LAPPEENRANTA
UNIVERSITY OF TECHNOLOGY

Marko Matikainen

DEVELOPMENT OF BEAM AND PLATE FINITE ELEMENTS BASED ON THE ABSOLUTE NODAL COORDINATE FORMULATION

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In memory of my grandfather

Preface

The research for this dissertation has been accomplished during the years 2005-2009, mainly in the Department of Mechanical Engineering at Lappeenranta University of Technology. The research has also been done during three and four months periods in 2005 and 2007, at the Institute of Technical Mechanics at Johannes Kepler University of Linz, Austria, and at the Department of Mechanical Engineering at Delft University of Technology, The Netherlands.

Firstly, I would like to give thanks to my supervisor Professor Aki Mikkola for suggesting me this interesting research topic and also for mainly organizing the financial support for this work. I am also grateful to my instructor Professor Raimo von Hertzen for giving encouragement and his time for my questions related to the technical mechanics. I would also like to give my best thanks to Dr. Johannes Gerstmayr, for his valuable advice and knowledge in the field of flexible multibody dynamics since 2005 and to Professor Arend Schwab, for an enjoyable cooperation and supervision during and after visiting in 2007. The valuable comments given by Dr. Kari Dufva and Dr. Kimmo Kerkkänen at the beginning of my research are also appreciated.

I would like to thank the reviewers Professor Daniel García-Vallejo from University of Seville and Professor Hiroyuki Sugiyama from Tokyo University of Science for their valuable comments and constructive advice.

Even though the research itself has been enjoyable, I am thankful to all colleagues and friends who suggested I should occasionally take a break from work by joining in coffee breaks or other social events more often.

Juha Laurinolli deserves special acknowledgments/thanks for his help with English in this dissertation.

The first two years of research was mainly funded by the Academy of Finland. In the beginning of 2007, I was accepted as a graduate student in the Finnish National Graduate School in Engineering Mechanics from where I also received the visiting scholarship to the Delft University of Technology in 2007. I am also grateful about the support for three months of visitation at Johannes Kepler University of Linz by the Austrian Science Fund and the Research Foundation of Lappeenranta University of Technology. In addition, completion of the dissertation was funded with a six-month scholarship supplied by the Research Foundation of Lappeenranta University of Technology. I am also grateful for the financial support offered by Emil Aaltonen Foundation, Walter Ahlström Foundation, Finnish Cultural Foundation, South Karelia Regional fund, Lauri ja Lahja Hotisen rahasto and a student travel scholarship in 2009 ECCOMAS Thematic Conference on Multibody Dynamics by Local Organizing Committee.

Finally, millions of thanks to Miia, who never once asked if I would ever able to complete my dissertation.

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Abstract

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Development of beam and plate finite elements based on the absolute nodal coordinate formulation
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The focus of this dissertation is to develop finite elements based on the absolute nodal coordinate formulation. The absolute nodal coordinate formulation is a nonlinear finite element formulation, which is introduced for special requirements in the field of flexible multibody dynamics. In this formulation, a special definition for the rotation of elements is employed to ensure the formulation will not suffer from singularities due to large rotations. The absolute nodal coordinate formulation can be used for analyzing the dynamics of beam, plate and shell type structures.

The improvements of the formulation are mainly concentrated towards the description of transverse shear deformation. Additionally, the formulation is verified by using conventional iso-parametric solid finite element and geometrically exact beam theory. Previous claims about especially high eigenfrequencies are studied by introducing beam elements based on the absolute nodal coordinate formulation in the framework of the large rotation vector approach. Additionally, the same high eigenfrequency problem is studied by using constraints for transverse deformation.

It was determined that the improvements for shear deformation in the transverse direction lead to clear improvements in computational efficiency. This was especially true when comparative stress must be defined, for example when using elasto-plastic material. Furthermore, the developed plate element can be used to avoid certain numerical problems, such as shear and curvature lockings. In addition, it was shown that when compared to conventional solid elements, or elements based on nonlinear beam theory, elements based on the absolute nodal coordinate formulation do not lead to an especially stiff system for the equations of motion.

Keywords: continuum based beam and plate elements, nonlinear finite element formulation, flexible multibody dynamics

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Tiivistelmä

Marko Matikainen
Palkki- ja laattaelementtien kehittäminen absoluuttisten solmukoordinaattien menetelmässä
Lappeenranta, 2009
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Tässä väitöskirjatyössä kehitetään joustavan monikappaledynamiikan tarpeisiin palkki- ja laattaelementtejä, jotka pohjautuvat absoluuttisten solmukoordinaattien menetelmään. Absoluuttisten solmukoordinaattien menetelmä on vastikään esitelty epälineaarinen elementtimenetelmä, joka soveltuu palkkimaisten, laattamaisten ja kuorimaisten rakenteiden dynamiikan analysointiin. Tämän epälineaarisen elementtimenetelmän erikoisuuksina ovat elementtien singulariteeteista vapaa kiertymän kuvaus sekä mahdollisuus poikkipinnan muodonmuutoksen huomioimiseen palkki-, laatta- ja kuorielementeissä.

Väitöskirjatyössä kehitetään sekä laatta- että palkkielementtien leikkausmuodonmuutoksen kuvausta. Lisäksi elementtejä verrataan aiemmin kehitettyihin epälineaaristen elementtimenetelmien elementteihin kuten tavanomaisiin tilavuuselementteihin ja teorialtaan geometrisesti eksaktiin palkkielementtiin. Leikkausmuodonmuutoksen huomioiminen absoluuttisten solmukoordinaattien menetelmään perustuvissa elementeissä edellyttää yleensä elementtien paksuussuuntaisten muodonmuutosten kuvaamisen. Varsinkin suhteellisen ohuilla elementeillä tästä seuraa korkeampien ominaistaajuuksien esiintymistä toisin kuin tavanomaisilla palkkielementeillä. Tämä tyypillisesti pidentää dynamiikan ratkaisuun käytettyä laskenta-aikaa.

Leikkausmuodonmuutoksen kuvausta korkea-asteisissa palkkielementeissä kehitetään käyttämällä vakioleikkausjännitysjakauman asemasta neliöllistä leikkausjännitysjakaumaa elementin paksuussuunnassa. Erityisesti kehitetyn kuvaustavan hyöty siirtymien laskennassa osoitetaan elastoplastisella materiaalimallilla, jolloin vertailujännitys määritetään koko poikkipinnan alueella. Osa tunnetuista puutteellisen kinematiikan kuvauksen tuottamista numeerisista ongelmista ratkaistaan käyttämällä laattaelementin poikittaisen leikkausmuodonmuutoksen kuvauksessa erillistä alennettua interpolaatiota sekä palkkielementin tapauksessa käyttämällä samanasteista interpolaatiota sekä leikkaus- että taivutusmuodonmuutokselle. Jälkimmäinen lähestymistapa pohjautuu pienten siirtymien approksimaatiosta tuttuun leikkausmuodonmuutoksen huomioivaan tasapainoyhtälöön.

Väitöskirjatyössä osoitetaan lisäksi, ettei absoluuttisten solmukoordinaattien menetelmä johda numeerisesti erityisen kankeaan liikeyhtälöryhmään verrattuna esimerkiksi epälineaariseen palkkiteorian tai tilavuuselementtien käyttöön elementtimenetelmässä.

Hakusanat: kontinuumipalkki- ja kontinuumilaattaelementit, epälineaarinen elementtimenetelmä, joustava monikappaledynamiikka

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CONTENTS

LIST OF PUBLICATIONS

This dissertation consists of an overview and the following publications, which are referred to as *Publication I*, *Publication II*, *Publication III*, *Publication IV* and *Publication V* in the text.

I Mikkola, A. M., Matikainen, M. K.
"Development of Elastic Forces for a Large Deformation Plate Element Based on the Absolute Nodal Coordinate Formulation", *Journal of Computational and Nonlinear Dynamics*, Vol. 1, No. 2, 2006, pages 103–108.

II Gerstmayr, J., Matikainen, M. K.
"Analysis of Stress and Strain in the Absolute Nodal Coordinate Formulation", *Mechanics Based Design of Structures and Machines*, Vol. 34, No. 4, 2006, pages 409–430.

III Gerstmayr, J., Matikainen, M. K., Mikkola, A. M.
"A geometrically exact beam element based on the absolute nodal coordinate formulation", *Multibody System Dynamics*, Vol. 20, No. 4, 2008, pages 359–384.

IV Mikkola, M., Dmitrochenko, O., Matikainen, M.
"Inclusion of Transverse Shear Deformation in a Beam Element Based on the Absolute Nodal Coordinate Formulation", *Journal of Computational and Nonlinear Dynamics*, Vol. 4, No. 1, 2009, pages 011004-1–011004-9.

V Matikainen, M. K., von Hertzen, R., Mikkola, A., Gerstmayr, J.
"Elimination of high frequencies in the absolute nodal coordinate formulation", *Journal of Multi-body Dynamics*, Vol. 223, No. 4, 2009.

AUTHORS CONTRIBUTION

The articles were written under the supervision of Prof. Aki Mikkola from Lappeenranta University of Technology and Dr. Johannes Gerstmayr from the Johannes Kepler University of Linz. This dissertation has been written under the supervision of Prof. Aki Mikkola.

In *Publication I*, the author is responsible for the development of the introduced plate element. In addition, the author has played a substantial part in the implementation and programming of the element. The first description of the developed element and numerical results can be seen in the author's Master's thesis [40]. Together, the author and A. Mikkola have finalized the article.

The research related to *Publication II* started during the author's three month visit in summer 2005 to the Institute of Technical Mechanics at the University of Linz for a course entitled "3D elasto-plastic robots". This research has been started in the field of elasto-plastic deformations in multibody systems in combination with the absolute nodal coordinate formulation. The modifications to the beam element investigated, which have already been implemented in the multibody code HOTINT [24, 25], were made by J. Gerstmayr. Together, J. Gerstmayr and the author implemented the elasto-plastic material laws to the beam element. While some sections of the paper, such as the description of the absolute nodal coordinate formulation and certain parts of elasto-plasticity were written by the author, the remaining sections came from J. Gerstmayr.

In *Publication III*, the author is partly responsible for the analytical and numerical solutions. The studied elements have been verified by the author and J. Gerstmayr independently with the author's research code and the multibody code HOTINT. J. Gerstmayr is mainly responsible for the first draft of the paper.

In *Publication IV*, the author is responsible for the implementation of the element and numerical solutions. The first draft of the paper was written by A. Mikkola, but the authors have finalized the paper together.

The idea to eliminate high frequencies with nonlinear constraints was suggested by A. Mikkola. The author has the main responsibility for *Publication V*, i.e. the programming of numerical procedures, the derivation and implementation of the studied elements. The first draft of *Publication V*, including numerical results, was written by the author before *Publication III*. Therefore, it also resulted in generating some ideas for *Publication III*. Finally, at the end of 2008, the author and co-authors finalized the paper together.

Some results of the publications, as well as some further research, have been presented in international conferences by author [42, 43] and co-authors [41, 27, 45, 28, 70].

