | 0,991261 | 0,824200 | 0,967052 | 3,35 % | 5,00 % | 4,97 % | | |
| 37 | | 2 | 0,969389 | 1,376678 | 0,930168 | 3,62 % | 5,00 % | 4,97 % | | |
| 38 | | 3 | 0,939461 | 1,747014 | 0,891492 | 3,83 % | 5,00 % | 4,97 % | | |
| 39 | | 4 | 0,904965 | 1,995259 | 0,852385 | 3,99 % | 5,00 % | 4,97 % | | |
| 40 | | 5 | 0,868210 | 2,161662 | 0,813693 | 4,12 % | 5,00 % | 4,97 % | | |
| 41 | | 6 | 0,830690 | 2,273205 | 0,775929 | 4,23 % | 5,00 % | 4,97 % | | |
| 42 | | 7 | 0,793349 | 2,347975 | 0,739389 | 4,31 % | 5,00 % | 4,97 % | | |
| 43 | | 8 | 0,756764 | 2,398094 | 0,704232 | 4,38 % | 5,00 % | 4,97 % | | |
| 44 | | 9 | 0,721277 | 2,431691 | 0,670532 | 4,44 % | 5,00 % | 4,97 % | | |
| 45 | | 10 | 0,687078 | 2,454211 | 0,638308 | 4,49 % | 5,00 % | 4,97 % | | |
| 46 | | 13 | 0,592864 | 2,486209 | 0,550253 | 4,60 % | 5,00 % | 4,97 % | | |
| 47 | | 16 | 0,511004 | 2,495846 | 0,474140 | 4,66 % | 5,00 % | 4,97 % | | |
| 48 | | 20 | 0,418967 | 2,499161 | 0,388704 | 4,72 % | 5,00 % | 4,97 % | | |
| 49 | | 25 | 0,326814 | 2,499887 | 0,303200 | 4,77 % | 5,00 % | 4,97 % | | |
| 50 | | 30 | 0,254922 | 2,499985 | 0,236502 | 4,81 % | 5,00 % | 4,97 % | | |
| 51 | | | | | | | | | | |

Picture 6.2: A screenshot of the data table used to construct the yield curve.

The infinitely long rate is calculated with equation (3) in section 2.2.4

$$R_{\infty} = \gamma - \frac{\sigma^2}{2\alpha^2}$$

It represents the yield of a bond with $T \rightarrow \infty$. It is useful to remind that the long-term equilibrium and the infinitely long rate are not equal. We can see from the formula however that if the pullback term is large, the second term becomes very small and the long-term equilibrium and infinitely long rate converge.

The next two diagrams illustrate the two other possible shapes of the term structure in a Vasicek model, namely the inverted and humped yield curves:

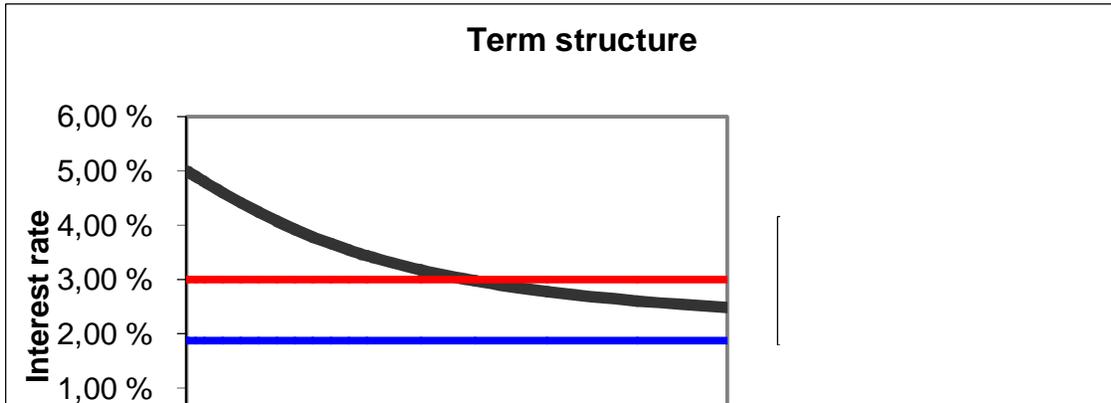


Figure 6.3: An inverted yield curve estimated with parameter values of $\alpha=0,2$ $\beta=3,0\%$, $\gamma=3,0\%$ $r(0)=5,0\%$

