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Demographic Modelling of Human Population Growth

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ABSTRACT

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This work presents models and methods that have been used in producing forecasts of population growth. The work is intended to emphasize the reliability bounds of the model forecasts. Leslie model and various versions of logistic population models are presented. References to literature and several studies are given. A lot of relevant methodology has been developed in biological sciences.

The Leslie modelling approach involves the use of current trends in mortality, fertility, migration and emigration. The model treats population divided in age groups and the model is given as a recursive system. Other group of models is based on straightforward extrapolation of census data. Trajectories of simple exponential growth function and logistic models are used to produce the forecast.

The work presents the basics of Leslie type modelling and the logistic models, including multi- parameter logistic functions. The latter model is also analysed from model reliability point of view. Bayesian approach and MCMC method are used to create error bounds of the model predictions.

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I dedicate this:

To Afia Yeboaah and Faustina Idun,
my mum's in all things,
with deepest love and gratitude

To Charlotte,
my sunshine, who makes me happy
when skies are gray.

Contents

1 Introduction

Population has been a controversial subject for ages. Charles Darwin ones said, in the struggle for life number gives the best insurance to win [10]. Let consider, therefore, a specific point of counsel stated by God. He said: `Be fruitfull and become many and fill the earth`.(Genesis 1:28) What does that mean? The word *fruitfull* implies that multiply your seed like the stars of the heavens and like the grains of sand that are on the seashore (i.e., $>> 10^{12}$).

Every government and collective sector always require accurate idea about the future size of various entities like population, resources, demands, consumptions and so on., for their planning activities. To obtain this information, the behaviour of the connected variables is analysed based on the previous data by the statisticians and mathematicians at first and using the conclusions drawn from the analysis they make future projections of the variable aimed. At present, there exist two major examples in statistics namely conventional and Bayesian in the interest of data analysis. The use of Bayesian methodology in the field of data analysis is relatively new and has found major support in last two decades from the people belonging to various disciplines. Apparently the main reason behind the increasing support is its flexibility and generality that allows it to deal with the complicate situations. In present study of population projection is based on the Bayesian approach of data analysis [20].

There are enormous concern about the consequences of human population growth for social, environment and economic development. Intensifying all these problems is population growth. World population has more than doubled in the past 45 years. The United Nations estimates that the figure was to be to 6.2 billion by the year 2000 and to 9.8 billion by 2050. The poorest areas of the world have the highest population growth rates. Roughly 90 million babies born in 1995, 85 million were born under less developed countries least able to provide for them. There are a lot of responsible for limiting population's extinction has increased tremendously. In recent times there have been big developments in analysis of population which we would considered in the proceeding chapters.

1.1 Definitions and estimation of population growth rate

The summary of parameter of any trends in population density or in abundance is known as population growth rate. In case of whether density and abundance are increasing or decreasing which inform as how fast they changes in terms of population growth rate. Population growth rate normally describes the per capita of growth of population $(1/P)dP/dt$ (In the absence of limitations to growth, food and territorial) as there is increasing the factors of population size increases per year. When we let $P = P(t)$ be the population size at time $t \geq 0$ and we assume that the initial population size $P(0) = P_0 > 0$. In other words given the symbols $\lambda = P_{t+1}/P_t$ or $r = log_e\lambda$, where r is growth rate and λ is finite growth rate respectively. In order to estimate population growth rate we normally use either population census data from period of time or from demographical data.

1.2 Importance in projection of future population sizes

To predict any future projection of population sizes of a given place it is important to know the population growth rate. Without density dependence the population growth then becomes exponential by using Euler-Lotka equation, the growth rate was calculated from demographic data from existing population. The projections of future population are normally based on present population. As we know population must to be articulated in order to make useful projections possible. Gone are the days, when we use human power to analysis all data to human population that would require thousands of people and millions of man power to complete. By the use of modern computers, this work done by human power can be cut down drastically.

1.3 Historical background of population study

The pivotal study of population growth rate was been recognised for long time. We can not discuss about historical background of population growth rate without mentioning names like [9] and [17], which some of the following will based. In (1798), Thomas Malthus wrote a paper on *An Essay on the Principle of Population.* Early late seventeenth century the table mortality with mathematical were analysis by Huygens and later Buffon among others. Surprise, Cole suggested that Newton outstandingly failed to comprehend the basic concept of expectancy was a function of age [12], the mathematical dependence of population growth rate was basically on age-specific birth rates and death rates, and he commends was that it ways comes back to these two principles, that of mortality and the fertility, which once they have been established for a certain place, make it easy to resolve all the questions which one could propose. Verhulst propose the logistic equation, then in our modern day the fathers of population growth rate of the ecology ([27], [14], [31],

[2] and [4]). The modern computers and with help of matrix methods for the analysis of life table, the importance of population growth rate in the study of population growth is becoming more widely raise the value of interest [6].

1.4 Scope and layout of the thesis

In endeavouring to constitute the central role population growth rate in population growth, we first consider basic definition of different models that depict the role of population dynamics, and we further examine the relationship between population growth rate and population model. Using the above ideas will help us to study the identification of good model for population growth. This thesis will emphasize the pivotal role of population growth rate and reviews the use of the data to test appropriate theory and models primarily for human populations. We conclude with a difference population models illustrating the ideas in practice and application to predict future population.

2 General about population study

Human populations have become the subject of changes in the number and age-structure. These changes normally take place through the followings processes of birth, deaths and the counter balance between immigration and emigration. The development of any country is base on a collective statistical information data. These has assist many countries to collect data available about it current populations, some of these are enormous in less developed countries. During the seventeenth century, however, men became very interested in the study of human population purely from scientific point of view. The first person was an Englishman, John Graunt (1620-1674) his work truely standard. He introduced the first life table and the studied of population of London in some detail. Many then follow his foot steps about the study of human population, was followed by thomas Malthus.

In 1798, in his work *An Essay on the Principle of Population*, Thomas Malthus painted a pessismistic picture of the future. He argued that the geometrical growth of the human population would soon outwit the arithmatical progression of the world's., leaving the world's population in dire strait which is different from that of Ghana. Let $P(t)$ be the population size of number people at time t and $a(t)$ be the concerntration of the rate-limiting substrate, then we have the simple hypothesis as

$$\frac{dP}{dt} = k(a)P(t) \tag{1}$$

where $k(a)$ is the specific growth rate of the population. Let us consider the following scenario: $P(t)$ is the population size, $a(t)$ be the concentration within an ecosystem. When population size $P(t)$ consumed more of $a(t)$, the rate of change of concentration would be less since the decreased in the concentration. The substrate is consumed by the population size and this was proposed by Monod's model, which defines the relation between the growth rate and the concentration of the rate-limiting substrate.

$$\frac{da}{dt} = -\frac{1}{Y}k(a)P \tag{2}$$

where Y is called the yield coefficient. From (1) and (2)

$$P(t) + Ya(t) = P(0) + Ya(0) = \overline{a}(say) \tag{3}$$

so that (1) gives

$$\frac{dP}{dt} = P(t)k\left[\frac{\overline{a} - P(t)}{Y}\right] \tag{4}$$

Integrating both sides of Equation (1), with constant k

$$\int_{P_0}^{P} \frac{dP}{P} = \int_0^t kdt \tag{5}$$

$$\log\left[P\right]_{P_0}^{P} = kt \tag{6}$$

$$\log P - \log P_0 = at \tag{7}$$

$$\log\left(\frac{P}{P_0}\right) = kt \tag{8}$$

$$\frac{P}{P_0} = e^{kt} \tag{9}$$

$$P = P_0 e^{kt} \tag{10}$$

where k is the productivity rate, the (constant) ratio of growth rate to population, P_0 is the population at whatever time is considered to be $t = 0$.

Basically all these can not be done without demography method or process. The mathematical way of modelling and statistical analysis of population is known as demography. In chapter three we would discuss the following factors fertility, mortality and migration. In population study, there are so many ways of projecting future population of a given country. We will then turn our attention to some them.

2.1 Scenarios of Future Population

What can we say about the future of world population? The simplest projection is to assume that current fertility rates continue to exist indefinitely into the future. Since the fertility rate is greater than one, population increases exponentially according to the formula $P(1+a)$, where P is the initial size of the population and a is the rate of increase; if P is 1million and a is 2, the population will be 2 million in 34.4 years, 4 million in 69 years and so forth. This is the Malthusian method of projecting population growth. This is the method shows that within a readily predictable amount of time there will be more human beings than there are atoms in the universe.

2.2 Conservative nature of assumptions

In addition, even these assumptions probably understate the possible variance contained in the current, really rather crude models. The Low assumption is based on a fertility rate of 1.7 and the High on a fertility rate of 2.5. But there's no reason at all to believe that these are the real low and high limits, since the current fertility rate in Italy is 1.2 and in East Africa is 6.1. Applying these numbers to (10) gives a truly titanic swing, everywhere from 1 to 200 billion.

2.2.1 Replacement rate immediately

The youthful nature of the earth's population means that the population will continue to increase for a while even if the fertility rate falls to replacement rate immediately, replacement rate being just over two children per woman on the average. This is because the number of women of childbearing age will continue to increase for several decades into the future. According to one illustrative UN projection, an immediate fall of fertility to the replacement rate would mean that the population would continue to increase until about 2100 and then stabilize at 8.4 billion.

2.2.2 Replacement rate and spread of estimates

This is an illustration of the importance of sensitive dependence on initial conditions. If instead of the replacement rate of 2.06, we substitute a rate of 1.96, one-tenth child per woman fewer, or five percent less than 2.06 the population would be 5.4 billion in 2150 and drop thereafter; if we assume a rate of 2.16-one-17 tenth child per woman more, or five percent more than 2.06 the population in 2150 is over 20 billion.

2.3 Implicit negative assumptions

These projections are basically linear projections and they depend on implicit negative assumptions, by which it means there are assumptions that nothing will change that will affect fertility. Let's name a few of these implicit assumptions: there will be linear economic development, the absence of epidemic disease, the absence of large-scale war, no basic changes in agricultural productivity, no truly crazy governments like the Khmer Rouge, no breakthroughs in energy technology, and so on and so forth. This very simple analysis throws in high relief the things that really are relevant to the population/resource relationship.