$\boldsymbol{b}$	vector of body forces
$\boldsymbol{e}$	vector of nodal coordinates at the current configuration
$\overline{\boldsymbol{e}}$	vector of nodal coordinates at the initial configuration
$\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$	base vectors of the inertial frame
$\overline{\boldsymbol{e}}_1$, $\overline{\boldsymbol{e}}_2$, $\overline{\boldsymbol{e}}_3$	base vectors of the reference configuration
$\boldsymbol{e}^{(i)}$	vector of nodal coordinates of element i
g_j	constraint function at node j
$\boldsymbol{g}$	field of gravity
k_s	shear correction factor
$\boldsymbol{n}$	unit normal vector
p, $\hat{p}$	position of arbitrary particle at the current and initial configurations
$\boldsymbol{r}$, $\overline{\boldsymbol{r}}$	position vector of arbitrary particle at the current and initial configurations
$\boldsymbol{r}_0$	position vector of an arbitrary particle of the mid-line or mid-plane
$\boldsymbol{r}_{0,x}$	gradient vector of the mid-line with respect to x
$\boldsymbol{r}_{,\alpha}$, $\overline{\boldsymbol{r}}_{,\alpha}$	gradient vectors with respect to α at current and initial configurations
$\boldsymbol{r}_{,x}^{EB}$	gradient vector with respect to x based on Bernoulli-Euler type discretization
$\boldsymbol{r}_{,y}^{(j)}$	gradient vectors with respect to y at node j
$\dot{\boldsymbol{r}}$	velocity vector of arbitrary particle in the current configuration
$\ddot{\boldsymbol{r}}$	accelerator vector of arbitrary particle in the current configuration
t	time
$\boldsymbol{t}_1$, $\boldsymbol{t}_2$, $\boldsymbol{t}_3$	base vectors of the moving frame
u_1, u_2	components of displacement field
$\boldsymbol{u}$	vector of displacement field
$\boldsymbol{u}^e$	elastic part of displacement field
$\boldsymbol{u}^p$	plastic part of displacement field
x, y, z	physical coordinates of the element
$\boldsymbol{x}$	vector of physical coordinates
A	area of the cross-section
$\boldsymbol{C}$	right Cauchy-Green deformation tensor
$\boldsymbol{D}$, ${}^4\boldsymbol{D}$	material stiffness matrix and tensor
$\boldsymbol{E}$	Green strain tensor

$\boldsymbol{E}^e$	elastic part of Green strain tensor
$\boldsymbol{E}^{ep}$	combination of elastic and plastic tensors
$\boldsymbol{E}^l$	linear (infinitesimal) strain tensor
$\boldsymbol{E}^{le}$	elastic part of linear strain tensor
$\boldsymbol{E}^{lp}$	plastic part of linear strain tensor
$\boldsymbol{E}^n$	nominal (Biot) strain tensor
$\boldsymbol{E}^p$	plastic part of Green strain
$\boldsymbol{E}_i^p$, $\boldsymbol{E}_{i+1}^p$	plastic strain at iteration steps i and $i+1$
$\dot{\boldsymbol{E}}^p$	plastic strain rate
F_y	Huber-von Mises yield condition function
$\boldsymbol{F}$	deformation gradient tensor
$\boldsymbol{F}_e$	vector of elastic forces
$\boldsymbol{F}_{ext}$	vector of external forces
H	height of the element
$\boldsymbol{H}$	non-symmetric strain matrix
I_z	second moment of area around the z-axis
$\mathcal{I}$	functional
$\mathbf{I}$	identity tensor
J	determinant of the deformation gradient tensor
J_2	second deviatoric stress invariant
M_z	applied moment around z-axis
$\boldsymbol{M}$	mass matrix
$N^{(i)}$	bilinear shape function i
$\boldsymbol{N}_i$	eigenvector i
Q_y	shear force
R_{ji}	values of rotation matrix
$\boldsymbol{R}$	rotation tensor
$\boldsymbol{R}_s$	rotation matrix for shear deformation
S_{yy}, S_{zz}	transverse normal stresses
S_{yz}	torsional shear stress
S_{xy}^S, S_{xz}^S	transverse shear stresses due to the shear force
S_{xy}^T, S_{xz}^T	transverse shear stresses due to the torsion
$\boldsymbol{S}$	stress tensor
$\boldsymbol{S}_m$	shape function matrix
$\boldsymbol{T}$	nominal stress tensor
$\boldsymbol{U}$	stretch tensor
V	volume of the element
W	width of the element

W_{ext}	energy due to external forces
W_{int}	strain energy
W_{int}^{p}	plastic part of strain energy
W_{int}^{tot}	strain energy based on total strains
W_{kin}	kinetic energy
W_{pot}	potential energy
ε_{xx}^{1}	bending strain
γ	magnitude of plastic strain
γ_{xy}, γ_{xz}, γ_{yz}	shear deformations
$\gamma_{xz}^{(i)}$, $\gamma_{yz}^{(i)}$	shear deformations at node i
γ_{xz}^{lin}, γ_{yz}^{lin}	linearized shear deformations
κ_x	twist
θ	time dependent angle of rotation
$\theta_{,x}$	rate of rotation of cross-section along the undeformed length of the beam
ρ	mass density
σ_{11}	normal stress
σ_y	yield stress
λ, μ	Lame's material coefficients
λ_i	eigenvalues
ν	Poisson's ratio
$\boldsymbol{\sigma}$	Cauchy stress tensor
ξ, η, ζ	local normalized coordinates of the element
$\boldsymbol{\xi}$	vector of local normalized coordinates
$\Delta\gamma$	increment of plastic strain
Φ	torsion function
Γ_1, Γ_1^0, Γ_2	strains at the moving base
Λ_1	value of the deformation gradient
Ψ	free energy function

CHAPTER 1

Introduction

During the past decades, the dynamic analysis of machines has become important in terms of advanced design. The increased computational power and enhanced formulations allow for the possibility to solve mathematical models that describe the dynamic performance of complex mechanical systems. These types of complex systems may consist of a number of interconnected rigid or flexible bodies, where an analytical solution may not be available. In the field of multibody dynamics, a considerable amount of formalisms are introduced for the dynamic analysis of mechanical systems [58, 60].

Multibody system dynamics offer a computer-based approach to treat and solve dynamic problems of mechanical systems. Multibody system dynamics rely on the description of system kinematics and can be used to solve static, as well as dynamic, equilibrium. This approach can be applied to a wide variety of engineering fields in which optimization and sophisticated design tools are required. Generally, a multibody system consists of a number of bodies that are connected together via constraints. Inherently, the bodies in the multibody system are assumed to be rigid, which may be an acceptable assumption for the analysis of motion and forces in many practical engineering problems. However, in some cases, deformation of the bodies should be taken into consideration in order to improve the accuracy of the numerical solution. The deformation of bodies can be described using a number of approaches. In simple approaches, linear strain-displacement as well as linear stress-strain relations are used by assuming that deformations are small and the material behavior is elastic. In some practical applications, the geometric change of a body may become significant in terms of the dynamic response, making it necessary to employ a nonlinear strain-displacement relation in the mathematical modeling. In addition to geometrical nonlinearity, advanced modeling approaches are capable of taking material nonlinearities into account by using a nonlinear stress-strain relation [21, 52]. Figure 1.1 shows helicopter blades, a belt pulley system and a tire. In these practical applications, the flexibility of mechanical components is significant, and the deformations of bodies should be accounted for in order to obtain accurate results from the mathematical model.

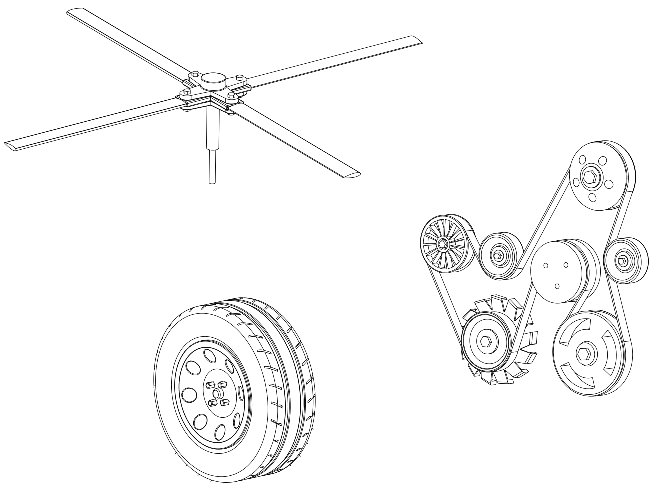

Figure 1.1. Multibody systems where geometrical nonlinearities and material nonlinearities may occur in some of the bodies.

1.1 Flexible multibody system dynamics

In the field of flexible multibody dynamics, the description of motion can be derived using numerous different formalisms. According to [60], the floating frame of reference formulation, the large rotation vector formulation and the absolute nodal coordinate formulation are widely used in the description of flexible bodies in multibody applications. These formulations differ from each other in a number of ways, although a common feature is that they produce exactly zero strain under rigid body movements. This could be considered as a minimum requirement to reach the energy balance in the case of lengthy multibody dynamic simulations. The nonlinear finite element formulations can also be used in flexible multibody system dynamics. Motion in the nonlinear finite element analysis can be represented using the total Lagrangian formulation, updated Lagrangian formulation or corotational formulation. It is important to note that formulations used in flexible multibody dynamics are often related to the finite element formulations. In the floating frame of reference formulation, elastic deformations can be approximated by employing conventional finite elements, whereas in the large rotation vector and the absolute nodal coordinate formulation, the total Lagrangian approach is employed.

Due to the fact that bodies of a multibody system undergo a different magnitude of deformations, it is common to combine different formulations in the multibody simulation. Accordingly, some bodies can be assumed to be rigid while some flexible bodies can be modeled using the floating frame of reference formulation with the assumption of small elastic deformation. For bodies that experience large deformations, such that geometrical and material nonlinearities are involved, the large rotation vector or absolute nodal coordinate formulation can be used in the modeling.

1.1.1 Beam and plate kinematics assumptions

In order to clarify the differences in assumptions associated with beam and plate modeling, some basic preliminaries, including the effect of pure moment and shear force, are explained in this section. In Figure 1.2 a, an undeformed beam with a rectangular cross-section is illustrated. It can be seen from the figure that under pure bending, the longitudinal edges are curved while the transverse edges are rotated (Figure 1.2 b). It is noteworthy that, under pure bending, the transverse edges remain straight and, due to the Poisson effect, the deformed cross-section does not deform out of plane. In the case of shear forces (Figure 1.2 c), the beam ends slide with respect to each other. In this loading condition, the cross-section becomes curved due to the non-uniform shear stress distribution produced by shear force [75, p. 170]. In the beam and plate theories, the Poisson effect is usually neglected and shear stress is assumed to be distributed uniformly. According to these assumptions, the cross-section is assumed to be inextensible.

The Bernoulli-Euler beam theory is based on the kinematics hypothesis according to which the cross-section remains straight, inextensible and normal to the mid-axis under deformation. These assumptions neglect the transverse shear and transverse normal strains and, in three-dimensional cases, shear due to torsion. In the case of plates and shells, the corresponding hypothesis is known as the Kirchhoff-Love hypothesis. Accordingly to the Kirchhoff-Love hypothesis,

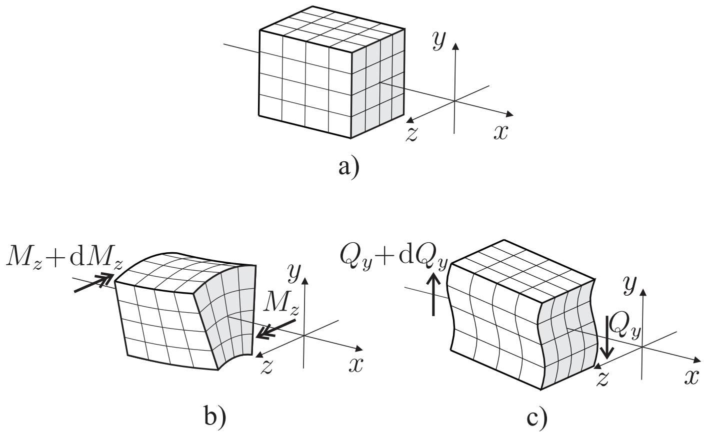

Figure 1.2. Rectangular cross-section. a) Undeformed b) Deformation due to bending moment M_z. c) Deformation due to shear force Q_y.

a transverse fiber of the plate remains straight, inextensible and normal to the mid-plane under deformation. In the Timoshenko beam theory [76], shear deformation is accounted for while the cross-section is still assumed to be straight and inextensible. Due to the description of shear deformation, the beam cross-section can rotate from the normal of the mid-axis. The Timoshenko beam theory assumes that shear strain is constant over the cross-section. The error due to this assumption can be compensated for through the use of a shear correction factor. Several definitions for the shear correction factor are introduced (see [10], for an example). The shear correction factor depends on geometry and material properties, as well as the boundary and loading conditions [78, p. 13]. To alleviate the assumption of constant shear strain and to avoid the use of a shear correction factor, higher order beam theories are introduced [54]. In the second order beam theory, displacement in the thickness direction is assumed to be quadratic, whereas in the third order beam theory, the displacement is correspondingly assumed to be cubic in the thickness direction [78, p. 13]. In the case of plates, the shear deformation can be accounted for by using the Reissner-Mindlin theory. Timoshenko's and Reissner-Mindlin's theories take the shear and rotational inertia effects into account, leading to an accurate modeling approach in case of dynamics for thick beams and plates. For the case of large strains, the nonlinear Reissner's beam theory is introduced in [55]. In this theory, the shear deformable element is described within an inertial frame. Reissner's nonlinear beam theory, as well as Timoshenko's beam theory, assume that the deformed cross-section remains straight i.e. deformation within the cross-section cannot exist. Consequently, the warping effect cannot be taken into account in the mentioned beam theories, for example. However, all theories mentioned above are restricted to linear elasticity, and furthermore, Kirchhoff-Love's plate theory and Timoshenko's beam theory are both based on kinematics simplifications. It is assumed that the thickness of beams and plates is small, such that there is no need to account for three-dimensional elasticity.