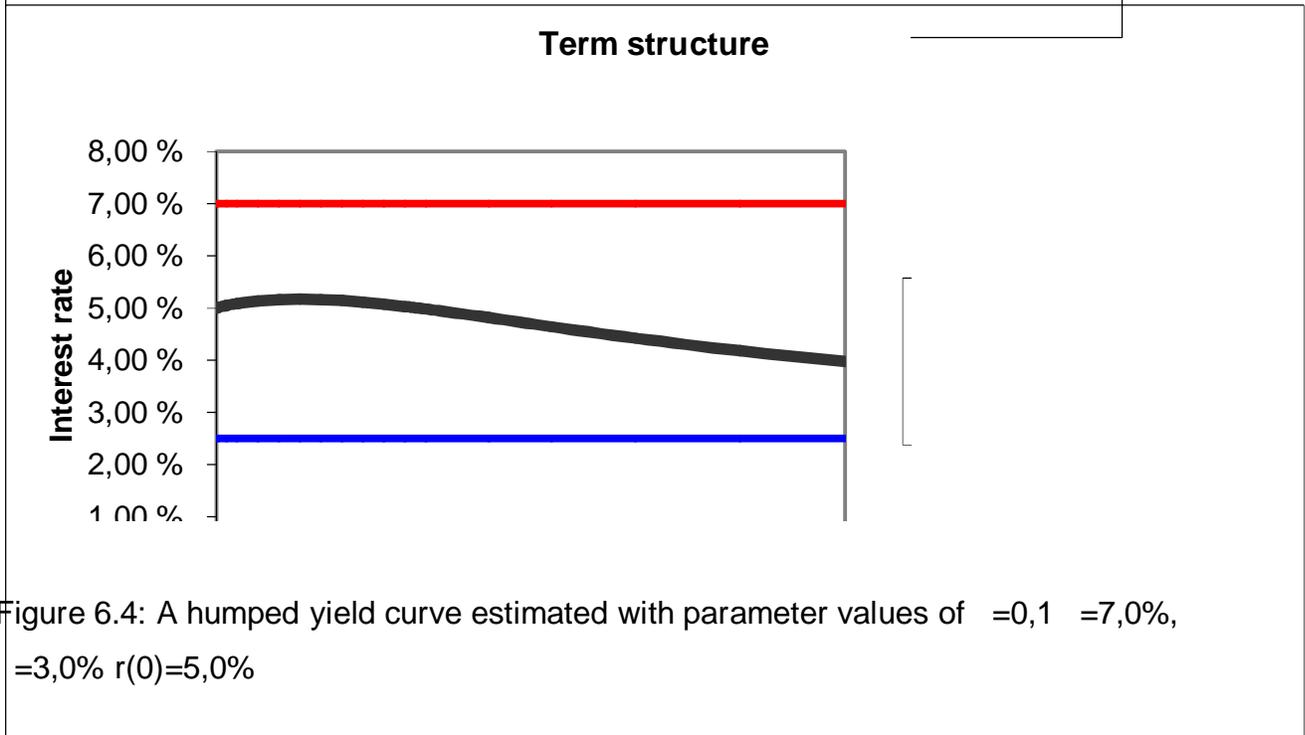


Figure 6.4: A humped yield curve estimated with parameter values of $\alpha=0,1$ $\beta=7,0\%$, $\gamma=3,0\%$ $r(0)=5,0\%$

If we were to draw the current term structure according to our parameter estimates, we would get basically a flat yield curve. This isn't however the situation currently prevailing in the market where the yield curve is upward sloping because of the extraordinarily large market price of risk, which isn't included in our bond pricing equations. Under normal market conditions the Euribor yield curve has been often flat.

7. CONCLUSIONS

In this paper we have developed easy to use Excel-programs, which can be used firstly to estimate the parameters of the Vasicek interest rate model, secondly to model the evolution of the instantaneous interest rate and lastly to model the actual term structure. These are the three main components of interest rate models. One could continue studying the model by trying to model the market price of risk, which plays a major role especially now because of the clogged up interbank markets. Another useful follow-up would be developing programs for valuing various kinds of fixed-income derivatives. One could also develop a program that would give the standard deviations of the estimates in order to evaluate their fit. This is a major drawback of our Excel-program but would require more complex modelling tools such as matlab or statistical programs with optimization tools to overcome. Hence we have stuck with Excel and opted for usability instead of improved accuracy.

In the process we have explained why interest rate models are needed and how dealers in the market use them. This paper is intended to act as a stepping-stone to getting acquainted with interest rate modelling and to give useful tools to implement the Vasicek model. We have explained the usual mathematical terms encountered, when one reads other papers on the subject to ease the process. It goes however without saying that to understand fully interest rate models, especially more complex ones than the Vasicek, one requires a lot of patience and tedious work.

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