Why does fertility fall? Will it fall in the developing world?. Most of the world's population now lives in countries in which the second half of the demographic transition the fall in fertility rates' is not finished, and in some of which it has not begun. This depends on why the fertility rates have fallen in Europe and Japan.

2.3.1 The demographic trap

One fear is that the fall in fertility depends on economic development, on the movement away from the high-family, largely rural society to the urbanized, small-family model of modern society. But that movement depends on the accumulation of wealth, which in turns depends on the possibility of savings, $Y = C + S$, where C represents the account and S represents the accumulation of the saving. But if population is so high that just keeping level with current consumption takes all the economic activity the country produces, no savings are possible and the movement to industrialization is checked. Or, to put it in relative rather than absolute terms, the amount of C will vary with the youthfulness of the population, and will make the accumulation of S that much more difficult; thus prolonging the time it takes to accomplish the demographic transition and leading to larger populations. Let as ask the following questions:

What do we know for sure? We know that zero population growth rates will eventually be achieved, because any long-term growth rate greater than zero (that is, any a greater than zero in the formula $P = P_0(1 + a)$ implies an exponential growth rate and a population that will grow until all the matter in the universe has been converted into human tissue, the only question is how long this process will take. But this limiting case is, again, not really useful for policy analysis.

So what is really useful? The number of variables and the complexities of their interactions increases rather than decreases in time, so that, at least in terms of the present level of science, the big questions of sustainability are unanswerable, and the attempts to answer it so far have not usually worked out. In these circumstances, the best we can do is to advance science; to find out what the relevant factors are and to begin to see how they fit together.

How can we ever predict the future population accurately? Only by controlling it. The only way that the future population of the earth can be predicted is if it is made to come out a certain way. But even this is not certain, because it depends not only on making a certain policy on a world scale but on being sure we stick to it. Chinese fertility rates have been undergoing wild swings precisely because the government has been trying to exercise conscious control over fertility, but the policy keeps changing; so the result of human intervention in fertility is to make the swings in fertility rates more dramatic, and probably more unpredictable, than they were before massive intervention began.

Nonetheless, the entire trend of human history is in favor of more control rather than less. Malthus believed that population would go on increasing until it was checked by famine, disease or war; because he believed that the human instinct for reproduction-for sex-could not be subjected to conscious control. This was fundamentally wrong, as the change in fertility rates over the past few decades' shows. Agriculture is the exertion of conscious control over the food supply, and in the past two centuries, pace Malthus, we have exerted conscious control over the size of our own families [1].

But we will consider one personality who has contribute to population demographic modeland such individual is [26]. Leslie Matrix Population Model why this model is current innovation in modern days mathematics, and it has been that found to be most useful in determining population growth. Matrix population models have been transformed into useful analysis of predicting population growth. Basically, servival and fertility assumptions of projection of population growth was intributed to [5]. In 1959, Leslie came out with modified form of projection of matrix that was allowed for the effect of the existence of other population members on the population grwoth. Details of this model will derive in chapter five.

2.4 Projections using the Leslie model

In order to understand the dynamics principle of population growth, we need to project a matrix model (Leslie). Appraisal will be made in the light of vital rates, which will depend on continuous survivorship and fertility functions.

Definition 2.1: The survivorship function is the chance of an individual surviving from birth to age x , and it can be rescaled to given a number of survivors from the initial cohort. It is mathematically denoted by $l(x)$. Where $l(x)$ is probability of the survivors.

Definition 2.2: The fertility function is the expected number of offsprings (female offspring) per individuals of age x at a unit time, and it is denoted by $m(x)$.

2.5 The birth-flow population

Under we would consider a series of snapshots of the future population.There exist other models which, in effect , give a movies of the future population, by giving predictions for all times in the future, but we would not deal with these continuous models here. Let $P_i(t)$ denote the number of females at time t in the *i*th age group, i.e., with ages in the interval $(i\Delta, (i+1)\Delta)$. We define the column vector $P(t)$ by

$$P(t) = \begin{pmatrix} P_0(t) \\ P_1(t) \\ \vdots \\ P_{w-1}(t) \end{pmatrix} \tag{11}$$

we called (11) *age distribution vector* for time t [39].

The probability of the surviors depends on the age of the individual within the population is from age x to $x+1$, and is given by

$$P_i = \frac{l(i+1)}{l(i)} \tag{12}$$

where the age is assumed to be known [24], on other hand if the age is not known [6], by considering the average within each age class over the interval $i-1 \leq x \leq i$, $l(x)$ can be estimated as,

$$P_i = \frac{l(i) + l(i+1)}{l(i-1) + l(i)} \tag{13}$$

The distribution of births and deaths in age structure depends fertility, which is given,

$$F_i = P_i m_{i+1} \tag{14}$$

The number of offspring born in the following year is multiplied by the survival probability.

2.6 Birth-pulse populations

Population is always limited during short breeding season which occurs on birthdays. The age distribution normally consist of two process when counting:

- prebreeding implies the limit as p goes to 1.
- postbreeding implies the limit as p goes to 0.

In case of survival probability, every individual is aged $i-1+p$. Thus, the probability of survival age within the interval $i-1+p$ to $i+p$, is given below

$$P_i = \frac{l(i+p)}{l(i-1+p)} \tag{15}$$

To determined survival probabilities, we use the formula below in cases of two related to the counting.

$$P_i = \begin{cases} \frac{l(i)}{l(i-1)} & \text{postbreeding } (p \to 0) \\ \frac{l(i+1)}{l(i)} & \text{prebreeding } (p \to 1) \end{cases} \tag{16}$$

Explanation: P_1 in postbreaking includes first-year mortality, on other hand in prebreeding is not true; the missing mortality is included into the fertility coefficients. The probability of surviving during next birthday of an individual for fraction $1-p$ is P_i^{1-p}. To count $n_i(t+1)$ for the individual, the survive fraction p of a time unit, the probability is then estimated by $l(p)$. The fertility of the birth-pulse population is considered by using,

$$F_i = l(p)P_i^{1-p}m_i \tag{17}$$

$$= \begin{cases} P_i m_i & \text{postbreeding } (p \to 0) \\ l(1)m_i & \text{prebreeding } (p \to 1) \end{cases} \tag{18}$$

2.7 Eigenvalue and the properties of the of the constant vector

Leslie model is base on square $n \times n$ matrix, from these we can deduced that there are n possible eigenvalues and eigenvectors which is represent as this

$$Av = \lambda v \tag{19}$$

where λ is eigenvalue and v is an eigenvector equivalent to λ. Basically, the study of change in a population over time in a dynamical system which gives useful biological understanding are base on eigenvalues and eigenvectors. The aim of this is to determine whether the population is increasing, decreasing or staying constant.

Moreover, the reason why λ is so important is that it definite definition of the rate of population growth. The meaning of the dominant eigenvalue is sustained by the Perron-Frobenius theorem for non-negative and net matrices, which has the following properties:

- There exists one eigenvalue that is greater than or equal to any of the other in magnitude, called the *dominant eigenvalue* of **A**,

- There exists an eigenvector such that their elements are non-negative,

- λ is greater or equals to the smallest row sum of **A** and less or equals to the largest row sum.

The eigenvalue may be a real or complex number, and the eigenvector may have real or complex entries. Equation(19) may be rewritten as

$$(A - \lambda I)v = 0, \tag{20}$$

which shows that the nonzero eigenvector v lies in the null space of the matrix $A - \lambda I$, where I is the identity matrix, the values obtained represent: When $\lambda = 1$, the population is stationary, $\lambda > 1$, over-population is experienced. Whenever this happened, the only to consider is harvesting as option to keep the population stable. When $\lambda < 1$, the population start to decrease. The yearly rate of increase of the population is given by the logarithm of the dominant eigenvalue,

$$r = log(\lambda) \tag{21}$$

3 Parameters of models used in population growth

The first mathematical model was attributed to Malthusian scheme for population growth is based on the work by Thomas R. Malthus (1766-1834). In his paper *The Principle of Population Essay* that was published in 1798, Malthus demonstrated in elementary and brightly in his theories of human population growth and the connection between over-population and misery. Population growth is aubiquitous feature of population in human.

3.1 Mortality

The process whereby death occur in population is known as mortality. The ratio of the number of death during a specific period (usually 1 year) of live-born infants who have not make their first birthday to the number of live births per unit time. Mortality rate is representatively expressed in deaths per 1000 individuals per year. With help of life table, ti is always hypothetical to enumerate various probabilities involving mortality. For instance, when we consider a life aged x. What will be the probability that this life will die between exact age $x + t$ and exact age $x + t + dt$?. The probability of this question will be $(l_{x+t} - l_{x+tdt})/l_x$. This function l_x is well- behaved and l_{x+t+dt} may expanded in Taylor series about the point $x + t$, as

$$\frac{l_{x+t} - l_{x+t+dt}}{l_x} = \frac{l_{x+t}}{l_x}\left\{-\frac{1}{l_{x+t}}\frac{d}{dt}(l_{x+t})\right\} + 0(dt) \quad (22)$$

$$= -{}_tP_x \mu_{x+t} dt + 0(dt), \text{ where} \quad (23)$$

$$\mu_x = -\frac{1}{l_x}\frac{d}{dt} l_x \quad (24)$$

$$= -\frac{d}{dx} log l_x \quad (25)$$

${}_tP_x$ is the probability that a life survives from age x to $x + t$, and $\mu_{x+t} dt$ is the probability that life age $x + t$ will die during the time element dt [32]. The life table functions l_x, μ_x

and ${}_tP_x$ are defined below

$$dx = l_x - l_{x+1} \quad \texttt{(deaths)}, \tag{26}$$
$$_nP_x = l_{x+n}/l_x, \tag{27}$$
$$P_x =_1 P_1, \tag{28}$$
$$_nq_x = 1 -_n P_x, \tag{29}$$
$$q_x = 1 - P_x \ \texttt{(mortality rate)}, \tag{30}$$
$$L_x = \int_0^1 l_{x+t} dt, \tag{31}$$
$$m_x = \frac{dx}{L_x} \ \texttt{(central mortality rate)}. \tag{32}$$

For details see [32]

3.2 Vertical and Horizontal Life Table

Demographers sometimes uses two different category of analysis in collecting data in the life table, that is the vertical and horizontal life table. The only difference between these, species that have short live span are called the vertical life table and other hand species that have long live span are called the horizontal life table [16].