1.1.2 The descriptions of motion in nonlinear finite element formulations

The description of large translations and large rotations in nonlinear finite element analysis can be determined using the total Lagrangian formulation, updated Lagrangian formulation or corotational formulation. In the total Lagrangian formulation, the motion of a body is defined with respect to the initial configuration, whereas in the updated Lagrangian formulation the motion is defined with respect to the latest configuration. According to [11, p. 136], the total Lagrangian formulation is an appropriate approach for modeling large rotations and small strains. It is noteworthy, however, that the formulation can also be used in cases of elasto-plastic material with small strains as well as large strains with the hyperelastic material model. The total Lagrangian formulation can be modified to be the updated Lagrangian formulation without any loss of accuracy in the solution [11, p. 146]. The total and updated Lagrangian formulations offer different manners of describing strains and stresses while leading to the same solution, provided that the correct constitutive relation is employed [5, p. 523]. When selecting between Lagrangian formulations, a possible incentive may be the computational efficiency and circumvention of the singularities from rotation used in the total Lagrangian formulation. It has been demonstrated that the use of the updated Lagrangian formulation is computationally more effective than the total Lagrangian formulation for nonlinear static and linearized dynamic problems [6]. It is important to note, however, that the computational efficiency is case-dependent, making it difficult to draw any general conclusions.

The corotational formulation was originally introduced in [79] after which several elements based on the formulation have been introduced, as well. This formulation provides a framework in which standard linear structural elements can be utilized, and therefore, it has become popular in many practical applications. The formulation relies on the decomposition of the total motion of a flexible body into reference rigid body motion and deformation at the co-rotational frame.

The decomposition into rigid body motion and the relative deformation can be accounted for, for instance, by using the polar decomposition theorem. According to [7, p. 185], corotational formulations can be divided into two different categories. In the first category, the embedded coordinate system is attached to each integration point of the element, allowing the formulation to be applied to large strains and large displacements. In the second category, the embedded coordinate system is attached to an element (see [3] for an example). The advantage of this formulation is that conventional finite elements can be used in multibody applications without extensive modification. Isoparametric elements can also be presented in the corotational framework [12, 49]. In [49], an eight-node linear brick element with incompatible displacement modes is used in the framework of corotational formulation to demonstrate the possibility of solving nonlinear problems in a computationally effective manner. In the linear brick element, hyperelasticity is used to describe large strains while demonstrating a close relationship between the Biot strain and the corotated engineering strains. The corotational formulations can not be considered as a geometrically exact approach because the rotations are assumed to be small with respect to a corotational coordination frame. Therefore, without using special treatments, the structural elements based on corotational formulations do not reproduce exactly zero strain under rigid body movements. For this reason, in multibody applications where lengthy simulation times are expected, the corotational formulation may lead to problems in the energy drift without the use of a special algorithm to guarantee energy balance. Nevertheless, the corotational formulation is found to be effective for the simulation of flexible mechanisms [13].

1.1.3 Floating frame of reference formulation

In the case of flexible multibody dynamics, a widely used formalism is the floating frame of reference formulation [62]. In the formulation, a non-inertial reference frame is used to describe large translations and large rotations with respect to inertial coordination. The deformation of a flexible body is defined with respect to a non-inertial reference frame using a set of elastic coordinates. In the floating frame of reference formulation, the deformations are usually assumed to be linear with respect to the non-inertial reference frame. Elastic deformation within the reference frame can be approximated by using the Ritz method, or by using the assumed deformation modes of the body. In [69], where the formulation is introduced for planar flexible mechanisms, the deformation of the body is approximated using the conventional Bernoulli-Euler finite beam elements that are interconnected by constraints. It is possible to obtain the deformation modes of the body through use of component mode synthesis [32, 31].

Component mode synthesis is a model reduction technique that can be used to decrease the degrees of freedom of the finite element model. The reduction makes the computation more effective and it may decrease the stiffness of the system but, unfortunately, also leads to a loss of accuracy. The usage of the reduction technique in the floating frame of reference formulation is explained in detail in [64, 63]. When component mode synthesis is used, the floating frame of reference is difficult to apply to geometrically or materially nonlinear problems. If nonlinearities are taken into account, using for example Ritz approximation for a displacement field, the elastic forces are nonlinear. It is demonstrated in [2] that material nonlinearities can be accounted for within the floating frame of reference formulation by using isoparametric finite elements and a body fixed reference frame. Due to the use of a body fixed reference frame, the approach is different than the traditional updated Lagrangian formulation [2].

The use of the floating frame of reference formulation leads to a simple description for strain energy with a constant representation of the stiffness matrix, and a highly nonlinear description for the kinetic energy. This is due to coupling between variables of reference and relative motion. In some cases, the constraint equations may become cumbersome to model due to the kinematics description of a flexible body. The main advantages of this approach are the exact description of rigid body motion and the possibility to decrease the number of degrees of freedom by employing component mode synthesis. It is also notable that the formalism is not limited to beam and plate type structures. However, due to the use of relative variables in the description of deformation, centrifugal and Coriolis terms will occur in the equations of motion.

1.1.4 Geometrically exact formulations

The geometrically exact beam element has been examined in numerous studies. In this theory, geometrical approximations, such as the linearization of rotation parameters, are not employed. This formulation is suitable for multibody applications in which large deformations, i.e. large displacements and large strains, need to be accounted for. When the theory is applied to practical applications, the element can be described within the concept of the total Lagrangian formulation. However, to overcome the singularity problem associated with Euler rotation angles in the total Lagrangian formulation, the updated Lagrangian formulation or quaternions can be used. Simo and Vu-Quoc present the geometrically exact beam formulation based on the Reissner theory with respect to the large rotation vector formulation in [67, 68]. In this approach, spatial basis functions are used in the element discretization procedure [68].

The large rotation vector formulation is a widely used approach and it has been extensively studied for two and three-dimensional beam elements [33]. Finite elements based on the large rotation vector formulation are discretized using position and rotational nodal coordinates. This discretization leads to a constant description of the mass matrix for two-dimensional elements. However, in three-dimensional cases, discretization leads the mass matrix to no longer be constant, regardless of the choice of rotational coordinates. It is important to note that the cross-section is described by an orthonormal moving basis leading to an orthogonal representation of the rotation matrix. This representation is favorable as it simplifies the element computation [66].

In [47], the beam element based on geometrically exact beam theory is introduced within the framework of the total Lagrangian formulation without singularity problems. The element is based on the Timoshenko-Reissner theory, and singularities of the rotation angles are avoided by varying parameterization on the rotation manifold. From a computational point of view, it is beneficial for the beam formulation to be presented in a manner in which the solution can be determined with a constraint free manifold, as it leads to a system of ordinary differential equations. In this formulation, the expression of the mass matrix is simple, but unfortunately, not constant. The three-dimensional element is defined using six degrees of freedom at a node. In the element, linear interpolation is used for displacements and rotations. This formulation appears to be effective since quaternions are not employed. It is noteworthy that the use of quaternions, such as Euler parameters, will result in one extra constraint and one extra rotation parameter at the node when compared to the use of Euler rotation angles. This type of total Lagrangian parameterization is also introduced for rigid bodies in [48].

1.1.5 Absolute nodal coordinate formulation

The absolute nodal coordinate formulation is a nonlinear finite element approach that is based on the use of global position and gradient coordinates. The formulation is designed for analysis of large deformations in multibody applications [60]. The absolute nodal coordinate formulation can be used for two or three-dimensional beams, plates and shells [51, 65, 46, 16, 15].

The kinematics description of an element based on the formulation does not include the rotational degrees of freedom. Therefore, the use of quaternions to avoid the singularity problem of finite rotations under three-dimensional rotations is not needed. In this formulation, gradient coordinates that are partial derivatives of the position vector are used to describe the cross-section or fiber orientations. Therefore, all nodal coordinates are described in an inertial frame allowing for the usage of the total Lagrangian approach, such as in the case of large rotation vector formulations and conventional solid elements. The use of the absolute nodal coordinate formulation leads to benefits including a constant mass matrix, which simplifies the description of the equations of motion. Due to the use of a global description of the element configuration, the estimation for contact surfaces and the description of geometric constraints, such as for a sliding joint, are straightforward - particularly when compared to the floating frame of reference formulation [73]. On the other hand, non-conservative forces, such as internal damping, are cumbersome to describe in the formulation [22]. Due to the use of positions and their derivatives, the Hermite base functions are usually employed in the elements based on the absolute nodal coordinate formulation.

In order to define an element into the framework of the absolute nodal coordinate formulation, the element should meet several requirements. All of these requirements should also be valid in three-dimensional cases and can be expressed as follows:

- Elements based on the absolute nodal coordinate formulation can be used for dynamic problems, such that the inertial forces are exactly described. Elements based on the absolute nodal coordinate formulation can be considered as geometrically exact because no geometrical simplifications are necessary.
- The mass matrix should be consistent and, as a trademark of the absolute nodal coordinate formulation, it should be constant. It is important to reiterate that the mass matrix is also constant for three-dimensional beam and plate elements based on the absolute nodal coordinate formulation.
- The element discretization is performed by using spatial shape functions with absolute positions and their gradients. Note that approximations for rotation parameters are not used in the formulation.

The elements based on the absolute nodal coordinate formulation can be categorized into conventional non-shear deformable elements [17] or shear deformable elements. In the formulation, shear deformation can be captured by introducing gradient coordinates in the element transverse direction. Elements that include transverse gradient vectors are often referred to as fully-parameterized elements. In this case, the elastic forces of the element can be defined by using three-dimensional elasticity or the elastic line approach. In case of three-dimensional elasticity, the strains and stresses are defined using general continuum mechanics. The elements based on three-dimensional elasticity relax some of the assumptions used in the conventional elements and they can account for the non-linear material models in a straightforward manner. It is important to note that the use of fully-parameterized elements allows cross-sectional or fiber deformation to be described. The transverse fibers of existing plate elements based on the absolute nodal coordinate formulation remain straight, but are extensible. This implies that plate elements can be used to account for shear deformation and deformation in the thickness direction. In some elements, the transverse Poisson contraction effect can also be taken into account. It is possible to describe geometrical and material nonlinearities in the element based on three-dimensional elasticity [51, 74]. Conventional elements based on the absolute nodal coordinate formulation are discretized using global positions and gradient coordinates in the element longitudinal direction. In the elements based on this approach, strains and stresses are described on the middle line or middle plane employing the elastic line approach.