3.2.1 Life Table

Normally life table analysis are based on the tabulating age-specific system survivorship and reproduction [6], [26]. An individual's chances of surviving and breeding are the most important two parameters of a population, and these are most cases depends on age factors.

Survival

Generally, survival is charatertised by three functions of independent of age:

- Survivorship function or the $l(x)$ curve

basically, is the probability of survival from birth to age x or clearly $\frac{P_x}{P_0}$. **Cohort method** (i.e the observation of individuals through along period of time) and **Static method** (i.e is assumption based on stable age distribution) are two methods used to determined $l(x)$.

- The distribution of age at death function or the $s(x)$ curve,
  mostly, the probability density function for which the age at individual die and the results values are then used to collate the risk of death for different age groups or simply $\frac{P_{x+1}}{P_x}$.
- The mortality rate or hazard function or the $\mu(x)$ curve
  which is given by $\frac{s(x)}{l(x)}$.

Reproduction

Reproduction is defined by the maternity function or the fecundity curve. The maternity function, $m(x)$ is quantified as female offspring per female of age x hence,

- $m(x)$ is the expected offspring per individual aged x per unit time or simply $\frac{1}{2}$ number of the offspring born to parent of x.
- In the absence of mortality in case of total life time reproduction is given by **Gross reproductive rate** is $\sum l(x)m(x)$ if all ecology limits are all remove for population, this becomes very important in potential growth.
- In terms of the offspring being average by the individual in it life time is given by **Net reproductive rate** R_0 is $\sum l(x)m(x)$ and the **replacement rate**, R_0 is given by $l(x)m(x)$.

Let as consider the following:

- When the population is shrinking, then $R_0 < 1$
- When the population is growing, then $R_0 > 1$
- When the population is stable, then $R_0 = 1$

The measuring of reproduction of the individual lifetime is based on R_0, on other hand this measure of the population growth is called **intrinsic rate of increase,** r is gíven by

$$r \approx \frac{lnR_0}{T}, \tag{33}$$

where T is the **generation time** is given by

$$T = \frac{\sum xl(x)m(x)}{\sum l(x)m(x)}, \tag{34}$$

where x is the age class, r is the birth per unit time minus death per time (mathematically, $r = b - d$) and T is the weighted average. Mostly T become long when the offspring of the mothers are old, and on other hand T become short when the offspring produced are young. Let as now consider the following:

- When the population is stable, then $R_0 = 0$, and $r = 0$.
- When the population is decrease, then $R_0 < 0$, and $r < 0$.
- When the population is growing, then $R_0 > 0$, and $r > 0$.

Note: When $R_0 \approx 1(r \approx 0)$, then the result of r is accurate. Hence, this can be deduced from Euler's equation as

$$1 = \sum e^{-rx} l(x)m(x). \tag{35}$$

By solving the above equation by iteration with approximation solution to estimate r and the error always determine with the comparism of the intrinsic rate matter in practical applications.

3.3 Modelling life expectancy

Life expectancy is a important parameter in determining the size of a population on account of a given birth rate and the number of people is proportional to it. In most of the developed world life expectancy changed from 1800, and started improving slowly. Medical doctoers, demographers and others are still contend to define the future of the process. With longevity by DNA, a clarification can be found. Dangers along the delay get as far as the final age. Notwithstanding, by abstracting the dangers through nutrition, hygiene, medicine, and various covering and protections, eventually one can arrive at an age corresponding to longevity. The fundamental methods of social development are the

same. Some of the developing countries, increase in life expectancy will add up quickly, helping the size of their populations on the end results of fertility. Life expectancy acts in long term on fixed multiplier on population and less important to fertility, which acts exponentially. The life expectancy is the average number of whole years lived after age x by a life who actions age x. This is denoted by e_x.
Distinctly,

$$e_x = \sum_{n=0}^{\infty} n \frac{d_{x+n}}{l_x} \tag{36}$$

$$= \frac{1}{l_x} \{(l_{x+1} - l_{x+2}) + 2(l_{x+2} - l_{x+3}) + 3(l_{x+3}) + ...\} \tag{37}$$

$$= \frac{1}{l_x} \sum_{n=1}^{\infty} l_{x+n} \tag{38}$$

$$= \frac{1}{l_x} \sum_{n=0}^{\infty} l_{x+n} - 1. \tag{39}$$

The complete expectancy of life is the average number of years of life lived after age x. The probability that a life age x will die at age $x + t$ is denoted by e_x^0. Clearly,

$$e_x^0 = \frac{1}{l_x} \int_0^{\infty} t l_{x+t} \mu_{x+t} dt \tag{40}$$

$$= -\frac{1}{l_x} \int_0^{\infty} t \frac{d}{dt}(l_{x+t}) dt \tag{41}$$

$$= \frac{1}{l_x} \int_0^{\infty} l_{x+t} dt. \tag{42}$$

It seems persuasive that e_x^0 should be greater that e_x by about half a year. When we applied this to formula 35,

$$e_x + 1 = \frac{1}{l_x} \int_0^{\infty} l_{x+t} dt + \frac{1}{l_x}(\frac{1}{2} l_x) - \frac{1}{12} \frac{1}{l_x}(l_x') + ... \tag{43}$$

$$= e_x^0 + \frac{1}{2} + \frac{1}{12} \mu_x + ... \tag{44}$$

Hence,

$$e_x^0 = e_x + \frac{1}{2} - \frac{1}{12} \mu_x \tag{45}$$

For information see [32]

3.4 Modelling fertility

Reproduction is the pivotal of life. The number of childern born to woman within period of time from 15 to 45 years at their reproductive age is refered to as fertility. Fertility is an meaningful measure of demograph. To estimate the number of children to woman is base on the meaningful of fertility. In fact, the problem related to population is not excessively people being born but few. The two important things that one must know about births are the total fertility rate and the replacement rate. The reciprocal number of woman of childbring age the number of birth is know as total fertility rate, which leads to the average number of births per woman.

On other hand, the total number of births divided by the total number of woman of childbring age in order to control the population steady in long term. For instance, one may think of two children per woman to grow the next generation. In this case half of the children are girls and the outcome will end up with one girl per woman on average and the same number of woman in the next generation, and so on into the future. This is relatively true but needs to be corrected to take into two considerations, we must know that the numbers of girls to boys at birth are unequal, in a ratio of 21 boys to 20 girls, secondly the fact that some will die before they reach their reproduction age. Base on this, the actual replacement rate will be 2.05 children per woman. One way of measuring fertility is the age specific fertility rate [1].

This method requires a complete set of data, birth according to the age of the mother and the distribution of the total population with the age and gender. The number of births which occurs during of specific age per 1000 of woman is known as age specific fertility toatl rate *(ASFR)*. The age specific fertility rate at age x is given by the below equation

$$ASFR(x) = \frac{\text{Total live in year to woman aged } x}{\text{Total mid-year population of woman aged } x} \tag{46}$$

The total live births of woman during their reproductive age within 15 and 49 is said to be total fertility rate, and is estimated below

$$TFR = \sum_{x=15}^{49} ASFR(x) \tag{47}$$

Since the age was 5-years interval, then aged will be multiply by 5 for each ASFR.

3.5 Modelling the Niche

Predicting of population may be made by methods that look at the accumulated numbers and disregard the mechanism. In animal societies, the growth of a given niche has algorithmic dynamic perfectly fitted by logistic equations with constant limit κ. The use of logistic models originated in the middle of last century from Europe and became very popular in the United States in the 1920's with the work of Pearl, Reed, and Lotka(multiplied in [35]; check also [22]). The work of these people did not end there, after the end of World War II, Putnam then continued this work [33]. Studies have found that logistics consistently fit well the growth of human population over a short period, but for time scale problem began to set in. Statisticians and mathematicians who have tried to look into these problems to find out alternative solution of these. They came out with various clarification and more sophisticated versions of logistic models, until it was no longer advantageous to do with these logistics because the same could be done with polynomials [3]. This means that the capacity to predict avoided the analysis. The logistics work well in animal population when they have constant size. When a population makes innovations or adopts new mechanisms, the perious logistic model is no longer valid and more complicated model is needed. This explains modelling and forecasting is very challenging. Logistic models have limitation is long time scale. This growth of the niche also occurs with human as well. Actually, homo faber keeps on innovating all the time, so that logistics have momentary limits.

4 Factors relation to population dynamics

In this chapter we will discuss factors that are related to population growth. There are so many factors, but we shall limited to the factors basically related to human population. How mankind have great impact on the earth by it affects.

4.1 Four evoluations in human growth

There were four outstanding changes that Cohen listed during his studies of growth rate of the human population. The **first** of these was the agricultural revolution, which took place around 8000 years ago in Southwest Asia and China. The **second** revolution is the global agricultural revolution, which was mainly Columbian Exchange. The **third** of these was the modern fall in death rate, which was especially in the decades after 1950. Finally, the download change in fertility rates in the last 30years. With all of these, the fourth is totally different from the first three, simply because they are all involved in increases in the rate of population growth whiles the fourth decreased in population growth.

4.1.1 The concerning of population boom and it agricultural revolution

First and foremost, agricultural revolution began around 8000 years ago. In the case, geography is one of most important topic, the how, when and why agriculture was originated. There have being a lot of transition in case of human history, from traditional to modern, mechanized society during the last 200 years ago. Many of people living in the olden days were surviving basically on hunting and gathering. This kind of system was pressure on the places they inhabited all these actually happened before agriculture. Agriculture is the fundamentally alternative sources of environment by human effort to increase their food production even beyond the limits help from anything else nature provides. Hence, the population then grows faster, larger, and much larger than under the hunter-gather regime. The modification of the environment by human has change the environments carrying capacity. On the other hand, the carrying capacity is not determined by the environment forces alone but also by human beings does with the environment. The question, how and when did agriculture begin? This is one of the difficult puzzles facing the history of geography. In one of his outstanding greatest work during his chairmanship, he published book known as *Agriculture Origins and Dispersals*[7]. In case of the originating of expect of agriculture Sauer responses was this:

Agriculture did not originate from a growing or chronic shortage of food. People living in the shadow of famine do not have the means or time to undertake the slow and leisurely experimental steps out of which a better and different food supply is to emerge in a somewhat distant future... The saying that necessity is the mother of invention is largerly not true. The needy and miserable societies are not inventive, for they lack the leisure for reflection, experimention and discussion.

From the above statement, we can say that agriculture was not a responsed to resource depletion. In terms of population effect of development, agriculture have led to increased in population in directly or indirectly ways. For one reason, better-fed food results in less susceptible to disease of human [1].

4.1.2 The global agricultural revolution

In this second of Cohen's four revolutions which is globalization of agriculture. This is classifying into two parts. The first is called the Columbia Exchange. The intentional part of this is that, the transplantation of food crops and domestic animals from one part of the world to another. This had every great impact on demography in biogeography. The other part of transformation of transportation in the mid nineteenth century has led to a drastic increase in the speedily and predictability of shipment and an equally drastic fall in all expects of transportation cost. There was effect in the distribution of food been produced due to the proximately between the area food grown and the people being fed. In the case of this, we can say that famine in other continent is completed different from the other continent. In order words, the size of the food production, the exchange and consumption will be more reasonably assessable to those who are near or across the global [1].

4.1.3 The modern fall in death rate

There comes the third transformation in the mid-twentieth century. The high birth rates and high death rates began during demographic transition of traditional society which was sort by cultural inertia that increases in high birth rate and falls in death rate that result with massive in population increase. There are so many reasons why death rate always falls. We shall consider every few lists of these reasons. **Firstly**, as we have already discussed previously, the transportation revolution of the local famine can be blocked by the import of food from areas producing a surplus. **Secondly**, the germ theory of disease was spread by dirty drinking water; this was due to in accurate knowledge of the dysentery. The average life expectancy of Thailand in 100 years ago was 25 of every single human being has dysentery. Now, the life expectancy is high in its 60's which is probably below ten percent. **Thirdly**, productive treatment for disease, and fourth, diseases vector like mosquito decline prevent malaria must be handle well.

There was high increase in world population in 50 years after World War II due to the drastic decrease in death rate. The world population grew from 2.8 billion to 4.4 billion, as increase of 57 percent in 1955 to 1980. The percentage rate increased to 2.1, indicating that the population doubling time was $69.3/2.1$ or 33 years in 5 years interval (1965-1970). Throughout this period the world supply of food was most doubled which resulted in the size of the world economy tripled. From 1970 to 1990 there was about decrease in hungry people from 941 million to 764 million of the population. As there was fast economic growth in developed world, this was contrasting in some parts of Africa. This is not the inability of people to grow enough food to sustain them.

The whole problem is due to greediness in political field. There were several countries which had famine problem like Ethiopia in the 1980's, Sudan and Zimbabwe are also facing the same problem due to civil war, barbarically and corrupt governments. The aftermath of these civil wars and corrupt governments turn back to their political structure being put in place by their colonial masters. Actually, the Europe colonial powers normally export from their colonised countries with port at the mouth of their rivers. For instance, there is north-to-south orientation alongside the West Africa coast in Ghana, Togo and Nigeria.

4.2 Effects of genetic variations on population size

There is several diversity of genetic that effected by variation which leads to survival of small and isolated populations. Small populations normally fall into category called extinction vortex. As results of genetic environmental and demographic factors sometimes effect small population extinct. The principle behind is that the factors of the number of individuals then becomes smaller of positive feedback until they get to extinction. The follows are scenario that leads to possible extinction:

- Population is compelled to a small to size due to habitat fragmentation
- There is a great effect of random genetic drift on genetic diversity of small populations
- The uncertainty in sex ratio that will equal in small population
- The population size of its effectiveness always approaches zero
- There is always change in the average fitness when there is effect in new inbreeding

With all the above scenarios seems to reduce both the census and effective population size and genetic diversity as well. These put great pressure on the small population. In the absent of genetic diversity, the present of organism in the population do not defence themselves in case of disease spreading throughout the group. Thus, the population then driven to extinction, to accesse these information see [19].

Metapopulations occurs when population of populations or a system of local populations connected by dispersing individuals. In metapopulation theory, we normally use this to establish a route for migration which improves the quality of population size, genetic diversity, and the survival connected by the local populations.

4.3 Human impacts on natural systems

Human has cost more harm to nature since his existence to many survival of species and even with it own species as well. About 83% of the earth has been affected by mankind which had been led to drastically change. There are several affects that mankind has demonstrated against the earth, but we will consider nine major activity that has alternated the nature.

- Fragmentation, destruction, and degrading of nature resources have been reduced by biodiversity.
- Human activity within the nature ecosystems in order of them to have food to sustain them, do this by clearing the land for planting food reducing the interaction. Whenever there is invasion of any pathogenic the spread of this speed up very quickly, due to this the costing time, energy, and money to control become very difficult.

- The third type of this alteration is the destruction of the earths after deduction primary productivity.
- Some of the pest species and disease-causing bacteria have inadvertently strengthened no matter how much intercession one make.
- There are some species which internationally removed from the ranching areas.
- Introduction of new species into the ecosystem has caused new alteration.
- Mankind have abused renewable resources by over-harvested, which has led overgrazing of grasslands, and using freshwater speedily to the recharge.
- Human activities has also caused intervene with normal chemical cycling and energy flows in the ecosystems.
- Mankind dominance within the ecosystems has progressively on non-renewable energy from fossil had been polluted to affect of the greenhouse gases into the atmosphere.

Human population can only increase effectively basing on reproduction, and other hand reproduction can take place depending on sun and other resources for the survival on the earth. In this we totally say that there is interconnection and interdependent between the mankind and the environment that itself in which most important[19].

5 Human Population Models

In population growth many process in biology and related to other fields illustrate S-shaped growth. These curves were well modelled with the logistic growth function which was first introduced by Verhulst in 1845[38]. This logistic curve has placed under severe criticised where the system is not remarkable, on other hand this has been proved useful in a wide range of phenomena. In recent time, Young assessed and distinguished growth curves used for concerning detail forecasting, containing the logistic function. All this was applied in case of single growth process managing in seclusion. We will then extend our discussion on logistic function cases on dual processes. In case of human system the carrying capacity is always restricted by the contemporary level of technology, which is uncertain. Several models have been used to determined the number of population growth of estimated country. We will consider some of them, which will give us theorical background of the proceeding models [40].

5.1 Leslie model

The Leslie matrix population model is a discrete (i.e., time goes in steps as opposed to continously) and age dependent model (contruction of the model consider only age). The Leslie matrix population model is widely used in population ecology and demography in order to determine the growth of the population, as well as the age distribution within the population over time. The aftermath of population inconstancy is mostly afflicted by the density dependency and stochasticity. P. H. Leslie put in place a suggestion of projection matrix model on the effect of population growth onto the other population members. In 1966, Pollard studied into the stochastic action towards his model[32]. The diagram below shown the discretization of the age classes and time, where class i corresponds to ages $i-1 \leq x \leq i$.

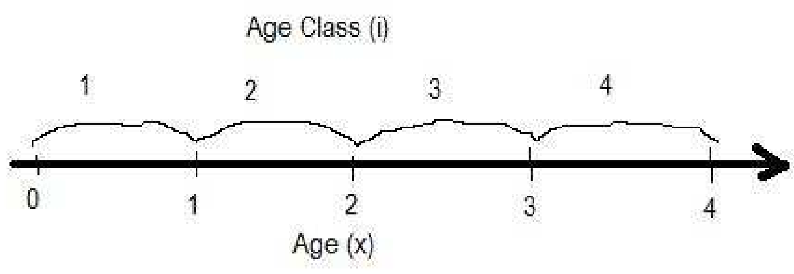

Figure 1: *Discrete age class i and continuous age x.*

5.1.1 Density Dependence in the Leslie

Leslie wrote a lot of papers in (1945, 1948, 1959) deal with the density as tally of all individuals in the population, no matter what the age. The population size is given by

$$N(t) = \sum n_i(t). \tag{48}$$

where $n(t) = \begin{pmatrix} n_1(t) \\ n_2(t) \\ \vdots \\ n_w(t) \end{pmatrix}$ and $n_i(t)$ = number of female of age i.

He defined the quantity of his postulating of the population density with each time interval of the different age group as this

$$q(t) = 1 + aN(t), \tag{49}$$

where a is the density parameter is given by

$$a = \frac{\lambda - 1}{\kappa}. \tag{50}$$

When $a = 0$ and $a < 0$, there will be no population density and with negative entries in the model respectively. In case of this, a must always be greater than 0 with the condition of take all age class in consideration of $q(t)$ must also be greater than 1. If the population is less than the carrying capacity κ, then the per capita growth rate is positive and the population increases, and after the population become stable and the total size of the population always remain constant. The $q(t)$ values are the diagonal elements of the matrix Q

$$Q(t) = \begin{pmatrix} q_1(t) & 0 & \cdots & 0 \\ 0 & q_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_w(t) \end{pmatrix} \tag{51}$$

We must note that the number of individuals in the constracting age groups at time t can now be mapped to time $t + 1$ as

$$n(t + 1) = AQ^{-1}n(t). \tag{52}$$

He introduced time-lag in 1959 by basic model. When these lags are then taken into consideration, the age groups in the population for each class i one has

$$q_i(t) = 1 + aN(t - i - 1) + bN(t), \tag{53}$$

where b is the effect of density at birth on the probability survival[34] at the later stage. Both a and b are > 0, and their magnitude is

$$\frac{b}{b+a} \tag{54}$$

Always the elements in the projection matrix are then divided into two depending on

- the size of the current population at time t , and
- the size of the population at time $t - i - 1$, which is the commencement of the interval where individuals were currently born of age i.