Contrary to the kinematics of conventional solid finite elements, the definition of higher order elements in the absolute nodal coordinate formulation does not fully describe the order of the displacement approximation. In the absolute nodal coordinate formulation, the fully-parameterized elements employ all gradient vectors at a nodal location [29]. In lower order elements based on the absolute nodal coordinate formulation (see [37] for an example) some of the gradients are omitted. Due to the fact that the inertial description is simple and interpolations of rotational parameters are not needed, the formulation has potential to be effective in large deformation multibody applications. Examples where the absolute nodal coordinate formulation is seen to be more effective than the floating frame of reference formulation are shown in [14]. Recently, the elements based on the absolute nodal coordinate formulation have been applied to practical applications, including the belt-drive and pantograph-catenary systems [36, 29]. Furthermore, in order to extend the usefulness of the absolute nodal coordinate formulation for applications that include fluid-structure interaction, a special pipe-element has been introduced [72].

1.1.6 Other formulations for continuum based beam and plate elements

In the study by Avello et al. [4], rotations and deformations of the cross-section are described by using nine parameters at a nodal location. In the study, the cross-section is forced to be rigid using constraint equations. The main advantage in this formulation is a constant mass matrix, although the constraints may be problematic in terms of the numerical solution.

In the dissertation by Rhim [56], the absolute motion of spatial beam elements is described using global shape functions. In the study, rotation at the nodal location is defined with two basis vectors along the cross-section, which creates nine degrees of freedom at the node. The strains and stresses are defined using a continuum mechanics approach. In this approach, locking problems - such as the Poisson locking - are observed and alleviated using a modified material model for continuum in which the Poisson effect is neglected; see also [57].

Continuum based beam and plate elements in two and three-dimensional applications are proposed. Elements are usually derived from solid elements to account for the kinematics assumption in beams or shells. Using this approach, which is also called the degenerated solid approach, only the degrees of freedom for nodes at a line or mid plane are used. For these elements, any of the continuum material laws can be used, provided the plane stress condition is valid. In the case of large rotations, quaternions must be employed [7, p. 514–550]. Elements based on the degenerated solid approach are straightforward to derive and capable of describing large deformations.

1.2 Objectives and outline for the dissertation

The objective of this study is to examine the descriptions of large deformations in multibody dynamics. Approximately a decade ago, the absolute nodal coordinate formulation was introduced for large deformations in multibody dynamics. One of the goals of this thesis is to present and solve problems associated with finite elements based on the absolute nodal coordinate formulation. In this study, properties of finite elements based on the absolute nodal coordinate formulation are investigated. Furthermore, comparisons between other formulations, aside from the absolute nodal coordinate formulation, are performed.

This study is organized as follows. In Chapter 2, the equations of motion for the absolute nodal coordinate formulation are shown. Elastic forces are presented using the general continuum mechanics approach, which leads to the possibility of using different material models known from continuum mechanics. In addition, difficulties associated with large strains are briefly discussed. In Chapter 3, an overview of the results and the background of each publication are briefly presented. Finally, Chapter 4 summarizes the conclusions reached in the dissertation.

1.3 Contribution to the absolute nodal coordinate formulation

According to the studies within this thesis, several improvements for beam and plate elements based on the absolute nodal coordinate formulation are proposed. The improvements are introduced in order to avoid slow convergence and inaccurate results. A study of beam elements in the field of elasto and elasto-plastic applications is carried out. This study is based on *Publication II*, *Publication IV* and partly on *Publication III*. In addition, the study of high frequencies due to transverse deformations and constrained numerical problems is performed and partly based on *Publication III* and *Publication V*. Based on these studies, previous claims that the absolute nodal coordinate formulation will lead to an especially stiff system or ill-conditioned matrices can be disproved. In *Publication I*, the improved plate element based on the absolute nodal coordinate formulation is introduced. In this plate, certain numerical lockings were avoided through the use of an improved description for kinematics.

CHAPTER 2

Absolute nodal coordinate formulation

The absolute nodal coordinate formulation is a finite element approach in which beam and plate elements are described with an absolute position and its gradients. Using the components of the deformation gradient instead of conventional rotational coordinates, the absolute nodal coordinate formulation leads to an exact description of inertia for the rigid body with a constant mass matrix. Transverse deformations can also be accounted for with the gradient components. Therefore, elements based on the absolute nodal coordinate formulation can be considered as more advanced than classical beam and plate elements. However, within fully-parameterized elements, different types of locking phenomena may occur due to low order interpolation in the transverse direction. In order to overcome this problem, alternative approaches are introduced to define the elastic forces, see for example [59, 29]. To clarify the absolute nodal coordinate formulation, a fully-parameterized element is described at the beginning of this section. These elements allow for the energy from kinetic, strain and external forces to be defined in a consistent manner.

2.1 Kinematics of the element

In elements based on the absolute nodal coordinate formulation, kinematics can be expressed using spatial shape functions and global coordinates. In Figure 2.1, the kinematics of the fully-parameterized beam element is shown. This isoparametric beam element includes two nodes, both of which are defined by 12 degrees of freedom.

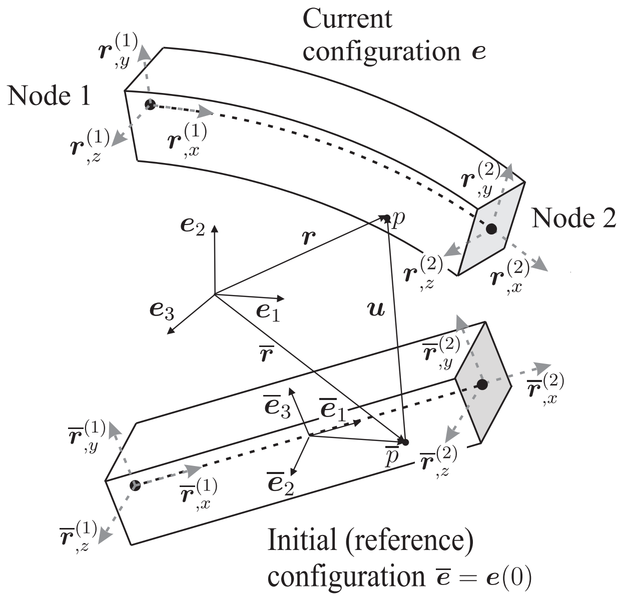

Figure 2.1. Description of the position of an arbitrary particle in the fully-parameterized beam element. Points $\overline{p}$ and p refer to the same particle at different configurations after displacement $\boldsymbol{u}$. The gradient vectors at nodes are shown by dashed arrows.

The position of an arbitrary particle p in the isoparametric element can be defined in the inertial frame as follows:

$$\boldsymbol{r} = \boldsymbol{S}_m(\boldsymbol{x})\boldsymbol{e} = \boldsymbol{S}_m(\boldsymbol{\xi}(\boldsymbol{x}))\boldsymbol{e}\,, \tag{2.1}$$

where $\boldsymbol{S}_m$ is a shape function matrix, $\boldsymbol{e} = \boldsymbol{e}(t)$ is the vector of nodal coordinates and vector $\boldsymbol{x} = x\overline{\boldsymbol{e}}_1 + y\overline{\boldsymbol{e}}_2 + z\overline{\boldsymbol{e}}_3$ includes physical coordinates. For the isoparametric elements, the shape functions can be expressed using physical coordinates $\boldsymbol{x}$ or local coordinates $\boldsymbol{\xi}$ in the range -1 ... +1. The kinematics of the element in the reference configuration at time $t = 0$ can be described as $\overline{\boldsymbol{r}} = \boldsymbol{S}_m(\boldsymbol{x})\overline{\boldsymbol{e}}$, where $\overline{\boldsymbol{e}} = \boldsymbol{e}(0)$. The vector $\boldsymbol{e}$ contains both translational and rotational coordinates of the element, and it can be written at node i of the three-dimensional fully-parameterized element as follows:

$$\boldsymbol{e}^{(i)} = \left[\begin{array}{cccc} \boldsymbol{r}^{(i)T} & \boldsymbol{r}_{,x}^{(i)T} & \boldsymbol{r}_{,y}^{(i)T} & \boldsymbol{r}_{,z}^{(i)T} \end{array} \right]^T , \tag{2.2}$$

where the following notations for gradients are used:

$$\boldsymbol{r}_{,\alpha}^{(i)} = \begin{bmatrix} r_{1,\alpha}^{(i)} \\ r_{2,\alpha}^{(i)} \\ r_{3,\alpha}^{(i)} \end{bmatrix} = \frac{\partial \boldsymbol{r}^{(i)}}{\partial \alpha} ; \qquad \alpha = x, y, z .$$

2.2 Equations of motion for the element

The weak form (variational form) of the equations of motion in the Lagrangian (material) description can be derived from the functional $\mathcal{I}$, see for example [30, p. 36], which can be written as

$$\mathcal{I} = \int_{t_1}^{t_2} \left(W_{kin} - W_{pot} \right) \, \mathrm{d}t , \tag{2.3}$$

where t_1 and t_2 are integration limits with respect to time t, W_{kin} is the kinetic energy of the element and W_{pot} is the potential energy which includes the internal strain energy W_{int} and the potential energy W_{ext} due to conservative external forces. The potential energy can be written as follows:

$$W_{pot} = W_{int} - W_{ext} . \tag{2.4}$$

In this study, non-conservative forces are not taken into account. The variation of the functional leads to

$$\delta \mathcal{I} = \delta \int_{t_1}^{t_2} \left(W_{kin} - W_{int} + W_{ext} \right) \, \mathrm{d}t = 0 . \tag{2.5}$$

The variations of the energies can be written as

$$\delta W_{kin} = \int_V \rho \dot{\boldsymbol{r}} \cdot \delta \dot{\boldsymbol{r}} \, \mathrm{d}V, \tag{2.6}$$

$$\delta W_{int} = \int_V \boldsymbol{S} : \delta \boldsymbol{E} \, \mathrm{d}V, \tag{2.7}$$

$$\delta W_{ext} = \int_V \boldsymbol{b} \cdot \delta \boldsymbol{r} \, \mathrm{d}V, \tag{2.8}$$

where $:$ denotes the double dot product, ρ is the mass density, $\boldsymbol{S}$ is the second Piola-Kirchhoff stress tensor, $\boldsymbol{E}$ is the Green strain tensor and $\boldsymbol{b}$ is the vector of body forces. In the special case of gravity, the body forces can be written as $\boldsymbol{b} = \rho \boldsymbol{g}$, where $\boldsymbol{g}$ is the field of gravity. The Green strain tensor can be written as

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T \boldsymbol{F} - \mathbf{I}) \, , \tag{2.9}$$

where $\mathbf{I}$ is the identity tensor and $\boldsymbol{F}$ is the deformation gradient tensor, which can be presented in terms of the initial and current configurations $\overline{\boldsymbol{r}}$ and $\boldsymbol{r}$ as follows:

$$\boldsymbol{F} = \frac{\partial \boldsymbol{r}}{\partial \overline{\boldsymbol{r}}} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}} \left(\frac{\partial \overline{\boldsymbol{r}}}{\partial \boldsymbol{x}} \right)^{-1} . \tag{2.10}$$

Integrating the variation of the kinetic energy in Equation (2.3) by parts within the time interval t_1 and t_2 yields

$$\Big/_{t_1}^{t_2} \int_V \rho \dot{\boldsymbol{r}} \cdot \delta \boldsymbol{r} \, \mathrm{d}V + \int_{t_1}^{t_2} \Big(- \int_V \rho \ddot{\boldsymbol{r}} \cdot \delta \boldsymbol{r} \, \mathrm{d}V - \int_V \boldsymbol{S} : \delta \boldsymbol{E} \, \mathrm{d}V + \int_V \boldsymbol{b} \cdot \delta \boldsymbol{r} \, \mathrm{d}V \Big) \, \mathrm{d}t = 0 \, , \tag{2.11}$$

where the term $\Big/_{t_1}^{t_2} \int_V \rho \dot{\boldsymbol{r}} \cdot \delta \boldsymbol{r} \, \mathrm{d}V = 0$ because the position vector is specified at the endpoints t_1 and t_2. The weak form of the equations of motion for an element can be written as follows:

$$\int_V \rho \ddot{\boldsymbol{r}} \cdot \delta \boldsymbol{r} \, \mathrm{d}V + \int_V \boldsymbol{S} : \delta \boldsymbol{E} \, \mathrm{d}V - \int_V \boldsymbol{b} \cdot \delta \boldsymbol{r} \, \mathrm{d}V = 0 \, . \tag{2.12}$$