Subsequently, sure number of projection will arrive at stability at time τ [26], $q(t) = \lambda \forall i$ thus

$$A(\tau) = AQ^{-1}(\tau) \tag{55}$$

$$= \lambda^{-1}A \tag{56}$$

accordingly the matrix becomes

$$A(\tau) = \begin{pmatrix} \frac{F_1}{\lambda} & \frac{F_2}{\lambda} & \frac{F_1}{\lambda} & \cdots & \frac{F_{w-1}}{\lambda} & \frac{F_w}{\lambda} \\ \frac{P_1}{\lambda} & 0 & \cdots & & & 0 \\ 0 & \frac{P_2}{\lambda} & & & & \\ . & & \frac{P_3}{\lambda} & & & \\ . & & & & & \vdots \\ . & & & \ddots & & 0 \\ 0 & . & . & \cdots & \frac{P_{w-1}}{\lambda} & 0 \end{pmatrix} \tag{57}$$

In population dynamics there is brainwork to decline the vital rate[6] due to competition, and other factors that affect the population growth. The entries of any elements of the density-dependence matrix are either compenastory, overcompensatory or depensatory.

This entry then becomes from the results of instructured populations:

$$N(t+1) = f(N) \tag{58}$$
$$= g(N)N, \tag{59}$$

where the function $g(N)$ is the rate per-capita, and $f(N)$ is the recruitment function[6]. The total population is given by $N = \sum n_i(t)$. When $N > 0$,

$$\frac{dg(N)}{dN} > 0. \tag{60}$$

Furthermore $g(N)$ is said to make depensation, and if

$$\frac{dg(N)}{dN} \leq 0, \tag{61}$$
$$\frac{df(N)}{dN} \geq 0 \text{ and} \tag{62}$$
$$\lim_{N\to\infty} f(N) = C > 0, \tag{63}$$

on other hand $g(N)$ becomes compensatory. When

$$\lim_{N\to\infty} f(N) = 0, \tag{64}$$

in all these $g(N)$ displays overcompensation.

5.1.2 Stochastic in the model

In stochastic model, F_i is defined to be the probability that a female in age group $i-$ at the time t will be give birth to a single daughter during the time interval $(t, t+1)$ and the this daughter will be alive at time $t+1$ to be enumerated in age group $0-$. The stochastic process is consistently estimated on the condition that if there is random variation over time, and this is in line with the Leslie model force $\mathbf{A}$ to be $\mathbf{A}_t$ ($\mathbf{A}$ is now a function of time). In case of the variation, there are physical or biological factors in the ecosystem. As a results of this, we can group these into two of stochasticity:

- Environment

- Demographic stochastic

In developing stochastic projection model there is one basic step that we need to consider. Absorption of variance to change from deterministic to stochastic and the matrix model will be given by

$$\mathbf{n}(t+1) = \mathbf{A}_t \mathbf{n}(t), \tag{65}$$

where $\mathbf{A}_t$ is the column stochastic transition matrix. $\mathbf{A}_t$ become homogenous when the environment is constant otherwise inhomogenous[6], [11]. In constructing a stochastic projection model, let us consider the female population at discrete interval, and let $n_i(t)$ be random variable, $e_i(t)$ be expected value, and $C_{i,i}(t)$ be the variance [6]. Thus, the covariance is given as $cov(n_i(t), n_j(t)) = E[(n_i(t) - \overline{(n_i(t))} * (n_j(t) - \overline{(n_j(t))})]$ and this is denoted by $C_{i,j}(t)$. When $n_i(t)$ and $n_j(t)$ becomes independent

$$cov(n_i(t), n_j(t)) = E[(n_i(t) - \overline{(n_i(t))}]E[(n_j(t) - \overline{(n_j(t))}] \tag{66}$$

$$= 0 \tag{67}$$

On other hand $cov(n_i(t), n_j(t))$ is not equal to zero if $n_i(t)$, $e_i(t)$ are correlated [32]. Leslie model can be constructed by the expectation of the variable $n_i(t)$, for instance when we consider the fact that the number of females of the age i at time t at fixed P_i and F_i, and mutually independent, thus $n_i(t+1)$ of it binomial variable $B(n_i(t), F_i)$ is given by

$$\mathbf{e}_{t+1} = A_t \mathbf{e}_t \tag{68}$$

Thus,

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_w \end{pmatrix}(t+1) = \begin{pmatrix} F_1 & F_2 & F_3 & \cdots & F_{w-1} & F_w \\ P_1 & & & & & \\ 0 & P_2 & & & & \\ . & & P_3 & & & \\ . & & & & & \\ . & & & \ddots & & \\ 0 & . & . & \cdots & P_{w-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_w \end{pmatrix}(t) \tag{69}$$

where $e(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_w(t) \end{pmatrix}$ and $e_i(t)$ = expected number of female of age i.

When we let $Q_i = 1 - P_i$ and $1 - F_i$ = G_i, the variance and convariance are given the below equations [32]

$$C_{i+1,i+1}(t+1) = P_i^2 C_{i,i}(t) + P_i Q_i e_i(t), \text{ for } i \geq 0, \tag{70}$$

$$C_{i+1,j+1}(t+1) = P_i P_j C_{i,j}(t) i, j \geq 0, i \neq j, \tag{71}$$

$$cov(n_1^i(t+1), n_i(t+1)) = F_i P_i C_{i,i}(t), i \geq 0, \tag{72}$$

$$cov(n_1^i(t+1), n_j(t+1)) = F_i P_i C_{i,j}(t), i \neq j, \tag{73}$$

$$cov(n_1^i(t+1), n_1^j(t+1)) = F_i F_j C_{i,j}(t), i \neq j, \tag{74}$$

$$var(n_1^i(t+1)) = F_i^2 C_{i,i}(t) + F_i G_i e_i(t), i \geq 0. \tag{75}$$

We can then concluded that

$$C_{1,1}(t+1) = \sum_{i=0}^{w} (F_i^2 C_{i,i}(t) + F_i G_i e_i(t)) + \sum \sum_{i \neq j} F_i F_j C_{i,j}(t); and \tag{76}$$

$$C_{1,j+1}(t+1) = \sum_{alli} F_i P_j C_{i,j}(t). \tag{77}$$

The recurrence affinity for the mean, variance, and covariance is defined well by Eq. (70), (71), (72), (77), and (78), which appears as linear recurrence. In matrix form it can be rewritten as

$$\begin{pmatrix} \mathbf{e} \\ \mathbf{C} \end{pmatrix} (t+1) = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{D} & \mathbf{A} \times \mathbf{A} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{e} \\ \mathbf{C} \end{pmatrix} (t), \tag{78}$$

where **A** is the Leslie matrix, $\mathbf{e}(t)$ is the vector of expectations, $\mathbf{C}(t)$ has te elements as the variance and the covariance.

5.1.3 Example

Illustration: The following example illustrates production of salmon population by using stochastic projection,there are specification that one to needs for the stochastic process. We let the stochastic process to be $y(t)$, and also a good year to be $y(t)$ equals to 1.5, and in a bad year $y(t)$ equals to 0.43. Then allow the good and bad year to occur ramdomly [6] by flipping a coin, and independently with probability 0.6. Therefore, **A** can be written as

$$\mathbf{A}_t = \begin{pmatrix} 0 & 4y(t) & 5y(t) \\ 0.53 & 0 & 0 \\ 0 & 0.22 & 0 \end{pmatrix} \tag{79}$$

The above example illustrates how the production of salmon population took place. From (68) where A_t is randomly chosen with $y(t) = \{1.5, 0.43\}$ with equal probability. Let suppose the initial population vector is $e^0 = [10 \ \ 10 \ \ 10]$. That is the population age distribution vectors for first ten years. We generate a sequence of matrix equations to find the production of salmon population as follows

$$e_1 = A_t e_0 \tag{80}$$

$$e_2 = A_t e_1 \tag{81}$$

$$e_3 = A_t e_2 \tag{82}$$

Below are graphs showing production of salmon population at different levels.

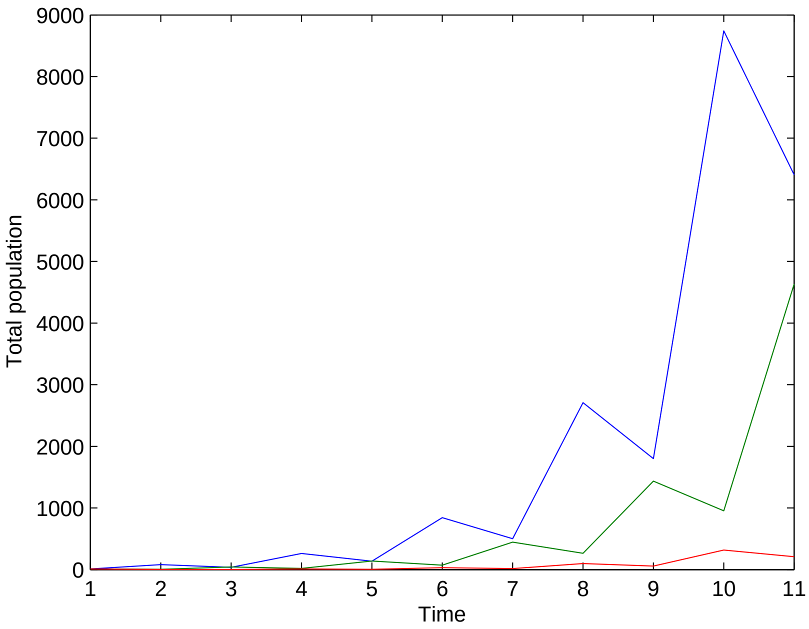

Figure 2: $e_1 = A_t e_0$

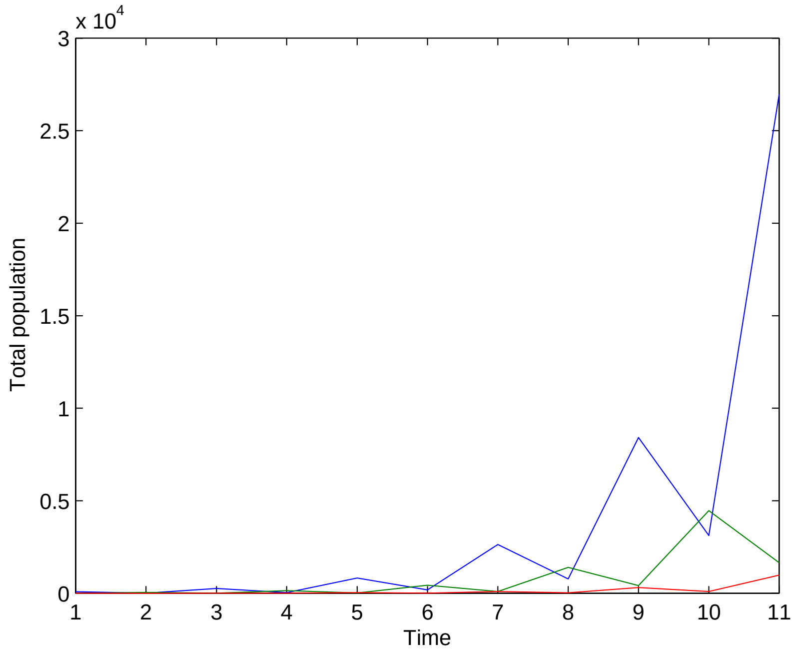

Figure 3: $e_2 = A_t e_1$

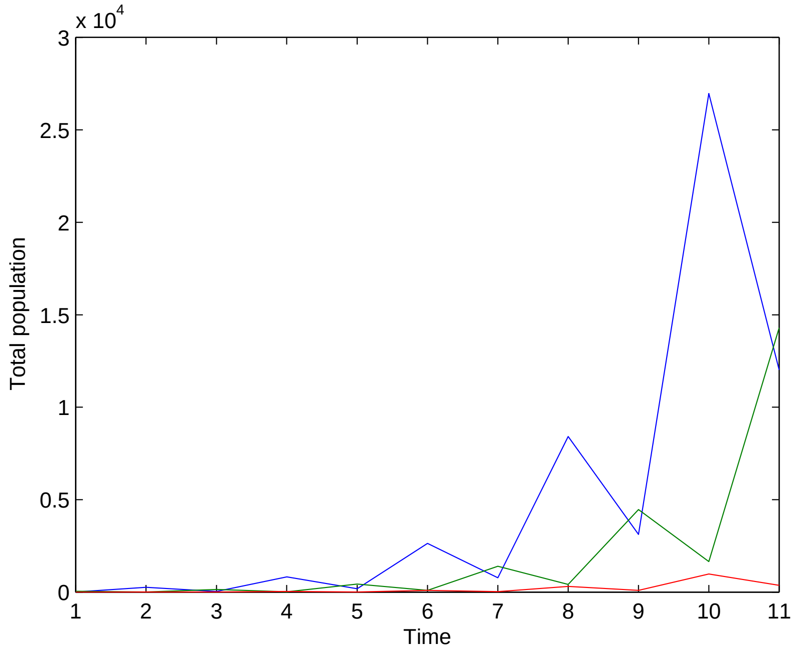

Figure 4: $e_3 = A_t e_2$

Figure 2, 3 and 4, depict projection of stochastic matrix, it is sometimes impossible to predict the dynamics of the population due to fluctuations and no sign of convergence is visible. In order to see the variations very clearly, one needs to iterate the stochastic model for long time say $t = 100$, but in illustration time was $t = 16$.

5.2 Population Growth Model Based on the Law of Teissier

Many expertises like actuaries and demographers are interested in the models of growth for human population for anticipating expected duration of life at various ages and for supposing future population trends. Teissier (1942) obtained from their experiment that law

$$k(a) = k_m \left[1 - exp\left(-\frac{alog2}{K} \right) \right] \tag{83}$$

fitted his data quite well.Where k_m is constant. From (4) and (13)

$$\frac{dP}{dt} = k_m \left[1 - exp\left(-\frac{\overline{a} - P}{KY} log2 \right) \right] P. \tag{84}$$

When integrate, we have

$$\int_{X_0}^{X} \frac{1}{X\left[1-exp\left\{-B(1-X)\right\}\right]} dX = \tau \tag{85}$$

where

$$B = \frac{\overline{a}log2}{KY}, X = \frac{P}{\overline{a}}, \tau = k_m t. \tag{86}$$

For instance, when we take $X_0 = 0.01$, and $B = 0.10, 0.15, 0.20, 0.25, 0.30$.

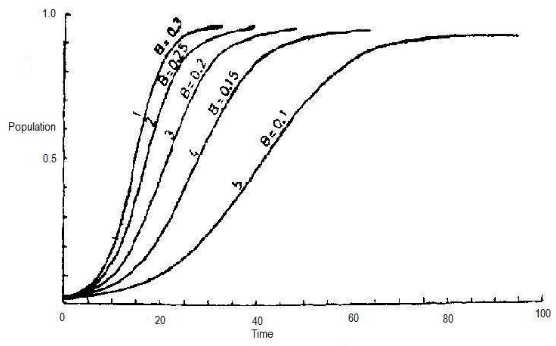

Figure 5: *Model based on the Teissier*

.

From the point of inflexion X^* is obtained by equating $\frac{d^2X}{d\tau^2} = 0$ so that

$$1 - (BX^* + 1)exp(-B(1 - X^*)) = 0. \tag{87}$$

This gives the table

B	0.10	0.15	0.20	0.25	0.30
X^*	0.506	0.509	0.512	0.515	0.518

Whenver B is very small, (87) gives on neglecting third and higer powers of B

$$1 - 2X^* - B\left[X^{*2} + \frac{1}{2}(1 - X^*)^2\right] = 0 \tag{88}$$

so that as $B \to 0$, $X^* \to \frac{1}{2}$. Let

$$f(X) \equiv 1 - (1 + BX)exp(-B(1 - X)) \tag{89}$$

then

$$f(0) = 1 - exp(-B) > 0, \tag{90}$$

$$f\left(\frac{1}{2}\right) = 1 - \left(1 + \frac{B}{2}\right)exp\left(-\frac{1}{2}B\right) > 0, \tag{91}$$

$$f(1) = -B < 0. \tag{92}$$

so that

$$\frac{1}{2} < X^* < 1. \tag{93}$$

Also as $B \to \infty$, $X^* \to 1$.

Hence for this model, a point of inflection always exists and occurs after half the final population size is reached [37].

5.3 The Model

There are alot of models that are use to predict population of a given population of a country. We will consider some models that are use to predict the future population of our given data from Ghana. We use population data that starts from year 1901.

5.4 Exponential function growth

From $P(t) = P_0 e^{at}$, we let $P(t) = P(t)$, $P_0 = \theta_1$, $a = \theta_2$ and $t = t_i - t_{1901}$. Hence,

$$P(t) = \theta_1 e^{\theta_2(t_i - t_{1901})} \tag{94}$$

Where $P(t)$ represents the total population size, θ_1 represents the initial population size, θ_2 represents the growth rate and t_i represents the year.

5.4.1 Single logistic growth

The logistic law of growth without proof state that systems always grows exponentially until carrying capacity characteristic in the system is approximated, at which the rate of grow slow and finally saturates, and generating the characteristic S-shape curve [36]. In the normally exponentially growth model, the rate of growth of population,$P(t)$, is proportional to population

$$\frac{dP(t)}{dt} = \alpha P(t). \tag{95}$$

In the well-known analytic form, α is a growth rate parameter and β is a location parameter that switch the curve horizoantally but do not change the shape

$$P(t) = e^{\alpha t+\beta} \tag{96}$$

When we add logistic model to exponential model (96), which slow the growth rate of the system of the carrying capacity when κ is attained. We then obtained this:

$$\frac{dP(t)}{dt} = \alpha P(t)\left[1 - \frac{P(t)}{\kappa}\right] \tag{97}$$

Equation (97) carefully looks like exponential growth when the values of $P(t) << \kappa$.When the population of $P(t)$ reaches κ, this leads to slow of rate of growth to zero, which is similar to symmetrical S-shaped curve. The logistic law of the growth from (97) after integration then becomes

$$P(t) = \frac{\kappa}{1 + e^{-\alpha t-\beta}} \tag{98}$$

where α is the rate parameter; β is the location parameter and κ is the asymptotic value that bounds the function and which gives the specifies level of the growth process being saturated [30]. When we define $\beta = -t_m\alpha$, we can replaced the location of parameter β by t_m. In population growth it is important to define a parameter Δt as the length between 10 to 90 percent of the saturation level κ of growth process. From (98), when $P(t)$ is equal to 0.1κ and 0.9κ, we then obtained $\Delta t = (ln81)/\alpha$. With the background of time-series data the 3 parameter of the logistic model (98) can then be define as

$$P(t) = \frac{\kappa}{1 + exp\left\{-\frac{ln(81)}{\Delta t}(t - t_m)\right\}} \tag{99}$$

Here, t_m is the midpoint (or inflection point) of the carrying capacity logistic. The first law of social dynamics states that *in the absence of any social, economic, or ecological force, the rate of change of the logarithms of a population, $P(t)$, of an* `organism` *is constant,*

$$\frac{d\,logP(t)}{dt} = constant(\alpha) \tag{100}$$

This (100) is similar to Newton's first law, which staes that *a particle in motion in the absence of any external force will travel in a straight line with constant velocity*. Equation (100) is also similar to exponential growth. When there is slow growth of unconstrained, the logistic growth can be considered as a canonical form of the system that is subjected. When more forces comes to together, the system then undergoes multiple of logistic growth pulse.