Using interpolation for the position vector $\boldsymbol{r}$, the variations of energy with respect to the nodal coordinates can be expressed. The variation of the kinetic energy can be represented as

$$\delta W_{kin} = \int_V \rho \ddot{\boldsymbol{r}} \cdot \delta \boldsymbol{r} \, \mathrm{d}V = \ddot{\boldsymbol{e}}^T \int_V \rho \boldsymbol{S}_m^T \boldsymbol{S}_m \, \mathrm{d}V \cdot \delta \boldsymbol{e} \,, \tag{2.13}$$

from which the mass matrix of the element can be identified as follows:

$$\boldsymbol{M} = \int_V \rho \boldsymbol{S}_m^T \boldsymbol{S}_m \, \mathrm{d}V \,. \tag{2.14}$$

As can be concluded from Equation (2.14), the mass matrix is constant as it is not a function of the nodal coordinates. This will save time on computation, especially when an explicit time integration method is used. However, this advantage may be marginal when implicit time integration is required. The virtual work for the externally applied forces can be written as

$$\delta W_{ext} = \int_V \boldsymbol{b}^T \delta \boldsymbol{r} \, \mathrm{d}V = \int_V \boldsymbol{b}^T \boldsymbol{S}_m \, \mathrm{d}V \cdot \delta \boldsymbol{e} \,, \tag{2.15}$$

where $\boldsymbol{b}$ is the vector of body forces. The vector of externally applied forces can be identified from Equation (2.15) as follows:

$$\boldsymbol{F}_{ext} = \int_V \boldsymbol{b}^T \boldsymbol{S}_m \, \mathrm{d}V \,. \tag{2.16}$$

The variation of the strain energy with respect to the nodal coordinates can be written as

$$\delta W_{int} = \int_V \boldsymbol{S} : \delta \boldsymbol{E} \, \mathrm{d}V = \int_V \boldsymbol{S} : \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{e}} \, \mathrm{d}V \cdot \delta \boldsymbol{e} \,, \tag{2.17}$$

The vector of elastic forces can be identified from Equation (2.17) as follows:

$$\boldsymbol{F}_e = \int_V \boldsymbol{S} : \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{e}} \, \mathrm{d}V \,. \tag{2.18}$$

2.3 Constitutive models

In this section, different constitutive models, primarily from *Publication I- Publication V*, are shortly described. Some of the constitutive models introduced in this section can be applied to large strain cases, allowing the absolute nodal coordinate formulation to be used for larger deformation problems.

2.3.1 Small strains

Among literature related to continuum elements based on the absolute nodal coordinate formulation, the St. Venant-Kirchhoff material model is the most widely occurring model. This is the simplest model for elasticity and is called hyperelasticity; it is also known as Green elasticity. In this material model, Green strain and Piola-Kirchoff stress tensors are used to define the constitutive relation. It is well known, however, that such a material model should only be used for small strains, and practical uses beyond the small strain regime are rare [9, p. 120].

Hyperelasticity is a constitutive theory where the elastic response is independent of the load history. In other words, the material response is assumed to be path-independent [9, p. 118]. This means that the material state can be uniquely defined by using the selected strain measure $\boldsymbol{E}$ [34, p. 469]. The free energy (elastic potential) for the St. Venant-Kirchhoff material model can be written as follows:

$$\Psi(\boldsymbol{E}) = \frac{1}{2}\lambda(\text{tr}\boldsymbol{E})^2 + \mu\boldsymbol{E} : \boldsymbol{E}\,, \tag{2.19}$$

where λ and μ are Lame's material coefficients. The stress tensor can be obtained through the use of the stress-strain relation as follows:

$$\boldsymbol{S} = \frac{\partial\Psi}{\partial\boldsymbol{E}} = \lambda(\text{tr}\boldsymbol{E})\mathbf{I} + 2\mu\boldsymbol{E}\,. \tag{2.20}$$

The stress and strain tensors are frame-indifferent (objective) under rigid body motion. It is important to reiterate that the St. Venant-Kirchhoff material is designed for small strains, and in the case of large strains, it will lead to unnatural solutions. This can be easily shown, for example, by representing the second Piola-Kirchhoff stress tensor as Cauchy stress. Cauchy stress is designated for large strains and it can also be determined from the second Piola-Kirchhoff stress tensor as follows:

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T \,, \tag{2.21}$$

where $J = \det\boldsymbol{F}$. For example, using the deformation gradient $\boldsymbol{F} = \Lambda_1\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$ in Equation (2.21), the result will be that normal stress $\sigma_{11} = \Lambda_1\mu(\Lambda_1^2 - 1)$. Obviously, it is not physically correct that $\Lambda_1 \to 0; \sigma_{11} \to 0$. The second Piola-Kirchhoff stress is objective, while in large deformation cases, the nominal stress (Biot stress) can be employed in the definition of elastic forces. To determine the nominal stress, the eigenvalues and eigenfrequencies of the right Cauchy-Green tensor $\boldsymbol{C}$ are defined as

$$\boldsymbol{C} = \sum_{i=1}^{3} \lambda_i^2 \boldsymbol{N}_i \otimes \boldsymbol{N}_i \,, \tag{2.22}$$

where $\boldsymbol{N}_i$ are the eigenvectors and λ_i^2 correspond to the eigenvalues of the right Cauchy-Green tensor $\boldsymbol{C}$. Therefore, understanding that $\boldsymbol{U} = (\boldsymbol{F}^T\boldsymbol{F})^{0.5} = \boldsymbol{C}^{0.5}$, the eigenvalues of the right stretch tensor $\boldsymbol{U}$ are λ_i. The deformation gradient can be expressed as a product of the rotation tensor and the right stretch tensor using the polar-decomposition theorem as follows:

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} \,. \tag{2.23}$$

In some cases, if the rotation tensor is known beforehand, then the stretch tensor can be straightforwardly obtained.

2.3.2 Large strains

For large strains, a nonlinear stress-strain relation should be employed. Some nonlinear constitutive models for large strains based on hyperelasticity are used with the absolute nodal coordinate beam elements in [39]. In the study conducted by Gerstmayr and Irschik [26], the hyperelastic material model for a Bernoulli-Euler type beam element based on the absolute nodal coordinate formulation is introduced for large strains. The authors propose that the strain energy should be defined using a nominal strain tensor instead of a nonlinear Green strain tensor. Therefore, with regard to work conjugation of the strain and stress, the nominal stress tensor must also be defined. The nominal strain is linear, and as a result, the element can be compared with Simo's geometrically exact beam element [67, 68] where axial and bending strains are also decoupled. The nominal strain tensor (Biot strain tensor) can be written using the stretch tensor as

$$\boldsymbol{E}^n = \boldsymbol{U} - \mathbf{I}\,, \tag{2.24}$$

and the stress tensor $\boldsymbol{T}$, which is the work conjugate to Biot strain, can be written as

$$\boldsymbol{T} = \frac{1}{2}(\boldsymbol{S}\boldsymbol{U} + \boldsymbol{U}\boldsymbol{S})\,. \tag{2.25}$$

The nominal strain is linearly-dependent on displacements, leading to a suitable expression for large strains.

2.3.3 Elastoplasticity

In this study, ideally perfect plasticity for small strains is taken into account. In the case of small and large strain plasticity, the displacement field $\boldsymbol{u}$ can be divided into elastic and plastic components as

$$\boldsymbol{u} = \boldsymbol{u}^e + \boldsymbol{u}^p\,, \tag{2.26}$$

where $\boldsymbol{u}^e$ and $\boldsymbol{u}^p$ are the elastic and plastic displacements, respectively. In case of deformation, the linear strain tensor $\boldsymbol{E}^l$ can be written as

$$\boldsymbol{E}^l = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}^T\right)\,. \tag{2.27}$$

When Equation (2.26) is inserted into Equation (2.27), the strain tensor can be divided into two symmetrical tensors as

$$\boldsymbol{E}^l = \boldsymbol{E}^{le} + \boldsymbol{E}^{lp} \equiv \frac{1}{2}\left(\frac{\partial \boldsymbol{u}^e}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{u}^e}{\partial \boldsymbol{x}}^T\right) + \frac{1}{2}\left(\frac{\partial \boldsymbol{u}^p}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{u}^p}{\partial \boldsymbol{x}}^T\right)\,, \tag{2.28}$$

where $\boldsymbol{E}^{le}$ and $\boldsymbol{E}^{lp}$ are the elastic and plastic parts of the linear strain tensor. It is noteworthy, however, that in *Publication II* the Green strain tensor is used in the definition of elastic forces to account for geometric nonlinearity, which makes it possible to account for the geometric stiffening effect, for example.

For more information about the effect of geometric stiffening in different multi-body formulations, see [35, 44]. The Green strain can be presented using the displacement field (assuming that the coordinations $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and $\{\overline{\boldsymbol{e}}_1, \overline{\boldsymbol{e}}_2, \overline{\boldsymbol{e}}_3\}$ are parallel) as follows:

$$\boldsymbol{E} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}^T + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}^T \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right) . \tag{2.29}$$

It can be shown that the decomposition of the Green strain into elastic and plastic parts is more complex than in the case of the linear strain tensor $\boldsymbol{E}^l$. Similarly to the decomposition of the linear strain tensor, the displacement field Equation (2.26) can be inserted into the Green strain which can be written as follows:

$$\boldsymbol{E} = \boldsymbol{E}^e + \boldsymbol{E}^p + \boldsymbol{E}^{ep} , \tag{2.30}$$

where $\boldsymbol{E}^e$ and $\boldsymbol{E}^p$ are the elastic and plastic parts of the Green strain tensor, and tensor $\boldsymbol{E}^{ep}$ includes components of both elastic and plastic parts. These tensors can be written as

$$\boldsymbol{E}^e = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}^e}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{u}^e}{\partial \boldsymbol{x}}^T + \frac{\partial \boldsymbol{u}^e}{\partial \boldsymbol{x}}^T \frac{\partial \boldsymbol{u}^e}{\partial \boldsymbol{x}}\right) , \tag{2.31}$$

$$\boldsymbol{E}^p = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}^p}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{u}^p}{\partial \boldsymbol{x}}^T + \frac{\partial \boldsymbol{u}^p}{\partial \boldsymbol{x}}^T \frac{\partial \boldsymbol{u}^p}{\partial \boldsymbol{x}}\right) , \tag{2.32}$$

$$\boldsymbol{E}^{ep} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}^e}{\partial \boldsymbol{x}}^T \frac{\partial \boldsymbol{u}^p}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{u}^p}{\partial \boldsymbol{x}}^T \frac{\partial \boldsymbol{u}^e}{\partial \boldsymbol{x}}\right) . \tag{2.33}$$