Collection of this method is use to estimate parameters depending on the connoted distribution of the measurement errors in the data. The best procedure to find out good measurement errors is autonomously and typically distributed with constant standard derivation. The best-fit parameters can then be found by minimizing the sum of the squares of the residuals. The difference between the time series data set (t_i, y_i) with m data points are the logistic model $P(t)$ is the residuals of the parameters. With the assistance from nonlinear regression technique this parameter can be estimated by minimizing the sum of the squares of the residuals

$$Residuals = \sum_{i=1}^{m} [P(t_i) - y_i]^2 \tag{101}$$

5.4.2 Multiple of logistic function

Combination of single logistic function leads to multiples of its logistic growth functions. The only different between the multiple logistic function and single logistic function is that, there are additional parameters added to the multiple logistics function. We will consider four parameter logstic function model. Here, we let X_i represent the population size of Ghana in the year t_i $(i = 1, 2, ..., 66)$, where present succesive statistical census data starting from 1901 for which $i = 1$. The four-parameter logistic growth model used for the projection may be described as folloows. For general assumption of regression equation:

$$X_i = P_i + \epsilon_1 \tag{102}$$

where, the population size X_i, in the year t_i was assumed to follow the normal distribution. For the four parameter logistic model, we assume that the deterministic part P_i as follows:

$$P_i = \frac{\beta_1 \beta_2}{\beta_1 + (\beta_2 - \beta_1)\, exp\left[-\beta_3 \left(\frac{t_i - mean(t)}{std(t)}\right)\right]} + \beta_4 \tag{103}$$

From the above logistic function, the upper asymptote of the curve approaches to $\beta_2 + \beta_4$ which is called the carrying capacity of the population and $-\beta_3$ is called the rate of growth of the population. Below is a diagram showing the various parameters in the above four parameter model.

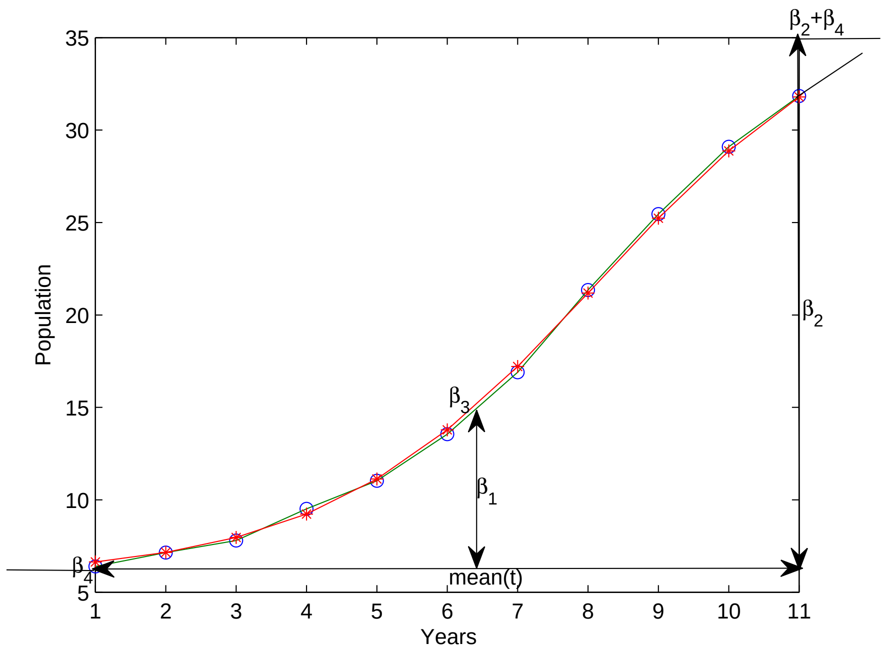

Figure 6: *Four Parameter Logistic Model.*

More specification: $\beta_4 = P(-\infty)$, $\beta_2 + \beta_4 = P(\infty)$, $\beta_1 = P(\bar{t})$ and β_3 determines the slope of $P(t)$ at $\bar{t}$.

6 Markov Chain Monte Carlo and Bayesian Method

Markov Chain Monte Carlo is a very big, and currently very rapidly developing, subject in statistical computation. There are many complex and multivariate types of random data, useful for calculating expectations and conditional expectations in Bayesian analysis, can be simulated in different ways. We will discuss some its algorithms theories, and give a computational example of some realistic complexity connected with the simulation of conditional distribution for random effects given the observed data in a random intercept logistic regression setting [29].

6.1 Bayesian approach

Bayesian approach is a statistical technique where observations are used to infer the probability that a hypothesis may be true by the use of Bayes formula. In the classical probabilities, If $P(D) = 0$, then the conditional probability is defined by [21]

$$P(A/D) = \frac{P(A \cap D)}{P(D)} = \frac{P(D/A)P(A)}{P(D)} P(A/D) \propto P(D/A)P(A) \qquad (104)$$

$P(D)$ is the normalising constant is used to make the total probabilities on the left sum equals to one. This is the marginal probability of the data which is defined by

$$P(D) = \sum_i P(D \cap A_i) = \sum_i P(D/A_i)P(A_i) \qquad (105)$$

On the conditon that, the initial outcome of A is true, and we observe some data D, then we can find the revised condition about A, in the light of D by the use of Bayes formula (105). Here $P(A)$, $P(D/A)$ and $P(A/D)$ are the prior distribution, the likelihood and the posterior distribution repectively. Therefore, posterior $\propto$ prior $\times$ likelihood. The prior distribution describes the previous information about the model parameters, the likelihood describes the probabilities of observing a set of parameter values and the posterior distribution defines the Bayesian soultion to the parameter estimation.

The Bayesian inference can be performed in the following:

- Enumerate all of the possible states of nature and choose a prior distribution,
- Establishing the likelihood function(which tells you how well the data we actually

observed are predicted by each hypothetical state of nature),

- Compute the posterior distribution by Bayes formula. Thus, the Bayesian approach is to choose a prior information that reflect the beliefs of observer about model parameters to be considered, and
- Then updating the beliefs on the basis of data observed, resulting in the posterior distribution.

We must note that the above intepretation are subjective.

For instance, when N coins are throw, the coins have heads and tails at both sides. A coin is selected at random and flipped k times resulting, all flippings, to obtain heads. Our interest here is the probability of getting a two headed coin. To find this, we let A_k be the event that a coin lands k-times, H_1 be the coin is two headed, and H_2 be the coin is fair. Thefore

$$P(H_1) = \frac{1}{q} \tag{106}$$

$$P(H_2) = 1 - \frac{1}{q} \tag{107}$$

The conditional probabilities are

$$P(A_k/H_1) = 1 \tag{108}$$

$$P(A_k/H_2) = \frac{1}{2^k} \tag{109}$$

From total probability formula,

$$P(A_k) = \frac{2^k + q - 1}{2^k q} \tag{110}$$

$$P(H_1/A_k) = \frac{2^k}{2^k + q - 1} \tag{111}$$

If $q = 1000000$ and $k = 1, 2 \cdots 30$, the graph of the posterior probabilities is given in Figure 5.

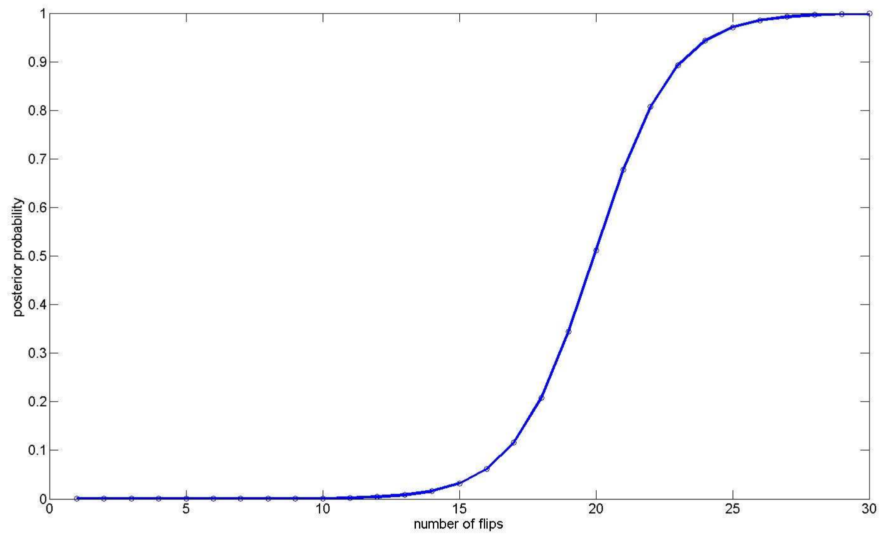

Figure 7: *Posterior Probability*.

6.1.1 Example

In a case of the Bayesian model $X \sim N(\theta, \sigma^2)$ and $\theta \sim N(\mu, \tau^2)$ with the following data set (2.9441, -13.3618, 7.1432, 16.2356, -6.9178, 8.5800, 12.5400, -15.9373, -14.4096, 5.7115). If $\sigma = 10000$, $\mu = 20$ and $\tau = 400$ such that the data are coming from $N(\theta, 10000)$ and the prior on θ is $N(20, 400)$, then the three densities are shown in Figure 6. The posterior is $N(6.8352, 6.6667)$

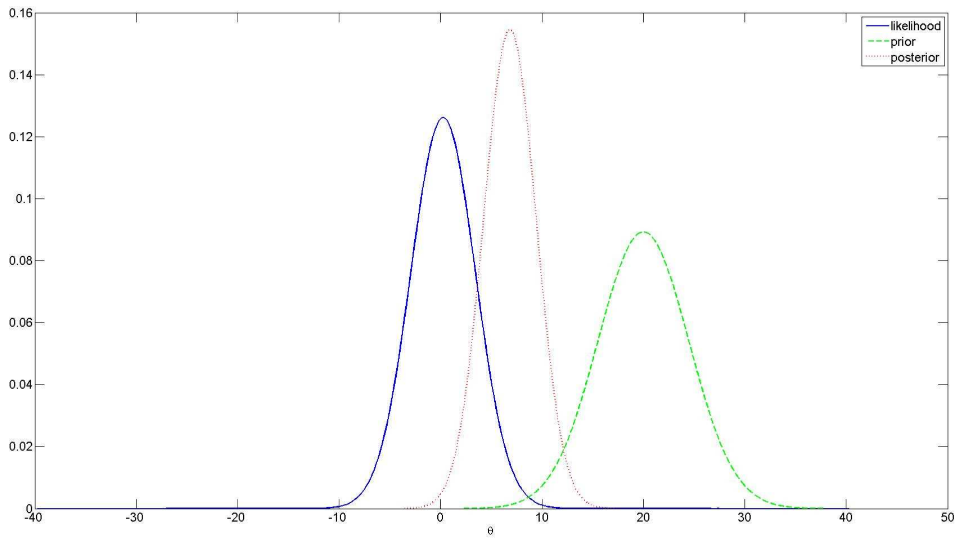

Figure 8: *Likelihood, Proir and Posterior*.