As can be seen from Equation (2.33), both elastic and plastic components are combined in the tensor $\boldsymbol{E}^{ep}$. Therefore, $\boldsymbol{E}^{ep}$ shows that it is simpler to use the displacement field decomposition for the linear strain tensor than for the nonlinear strain tensor. As a result, it is not possible to determine the elastic and plastic components of the strain tensor. This may become problematic when using this approach for large strain plasticity. Therefore, the multiplicative decomposition of the deformation gradient for defining the elastic and plastic components [9, p. 232] or, alternatively, the hypoelastic material model based on Eulerian rate theory [80] can be used. In addition, the behavior of the materials, such as metal under the influence of large strains, does not belong to the theory of hyperelasticity [9, p. 231]. In *Publication II*, it is assumed that the combination of elastic and plastic components can be neglected, i.e. $\boldsymbol{E}^{ep} = 0$. Therefore, in this study, the additive decomposition of the Green strain is used as

$$\boldsymbol{E} = \boldsymbol{E}^{e} + \boldsymbol{E}^{p}\,. \tag{2.34}$$

Numerical examples in *Publication II* are chosen in such a manner that the elastic and plastic strains are relatively small. The stress $\boldsymbol{S}$ depends on the elastic component of strain only, and it can therefore be defined as

$$\boldsymbol{S} = {}^{4}\boldsymbol{D} : \boldsymbol{E}^{e} = {}^{4}\boldsymbol{D} : (\boldsymbol{E} - \boldsymbol{E}^{p})\,. \tag{2.35}$$

In case of perfect plasticity, the Huber-von Mises yield condition is expressed as

$$F_y = \sqrt{3J_2} - \sigma_y \leq 0\,, \tag{2.36}$$

where $J_2 = \frac{1}{2}\mathrm{dev}(\boldsymbol{S}) : \mathrm{dev}(\boldsymbol{S})$ is the second deviatoric stress invariant and σ_y is the yield stress. The stress deviator tensor is defined as

$$\mathrm{dev}(\boldsymbol{S}) = \boldsymbol{S} - \frac{1}{3}\mathrm{tr}(\boldsymbol{S})\mathbf{I}\,. \tag{2.37}$$

The plasticity theory based on the Huber-von Mises criterion is often referred to as J_2-plasticity. The plastic strain rate $\dot{\boldsymbol{E}}^{p}$ is presented with the associative flow rule as

$$\dot{\boldsymbol{E}}^{p} = \gamma \frac{\partial F_y}{\partial \boldsymbol{S}} = \gamma\, \mathrm{dev}(\boldsymbol{S})\,, \tag{2.38}$$

where γ is the magnitude of the plastic strain and $\frac{\partial F_y}{\partial \boldsymbol{S}}$ is the gradient that defines the direction of the flow. If the yield condition satisfies $F_y = 0$, the behavior is plastic and, consequently, the increment of plastic strain can be computed from Equation (2.38). On the other hand, if the behavior is elastic, then the yield condition gives $F_y < 0$ and the increment of plastic strain is zero. These relations are unified in the so-called Kuhn-Tucker complementary conditions as follows:

$$\gamma \geq 0, \qquad F_y \leq 0 \qquad \text{and} \qquad \gamma F_y = 0\,. \tag{2.39}$$

In case of perfect plasticity, the elastic forces can be determined by dividing the volume integral into two parts based on total and plastic strains as follows:

$$\begin{aligned} \delta W_{int} &= \delta W_{int}^{tot} + \delta W_{int}^{p} \\ &= \int_V {}^4\boldsymbol{D} : \boldsymbol{E} : \delta \boldsymbol{E} \, \mathrm{d}V - \int_V {}^4\boldsymbol{D} : \boldsymbol{E}^p : \delta \boldsymbol{E} \, \mathrm{d}V \,, \end{aligned} \tag{2.40}$$

where δW_{int}^{tot} can be integrated by using Gaussian quadratures and δW_{int}^{p} can be defined by using plastic cells (subregions, sub volumes). The plastic cells have equal size and the plastic strain is constant over the cell. Instead of the plastic cells, one can use the plastic integration points and interpolate between them. In order to find out the accurate distribution of plastic strains, the number of plastic cells is higher than that of the needed integration points. The number of plastic cells is defined so that the plastic strains are converged. The plastic strain distribution can be defined with an iterative procedure where the plastic strain for every cell at step $i+1$ is determined by integrating Equation (2.38) over time step $[t_i, t_{i+1}]$ as follows:

$$\begin{aligned} \boldsymbol{E}_{i+1}^{p} &= \boldsymbol{E}_{i}^{p} + \int_{t_i}^{t_{i+1}} \gamma \, \mathrm{d}t \, \mathrm{dev}(\boldsymbol{S}) \\ &= \boldsymbol{E}_{i}^{p} + \Delta\gamma \, \mathrm{dev}(\boldsymbol{S}) \,, \end{aligned} \tag{2.41}$$

where $\boldsymbol{E}_{i}^{p}$ and $\boldsymbol{E}_{i+1}^{p}$ are plastic strains at steps i and $i+1$, and $\Delta\gamma$ is an increment of plastic strain. At the beginning of the first time step of the iteration, the plastic strains are initialized to zero, and therefore, the generalized coordinates are solved using the total strain distribution $\boldsymbol{E}$. At the end of the first sub step, the yield condition is checked and the plastic strains are defined at every plastic cell for the next sub step. The yield condition is verified at the end of every sub step and the computation of plastic strains is adopted in the iteration step of Newton's iteration. As the initial value for the plastic strain distribution at the following time steps, the converged plastic strain distribution of the previous step is used. (See more details about the algorithm used and the plastic cells [77].)

2.3.4 Other models

It is shown for continuum beam elements, where the cross-sectional deformation and rotations are defined by transverse gradient vectors, that general continuum mechanics may yield an incorrect result [57]. According to the work by Rhim and Lee, these incorrect results can be avoided by neglecting the Poisson effect for continuum beam elements. This locking phenomenon is also found to be problematic in case of fully-parameterized elements based on the absolute nodal coordinate formulation [71, 29]. To neglect the Poisson effect, one can use the diagonal material stiffness matrix, which leads to similar results as obtained with conventional beam theory. A similar constitutive relation among the diagonal material stiffness matrix $\boldsymbol{D} = \mathrm{diag}(E, E, E, \mu, \mu, \mu)$ is also applied for beam elements based on the absolute nodal coordinate formulation [37, 19, 18]. Although this type of modified constitutive relation may be helpful in avoiding Poisson and thickness lockings for certain beam elements based on the absolute nodal coordinate formulation, it is not generally recommended for continuum elements. In case of plate elements, such a simplified constitutive relation will clearly lead to an incorrect solution, which can be seen from the convergence of the results in *Publication I*. More discussion and numerical results related to this plate element can be found in [43]. In the case of the continuum plate elements, different modified material stiffness matrices are introduced to avoid thickness locking, see for example [38]. When using the special case with Poisson's ratio ν=0 in three-dimensional elasticity, or when using a modified material stiffness matrix, thickness locking can be avoided. The study [43] also shows that it can be used to approximate three-dimensional elasticity for thin plates modeled with fully-parameterized plate elements.

CHAPTER 3

Summary of the findings

The main motivation of this dissertation is to propose numerical procedures in order to overcome and highlight problems associated with the finite elements based on the absolute nodal coordinate formulation. The publications in this dissertation focus, in part, on descriptions of the shear and transverse deformations in the finite elements based on the absolute nodal coordinate formulation. For the plate element studied in *Publication I*, the shear deformation is accounted for by employing bilinear shape functions. This linearization of shear deformations is used to avoid shear locking. In *Publication I*, a simplified material model was used in order to improve the numerical performance of the plate element. A comparison of fully-parameterized plates, where the St. Venant-Kirchhoff material model is employed, is shown in [43]. In *Publication II*, improvements to the stress and strain description in the higher order beam element are introduced. In the study, the shear stress distribution is defined by using a quadratic distribution in the transverse direction and linearly along the longitudinal direction of the beam. Usually, shear stress distribution is defined directly from the deformation field. In *Publication III*, Reissner's nonlinear rod theory is implemented into the framework of the absolute nodal coordinate formulation using strain and kinetic energies, similar to the large rotation vector formulation by Simo and Vu-Quoc. The transverse deformation is taken into account using the strain energy and is somewhat similar to the elastic line approach. In *Publication IV*, the transverse shear deformation is included in the Bernoulli-Euler type beam element based on the absolute nodal coordinate formulation by using the strain-displacement relations of the Timoshenko beam theory. This approach leads to a low amount of degrees of freedom, but suffers from slow convergence under large deformations.

This same problem can also be seen in [70], where the Timoshenko beam theory with nonlinear strain-displacement relations are employed into the Bernoulli-Euler type beam element based on the absolute nodal coordinate formulation. In *Publication V*, high frequencies due to transverse deformation are studied, and a linear continuum beam element is improved so that the Poisson phenomenon is accounted for without Poisson or thickness type locking. It is also shown that by eliminating the transverse frequencies, only minor improvements to computational efficiency can be reached.

In this section, the main advantages and disadvantages of the studied elements based on the absolute nodal coordinate formulation are described. The first part of this section concentrates on studies of the beam elements and the second part on plate elements. Additionally, numerical problems due to the locking phenomena are briefly discussed in the both sections.

3.1 Studies of beam elements

Publication II

In *Publication II*, improvements to the description of the stress and strain distribution in the higher order beam element are introduced. The higher order beam element based on the absolute nodal coordinate formulation is introduced in [29], and it avoids shear locking phenomena due to the usage of higher order terms in the polynomial expansion. However, the shear deformation is directly defined from the deformation field and leads to constant transverse shear deformation. This is due to the linear interpolation along the cross-section of the beam. It is noteworthy that the linear displacement field in the transverse direction may lead to incorrect results, provided that the shear correction factor is not employed. An accurate description for shear strains in the transverse direction is particularly important in case of nonlinear material or fatigue analysis. In the element, shear stresses are interpolated linearly along the longitudinal direction of the beam because it improves the accuracy of the shear stress. In *Publication II*, the improved shear stress is based on beam theory for transverse and torsional shear stresses. In order to define the shear stress due to shear force at any point in the cross-section of the beam, the quadratic distribution can be used as follows:

$$S_{xy}^{S} = \frac{6Q_y}{WH^3}\left(\frac{H^2}{4} - y^2\right) , \tag{3.1}$$

where W and H are width and height of the beam element, and shear force $Q_y = \mu A \gamma_{xy}$ and shear deformation $\gamma_{xy} = \boldsymbol{r}_{,x} \cdot \boldsymbol{r}_{,y}$. The shear stress S^S_{xz} is derived analogously to S^S_{xy}. According to the beam theory, only three components of the symmetric stress tensor make significant contributions to the beam performance. Therefore, the transverse normal stresses S_{yy} and S_{zz} are neglected as they are significant only in case of large surface loadings of the beam. In addition, it is assumed that the shear stress S_{yz} is neglected in the cross-section. This is due to the assumption that the cross-section is not able to distort. It shall be noted that a trapezoidal deformation of the cross-section does not exist in the beam elements based on the absolute nodal coordinate formulation, please see possible modes for example in [29]. The kinematic assumptions are fundamentally the same as those used in the Timoshenko theory [76]. In the improved beam element, the transverse shear stresses due to the torsion are defined by using the torsion function as follows:

$$S^T_{xy} = -2\mu\kappa_x \frac{\partial \Phi}{\partial z}; \quad S^T_{xz} = 2\mu\kappa_x \frac{\partial \Phi}{\partial y}\,, \tag{3.2}$$

where $\Phi = \Phi(x, y)$ is the torsion function and κ_x is the twist approximated using the definition in [59]. Due to the use of the torsion function, the axial stress due to torsion is neglected. A good approximation for the torsion function was found by using a moderate number of terms. For example, when using four terms for the polynomial expansion of torsion, the error of shear components S^T_{xy} and S^T_{xz} is approximately 1% at the integration points. In the improved beam element, the axial normal stress due to the axial and bending deformation is defined by using the theory of St. Venant and Kirchhoff, where the transverse normal strains are neglected and Poisson locking is therefore avoided.

Several numerical examples were used to demonstrate the accuracy of the proposed stress definitions. The reference solutions were obtained by using solid elements in the commercial software ABAQUS. In the case of the cantilever beam problem, it was found that maximum shear stresses (near the clamped end) differ by only about 10% from the reference solution. Accordingly, the proposed definitions for the stress and strain distribution improve the accuracy significantly when compared to the original beam element where shear stress is constant in the transverse direction. In elasto-plastic cases, the computational efficiency of higher order beam elements is highly dependent on the number of plastic cells, where comparative stress is defined as well as the order of integration for the elastic forces. In the case of the plane pendulum, each element has 32 $\times$ 32 $\times$ 8 plastic cells for height, width and length. Such discretization arrangement was needed to obtain accurate results for deformations and stresses. It was found that in this case, the computational efficiency was improved by 44.9%.

Publication III

In *Publication II*, the shear stress distribution was improved by using the beam theory for transverse and torsional shear stress, which is only valid for moderately thick beams and small strains. In *Publication III*, the geometrically exact beam element was implemented into the framework of the absolute nodal coordinate formulation. A geometrically exact beam theory such as Reissner's nonlinear beam theory can be used for large strains. The implementation into this framework was done to demonstrate that the elements based on the absolute nodal coordinate formulation could be seen as a geometrically exact formulation, according to Reissner's nonlinear beam theory. It shall be noted that in the formulation the transverse deformation can be restrained using constraints or by using additional terms for strain energy. However, in the numerical examples used for the verification of theories, the transverse deformation plays a minor role. In *Publication III*, two modifications of the original strain energy as introduced by Omar and Shabana were proposed.

In the first approach, Poisson locking was eliminated by using plane stress assumptions with an integration procedure. In the second approach, two different implementations based on strain energy used in the large rotation vector formulation by Simo and Vu-Quoc are presented. Therefore, the strains are described in the orthogonal moving frame $\{\boldsymbol{t}_1, \boldsymbol{t}_2\}$ attached to the cross-section of the beam. In this approach, the position vector of an arbitrary point in the inertial frame $\{\boldsymbol{e}_1,\boldsymbol{e}_2\}$ is defined as follows:

$$\boldsymbol{r} = \boldsymbol{r}_0 + y\boldsymbol{t}_2\,, \tag{3.3}$$

where $\boldsymbol{r}_0$ is an arbitrary position on the elastic line (Figure 3.1) which is defined as follows:

$$\boldsymbol{r}_0 = (x + u_1)\boldsymbol{e}_1 + u_2\boldsymbol{e}_2\,, \tag{3.4}$$

where u_1 and u_2 are the components of displacement vector $\boldsymbol{u}$. The moving vectors can be written in the inertial frame $\{\boldsymbol{e}_1,\boldsymbol{e}_2\}$ as follows:

$$\boldsymbol{t}_i = R_{ji}\boldsymbol{e}_j\,, \tag{3.5}$$

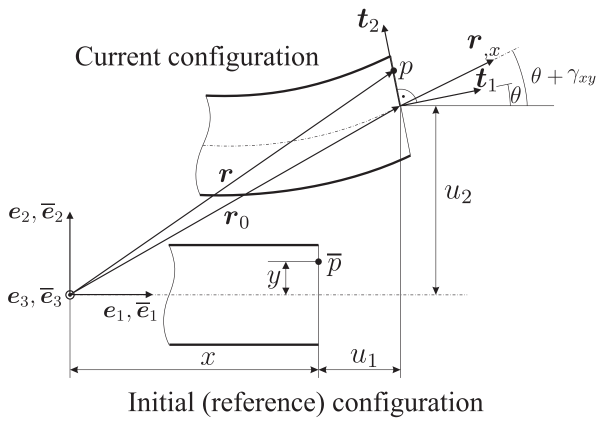

Figure 3.1. Cross-section and elastic line in the beam element based on the absolute nodal coordinate formulation in the framework of the large rotation vector formulation. Points $\overline{p}$ and p refer to the same particle at different configurations after displacement $\boldsymbol{u}$.

where values of the R_{ji} can be present in the matrix form as follows:

$$\boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} , \tag{3.6}$$

where θ is the time dependent angle of rotation that is defined by using the gradient vector $\boldsymbol{r}_{,y}$ in *Publication III*. It is shown in *Publication III* that the strains can be derived by using a pseudo-polar decomposition of the deformation gradient [23, p. 110]. Unlike the case for classical polar decomposition, the rotation must be determined beforehand for pseudo-polar decomposition, and it can be defined as

$$\boldsymbol{F} = \boldsymbol{R}(\mathbf{I} + \boldsymbol{H}) , \tag{3.7}$$

where $\boldsymbol{H}$ is a non-symmetric strain matrix. The deformation gradient can be written by using the relation $\boldsymbol{t}_{2,x} = -\boldsymbol{t}_1\theta_{,x}$ as follows:

$$\boldsymbol{F} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}} = \left[\; \boldsymbol{r}_{0,x} - y\theta_{,x}\boldsymbol{t}_1 \quad \boldsymbol{t}_2 \; \right] . \tag{3.8}$$

The strain matrix can be written as

$$\boldsymbol{H} = \boldsymbol{R}^T \boldsymbol{F} - \boldsymbol{I} = \begin{pmatrix} \boldsymbol{t}_1{}^T(\boldsymbol{r}_{0,x} - y\theta_{,x}\boldsymbol{t}_1) - 1 & 0 \\ \boldsymbol{t}_2{}^T(\boldsymbol{r}_{0,x} - y\theta_{,x}\boldsymbol{t}_1) & 0 \end{pmatrix} \equiv \begin{pmatrix} \Gamma_1 & 0 \\ \Gamma_2 & 0 \end{pmatrix} , \qquad (3.9)$$

where $\theta_{,x}$ is the rate of rotation of the cross-section along the undeformed length of the beam, Γ_1 is the strain distribution due to axial forces and the bending moment, and Γ_2 describes the transverse shear deformation. The strains at the orthonormal moving base $\{\boldsymbol{t}_1, \boldsymbol{t}_2\}$ can be written as

$$\Gamma_1 = \boldsymbol{t}_1{}^T \boldsymbol{r}_{,x} - 1 - y\theta_{,x} = \Gamma_1^0 - y\theta_{,x}, \qquad (3.10)$$

$$\Gamma_2 = \boldsymbol{t}_2{}^T \boldsymbol{r}_{,x} , \qquad (3.11)$$

where Γ_1^0 is the axial strain of the elastic line. It is noteworthy that the strain energies were described in the total Lagrangian scheme while no geometrical approximations were used. In these implementations, an elastic line approach [59] with a modified strain energy was used. Such an elastic line approach suffers from shear locking, and therefore, shear strain Γ_2 was modified by using linear interpolations.

Based on *Publication III*, it can be concluded that beam elements based on the strain energy by Simo-Vu-Quoc produce the same convergence as the original beam element by Simo and Vu-Quoc, provided that the order of interpolations for the position and angle are identical in both approaches. In addition, compared to the classical nonlinear beam theory, it was determined that the numerical inefficiency of the absolute nodal coordinate formulation was not due to the transverse eigenfrequencies, and accordingly, the description of transverse deformation will not lead to ill-conditioned finite elements.

Publication IV

In the fully-parameterized elements based on the absolute nodal coordinate formulation, shear deformation is usually captured through the introduction of additional gradient vectors in the element transverse direction. In *Publication IV*, a different approach to account for the transverse shear deformation is introduced. The main benefit of the introduced approach is that shear deformation can be accounted for with a low number of degrees of freedom, leading to reduced computational effort. The definition of the transverse shear deformation is based

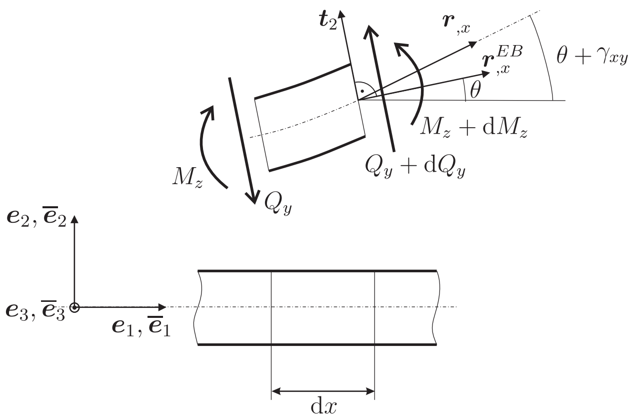

Figure 3.2. Deformation of the infinitesimal section of the beam element based on the absolute nodal coordinate formulation in the framework of the Timoshenko theory.

on the Timoshenko theory [76]. In *Publication IV*, shear deformation is implemented into the two-dimensional Bernoulli-Euler type beam element based on the absolute nodal coordinate formulation using a well known approach to classical beam elements [53, 50], where small displacement theory is used. In this formulation, there is no need for the use of transverse gradient vectors to capture the shear deformation, which is the case for the previous beam elements based on the absolute nodal coordinate formulation. The gradient vector $\boldsymbol{r}_{,x}$, where shear deformation is included, is determined by using the gradient vector based on Bernoulli-Euler type discretization $\boldsymbol{r}^{EB}_{,x}$ and the linearized moment equilibrium based on the Timoshenko beam theory. In Figure 3.2, the deformation of the infinitesimal element and corresponding gradient vectors are shown. The shear deformation is based on the linearized moment equilibrium and can be written as

$$EI_z \frac{\partial \varepsilon^1_{xx}}{\partial x} + k_s \mu A \gamma_{xy} = 0\,, \tag{3.12}$$

where E is Young's modulus, I_z is the second moment of area around the z-axis, ε^1_{xx} is the bending deformation and k_s is the shear correction factor. The bending deformation can be described by using the definitions shown in [59] or in *Publication III*, for example. In *Publication IV*, the shear deformation γ_{xy} is assumed to be constant along the longitudinal axis of the beam, and it can be straightforwardly solved from Equation (3.12). Due to the lack of transverse gradient vectors, the cross-sectional deformation cannot occur in the thickness direction, as is the case for fully-parameterized beam elements. Due to the neglected cross-sectional deformation in the thickness direction, the spread of eigenfrequencies is decreased compared to the fully-parameterized beam element. The strains are defined by using a linear strain-displacement relationship at the corotated frame. It shall be noted that this approach employs geometrical approximations, which is usually not the case in geometrically exact beam theories such as Reissner's beam theory and shear deformable beam elements based on the absolute nodal coordinate formulation. In *Publication IV*, the implementation follows the paper [50], where the approach is applied to a classical finite element, and the shear strain is assumed to be constant on the longitudinal axis of the beam.

It is important to note that the strains can be presented by using the nonlinear strain-displacement relationship. However, the approach is only valid in a small strain regime and uses geometrical approximations. In the approach based on *Publication IV*, the mass matrix is not constant, which is one of the trademarks of the absolute nodal coordinate formulation. The introduced approach does not suffer from shear locking because the order of the polynomials of vectors $\boldsymbol{t}_2$ and $\boldsymbol{r}_{,x}$ is the same. Another implementation of this approach can be found in [70], but it suffers from slow convergence under large deformations. It shall also be noted that the approaches presented in *Publication IV* and [70] are based on the linearized moment equilibrium where the effect of the axial force is neglected. Therefore, the differences between the approach introduced and the geometrically exact beam theory for problems with large deformation are understandable [70]. However, it was found that the approach introduced agrees well with the Timoshenko theory when geometrical nonlinearities and strains are moderate.

Publication V

When thin structures are modeled by a continuum element, high frequencies may lead to problems in terms of a large spread of frequencies. This, in turn, can increase the computational time and the accuracy of the solution. Therefore, the use of explicit integration methods may be restricted by the use of small time steps. This is one reason why structural elements with line or surface parameterization, such as beams, plates and shells are useful in engineering analysis [7, p. 509]. The problem described above concerns fully-parameterized elements based on the absolute nodal coordinate formulation. However, this case is not as severe as it is for solid elements because the discretization cannot be increased in the transverse direction in elements based on the absolute nodal coordinate formulation. In *Publication V*, the elimination of high frequencies due to transverse deformation was studied. Transverse deformations of the beam cross-section were constrained at node j by using the following algebraic equation:

$$g_j = \left\| \boldsymbol{r}_{,y}^{(j)} \right\| - 1 = 0; \quad j = 1, 2\,. \tag{3.13}$$

The constraints were applied to a simple linear beam element based on the absolute nodal coordinate formulation. It is noteworthy that the constraints cannot restrict the transverse deformation at the mid-span of the beam element. This shrinking due to bending deformation is typical for continuum elements and can be avoided by using linearly interpolated transverse deformations. In *Publication V*, the linear two-dimensional beam element was modified in order to demonstrate the effect of eliminating high frequencies. Poisson locking of the beam element was avoided by adjusting selective reduced integration.

It was found that minor improvements to the computational efficiency of dynamic problems can be achieved by constraining the transverse deformation. The reason behind this is that frequencies associated with transverse deformation are only slightly higher than frequencies associated with shear deformation. It was shown in *Publication III* that in case of one beam element with Hookean linear material, frequencies associated with transverse deformation are approximately two times higher than frequencies associated with shear deformation. It was found in *Publication V* that the improvement in computational efficiency depends on the thickness-to-length relation and the discretization used. When fine discretizations were used, the other eigenfrequencies, such as axial and shear bending, dominated. It shall be pointed out that in the elastic forces of the linear element used, the Poisson effect is accounted for in the case of axial and transverse deformation, but is omitted in the case of bending deformation. Therefore, the use of transverse constraints leads also to the restriction of axial deformation when Poisson's ratio $\nu \neq 0$. To avoid this problem, the less general elastic line approach can be used in the definition of elastic forces. In the case of a small thickness-to-length relation, computational efficiency can be improved. The frequencies associated with shear deformations were also verified using the large rotation vector formulation of Simo and Vu-Quoc.

3.2 Studies of plate elements

Nonlinear continuum plate and shell elements have been under active research for more than four decades. Usually, these conventional continuum plate and shell elements utilize rotation parameters instead of gradient vectors. It has been previously proposed that continuum elements with fully three-dimensional stresses and strains can be degenerated to shell elements behavior so that the kinematics and constitutive assumptions of shells are acceptable; see for example [1]. The isoparametric continuum shell element introduced in [1] (known as the A-I-Z shell element) is based on the Reissner-Mindlin hypothesis. However, it is known that the A-I-Z shell element suffers from shear locking, which can be alleviated by introducing independent linear interpolations for transverse shear deformations in a four node shell element (known as the MITCH4 shell element) [20]. The original MITCH4 element is derived from the A-I-Z shell element using five nodal parameters; the only difference is that shear locking is avoided by using mixed interpolation. To be able to use general three-dimensional elasticity without making any modifications, it is worth the addition of thickness deformation in the continuum shell formulation.

Publication I

The first plate element based on the absolute nodal coordinate formulation was developed by Shabana and Christensen [61]. This plate element was based on the classical Kirchhoff-Love plate theory in which rotation parameters were used only to describe bending deformation. In order to account for the shear deformation and thickness deformation in the case of thick plates, a fully-parameterized quadrilateral plate element was developed [46]. However, this plate element suffers from slow convergence due to different locking phenomena. The plate element especially suffers from shear locking because the transverse gradient vector and in-plane gradient vectors contain different orders of polynomials. This means that in case of fully-parameterized plates, the rotation of a transverse fiber is described with linear interpolation using in-plane coordinates, and the rotation of the mid-plane is described using quadratic interpolation. The unbalance of the base functions leads to overly large shear strain, which can be alleviated by linear interpolation for transverse shear deformations [20]. The main motivation for developing the plate element in *Publication I* was to overcome shear locking. Additionally, due to the kinematic description, curvature locking (shrinking effect) can also be avoided.

In this approach, the position of an arbitrary particle p (Figure 3.3) is defined as

$$\boldsymbol{r} = \boldsymbol{r}_0 + z\boldsymbol{t}_3; \quad \boldsymbol{t}_3 = \boldsymbol{R}_s(\gamma_{xz}^{lin}, \gamma_{yz}^{lin})\boldsymbol{n}\,, \tag{3.14}$$

where $\boldsymbol{r}_0$ is a vector of an arbitrary position of the mid-plane, $\boldsymbol{R}_s(\gamma_{xz}^{lin}, \gamma_{yz}^{lin})$ is a rotation matrix which is used to describe the effect of shear deformation, and $\boldsymbol{n}$ is a unit normal vector to the mid-plane.

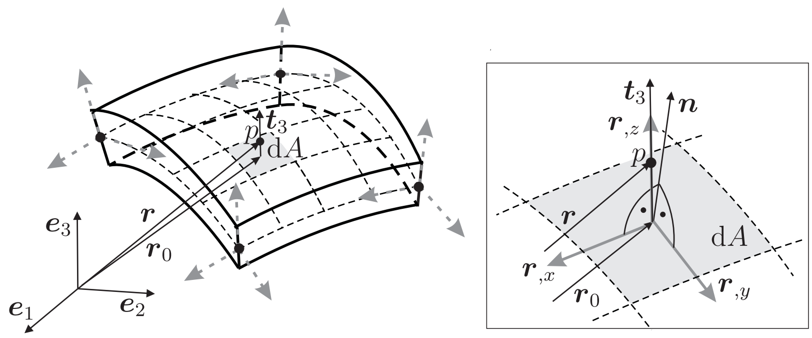

Figure 3.3. Description of the position of an arbitrary particle p in the fully-parameterized plate element with linearization of shear deformation. The gradient vectors at nodes are shown by dashed arrows.

Due to the use of normalization, the thickness deformation cannot be defined. Therefore, the thickness deformation is accounted for by using the original description for kinematics. In *Publication I*, shear locking is avoided by using a similar approach as in the MITCH4 element [20], but using the nodal values instead of sampling points [8] to guarantee that parasitic strain distribution is zero. The shear deformation is defined as follows:

$$\gamma_{xz}^{lin} = \sum_{i=1}^{4} N^{(i)}\,\gamma_{xz}^{(i)} \quad \text{and} \quad \gamma_{yz}^{lin} = \sum_{i=1}^{4} N^{(i)}\,\gamma_{yz}^{(i)}\,, \tag{3.15}$$

where $N^{(i)}$ are bilinear shape functions at the mid-plane, and shear deformation γ_{xz} and γ_{yz} are defined by using gradient vectors as follows:

$$\sin\gamma_{xz} = \frac{\boldsymbol{r}_{,x}\cdot\boldsymbol{r}_{,z}}{\|\boldsymbol{r}_{,x}\|\,\|\boldsymbol{r}_{,z}\|} \approx \gamma_{xz} \quad \text{and} \quad \sin\gamma_{yz} = -\frac{\boldsymbol{r}_{,y}\cdot\boldsymbol{r}_{,z}}{\|\boldsymbol{r}_{,y}\|\,\|\boldsymbol{r}_{,z}\|} \approx \gamma_{yz}\,, \tag{3.16}$$

where $\boldsymbol{r}_{,x}$ and $\boldsymbol{r}_{,y}$ are in-plane gradient vectors (Figure 3.3). In the introduced plate element, shear angles are used in the definition for elastic forces. However, this definition is computationally heavy, and therefore, the shear angles can be approximated such that $\gamma_{xz} = \boldsymbol{r}_{,x} \cdot \boldsymbol{r}_{,z}$ and $\gamma_{yz} = \boldsymbol{r}_{,y} \cdot \boldsymbol{r}_{,z}$. In case of this improved plate element, the mass matrix would no longer be constant due to the kinematics description where local coordinations are used. However, the mass matrix can be assumed to be constant using an inconsistent kinematic definition in strain and kinetic energies. The influence of the assumption decreases in value when using finer meshes.

It was found that the introduced plate element leads to an improved convergence since the use of linearized shear deformations overcomes shear locking and the use of local coordination for rotation overcomes curvature locking associated with a fully-parameterized quadrilateral plate. It shall be noted that the simplified constitutive relation applied will lead to an incorrect solution, which can be seen from the results. However, the introduced plate element does not suffer from Poisson locking because it includes the trapezoidal deformation mode of the cross-section. Although, it still suffers from thickness locking which is a problem when three-dimensional elasticity is used. In order to shed light on this matter, fully-parameterized plate elements based on three-dimensional elasticity are carefully compared in [43].

CHAPTER 4

Conclusions

The objective of the publications included in this dissertation has been to present and overcome problems associated with the beam and plate finite elements based on the absolute nodal coordinate formulation. The elements based on the absolute nodal coordinate formulation are designed for multibody applications and they can be described in the total Lagrangian scheme using the components of the deformation gradient as generalized coordinates instead of finite rotations. The use of this set of generalized coordinates leads to a singularity-free description in large rotation problems and also to a constant mass matrix, which can be considered as the main advantages of this formulation. Elements based on the absolute nodal coordinate formulation can be categorized based on their kinematic description. In the first category, elements are described in a conventional manner using parametrization for the mid-line in case of beams, or for the mid-plane in case of plates. In the second category, elements are described as a continuum using full parameterization. The strains and stresses in the fully-parameterized elements can be defined using general continuum mechanics, where three-dimensional elasticity can be included in the formulation, or alternatively, using the elastic line approach, where strains and stresses are defined as lines or planes. The absolute nodal coordinate formulation elements based on three-dimensional elasticity resemble finite solid elements. However, they are applied to beam, plate and shell structures by using a different order of polynomials in the longitudinal and transverse directions. The main benefit for using continuum elements with a full elasticity description is that all material laws known from general continuum mechanics can be employed in the formulation. However, the straightforward use for this approach may lead to different locking phenomena. In the elements included in this thesis, the strains and stresses are described by using continuum mechanics or elastic line approaches.

The publications included in this dissertation mainly focus on the different descriptions for the shear and transverse deformations in the finite elements based on the absolute nodal coordinate formulation. In addition, the absolute nodal coordinate formulation was compared to conventional isoparametric solid finite elements, as well as to the large rotation vector formulation. The previously introduced plate element based on the absolute nodal coordinate formulation was improved so that shear locking, as well as curvature locking, were avoided. Nevertheless, similarly to the continuum plate elements known from finite element literature, the introduced plate element suffers from thickness locking when three-dimensional elasticity is used. The introduced quadratic shear distribution for the beam element in the transverse direction is important, particularly in the case of elasto-plastic material, as the comparative stress is needed. Compared to the solution with finite solid elements in commercial software, clear improvements in computational efficiency were found. However, the elements based on the absolute nodal coordinate formulation include more degrees of freedom than the well-known nonlinear Reissner's theory. Therefore, in order to improve computational efficiency in a small strain regime, the degrees of freedom of the shear deformable element were decreased by using the conventional Timoshenko theory in the formulation. One may think that avoiding high frequencies due to thickness deformation will also improve computational efficiency. However, the fact is that this may not always be the case. It was shown that the fully-parameterized elements based on the absolute nodal coordinate formulation lead to only slightly higher eigenfrequencies due to the use of the transverse gradient vector than the beam element based on Reissner's theory. However, it is also shown in this dissertation that when thickness deformation is neglected, the computational efficiency can be slightly improved.

It can be concluded that the fully-parameterized elements based on the absolute nodal coordinate formulation are promising elements due to the possibility for the usage of three-dimensional elasticity. However, the formulation still suffers from lockings, mainly due to the assumptions for kinematics, such as the assumption according to which the cross-section remains plane during deformation. For the most objective comparison, the fully-parameterized element should be compared to finite solid elements or beam/plate theories by using the same material model. Many different elements based on the absolute nodal coordinate formulation were introduced in the past, whereas not that many studies have focused on the usage of these elements in a large strain regime. Clearly, some of these elements are only suitable for small strain problems. Therefore, in future work, verification with a different material model in the absolute nodal coordinate formulation is needed. Additionally, the objective efficiency comparison between different nonlinear finite element formulations under large deformations is also required.

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