In case of the probability densities, the formulae are similar; if θ represents the parameters of the density and x is the observed data, therefore

$$P(\theta/x) = \frac{P(\theta \cap x)}{P(x)} \qquad = \frac{P(x/\theta)P(\theta)}{P(x)} \tag{112}$$

Estimating the marginal probability of the data is the dedious practical problem of Bayesian inference because the integrals are over high-dimensional spaces. Number of researches in Bayesian statistics contributed tremendous of broadening the scope of Bayesian models such that the models that were not handled are now routinely solved by other methods such that MCMC methods.

6.2 The Metropolis-Hastings Algorithm

Monte Carlo refers that the randomness involved(as there is at casinos in Monte carlo). The Markov Chain refers to having a chain of samples with Markovian property that each element depends only on the one preceeding it. The MCMC algorithms have become extremely popular in statistics studies. They are the way of approximating sampling from complicated and higher dimensional probability distributions. The MCMC algorithms have transformed Bayesian inference, by allowing practitioners to sample from posterior distributions of complicated models. The MCMC algorithms involve Markov chains, with a complicated stationary distribution. The Metropolis algorithm is the most fundamental to MCMC development. For instance, suppose the target distribution is known, we need a chain π as its stationary distribution. The MH is based on accept-reject methodology. In this a new candidate point $proposal\theta^*$ is drawn from the uer-pre-defined proposal distribution $q(.|\theta)$. The proposal distribution reduces of having a pre-defined convariance. In order to construct a Markov chain of the Metropolis-Hasting algorithm the following are the steps to be followed are considered:
1. Initialise:

- Choosing a starting point θ_0.
- Set $\theta_{old} = \theta_0$.
- Set `chain(1)`=θ_0 and $i = 2$.

2. Sample a candidate from the proposal distribution:$\theta^* \rightarrow q(.|\theta_{old})$.

3. Accept the candidate with probability

$$\alpha = min\left(1, \frac{\pi(\theta*)q(\theta_{old}|\theta*)}{\pi(\theta_{old})q(\theta^*|\theta_{old})}\right). \tag{113}$$

- If accepted set `chain(i)`=θ^* and $\theta_{old} = \theta^*$.
- If rejected set `chain(i)`=θ_{old}.

4. Set $i = i + 1$ and go to 2.

The algorithm produces achain, `chain`, of samples from the posterior distribution. The normalizing constant is canceled out in(113) and hence the problem of integrating the normalizing constant is removed.

The choice of the proposal distribution depends the nature of components. For example, for discrete components, in order to get uniform distribution over the state space, is common to choose an alternative that is use as distribution(uniform) over all values except the current one. For the case of continuous components, the Gaussian distribution (or multivariate Gaussian for compound components) centered on the current value is choosen and its alternative is the Cauchy distribution (or multivariate Cauchy), with heavier tails allowing occasional large jumps in the Markov chain. There are many ways of choosing different classes of proposal density namely: Symmetric Metropolis Algorithm, Random walk Metropolis-Hastings, Independence sampler, and Langevin algorithm.

6.3 The Gibbs Sampler

The Gibbs sampler is a popular MCMC algorithm for its computational simplicity and it is a special case for the MHA. Ihe aim is to make sampling from a high-dimensional distribution more tractable by sampling from a collection of more manageable smaller dimensional distributions because of the problem of finding a proposal distribution for higher dimensional models. The idea behind is that we can set up a Markov chain simulation algorithm from the joint posterior distribution by simulating parameters from the set of conditional distributions.

Let θ_{-i} be the set $\theta \mid \theta_i$ and $\pi(\theta_i) = \pi(\theta_i \mid \theta_{-i}), i = 1, \cdots, d$ be conditional distribution. Then

- Start with the arbitrary $\theta^{(0)} = \theta_0^{(0)}, \cdots, \theta_d^{(0)}$ for which $\pi(\theta^{(0)}) > 0$
- Obtain $\theta_1^{(t)}$ from the conditional distribution $\pi(\theta_1 \mid \theta_2^{(t-1)}, \cdots, \theta_d^{(t-1)})$
- Obtain $\theta_2^{(t)}$ from the conditional distribution $\pi(\theta_2 \mid \theta_1^{(t)}, \theta_3^{(t-1)}, \cdots, \theta_d^{(t-1)})$
-
-
- Obtain $\theta_p^{(t)}$ from the conditional distribution $\pi(\theta_p \mid \theta_1^{(t)}, \cdots, \theta_{p-1}^{(t-1)}, \theta_{p+1}^{(t-1)}, \cdots, \theta_d^{(t-1)})$
- Repeat from second step.

The running of these mutiple chains is also a way to check the convergence of MCMC simulations that are attained, as the convergence is reached, the resulting value $\theta^{(j)}$ is drawn from $\pi(\theta)$. The main requirement is that the sampling process is ergodic (it is possible to reach every state). An ergodic process always converge to the correct distribution by given enough time. As the number of iterations become large, the Gibbs sampling algorithm converges.

The distinguishing feature of the Gibbs sampler is that first, sample one variable conditioned on all the others, then a second variable, then a third variable, and so on, always conditioning on the most current values of the other variables. However, you need to be able to draw a sample from each of the conditional distributions, otherwise we can not use exact Gibbs. Therefore, this algorithm assumes that the conditional distributions are known, and the points created are accepted. There are techniques for doing this under some circumstances, such as importance sampling and slice sampling.

6.4 Adaptive MCMC Algorithms

For the past decade now, the theory of adaptive method has extremely well. Moreover, it is very complex, its applications are very straightforward. The adaptive methods in MCMC software is very difficult to work with it. The MCMC algorithms, such as the MHA, are used in statistical inference, to sample from complicated high-dimensional distributions. However, it is difficulty to find a proposal that fit the target distribution due to time-consuming, trial-and-error tuning of the proposal. For instance, when dealing with the Gaussian proposal, tuning of associated parameters, proposal variances, is crucial to achieve efficient mixing, but can also be very difficult. The obstructions of finding a suitable sized proposal distribution is overcome by adapting the proposal spontaneously during run of the algorithm. They do not need to determine the recommended distribution of variables in advance, they use the history to tune the proposal distribution suitably. Accordingly, AM attempts to adaptively to tune the algorithms as it continuously for the purpose of elaborating the performance of the algorithm.

In the Adaptive Metropolis method of [15] the proposal covariance is adapted by using the history of the chain generated so far. The algorithm for AM is given below [25].

- Start from an initial value θ^0 and initial proposal covariance $C = C_0$. Select a covariance scaling factor s, a small number ϵ for regularizing the covariance, and an initial non-adapting period n_0
- At each step, we then propose a new θ^* from a Gaussian distribution centred at the current value $N(\theta^{i-1}, C)$
- Accept or reject θ^* according to the MH acceptance probability
- After an initial period of simulation, let say for $i \geq 0$, adapt the proposal covariance matrix using the chain generated so far by $C = cov(\theta^0, .., \theta^i)s + I\epsilon$. Adapt from the beginning of the chain or with an increasing sequence of values. Adaption can be done at fixed or random intervals
- Iterate from step 2 until enough values have been generated.

7 Case study of Ghana Population

The Republic of Ghana is bounded on the north and north west by Burkina Faso, on the east by Togo, on the south by the Atlantic Ocean, and on the west by Cote d'Ivoire. Formerly a British colony known as the Gold Coast, Ghana was the first nation in sub-Saharan Africa to achieve independence (1957). The country is named after the ancient empire of Ghana, from which the ancestors of the inhabitants of the present country are thought to have migrated. The total land area is 92,100 square miles (238,537) square kilometres. The country can be roughly divided into three vegetation zones, namely coastal Savannah characterised by shrubs and mangrove swamps, a forest belt that gradually thins out into a dry Savannah as one moves northwards. Ghana has ten administrative regions, these are Greater Accra, Eastern, Western, Ashanti Central, Brong-Ahafo, Northern, Volta, Upper East, and Upper West Regions which are further divided, into 110 districts; these form the basic units of political administration. The capital town of Ghana is Accra. The Ghanaian population is made up of many ethnic groups. The largest is the Akans, accounts for 44% of the population. Other major ethnic groups are the Mole-Dagdani (16%), Ewe (13%), Ga-Adamgbe (18%), Gruma (4%) and Grussi (2%) a number of smaller ethnic groups make up the remainder.

The case study is based on the results of national census statistical data. Ghana then became the third country in Sub-Saharan Africa to come out with a comprehensive population policy in 1969 after Mauritius (1958) and Kenya (1967). The policy was meant to affect the growth, structure or distribution of the country's growing population. After all this effort of the government, there are still inconsistencies in data [18]. Obtaining data or collecting from Africa is not relatively as perfect as in Europe. For instance, census data in Finland are relatively accurate as even compared to part of europian countries. Here, we will analyse the numerical response, and show that some of negative effects of population growth and also population denstity are due to rate of growth and this drives many factors like competition of food, diseases(AIDS, malaria and so on) and instability of African countries like wars.

7.1 Methods

A different model design was used to project future of the nation. Calculations are done using MATLAB 7.3.0 (R2007a) on 2x3GHz Dual Core Xeon 8Gb desktop PC. For many years, many population projections use mathematical methods. This method expresses population as a function of time, with various exponential function being the most usual function used. Even though trustworthy for short-term projections, mathematical methods give practical values for long-term projections as well. A majority of population projections today use the cohort-component method. This method makes separate and independent projections of fertility, mortality and migration. We will consider special exponential, multiple of logistic equations and many more.

7.2 Results and Analysis

7.2.1 Case 1

The results are based on the model $P(t) = \theta_1 e^{\theta_2(t_i - t_{1901})}$, the following values taken into consideration, the initial population value $\theta_1 = 1486$ and $\theta_2 = 0.03$ reprectively. Adaptive metropolis codes for the model are given in the appendice 1. During the implementation of the program, we have taken two chains for each parameter.

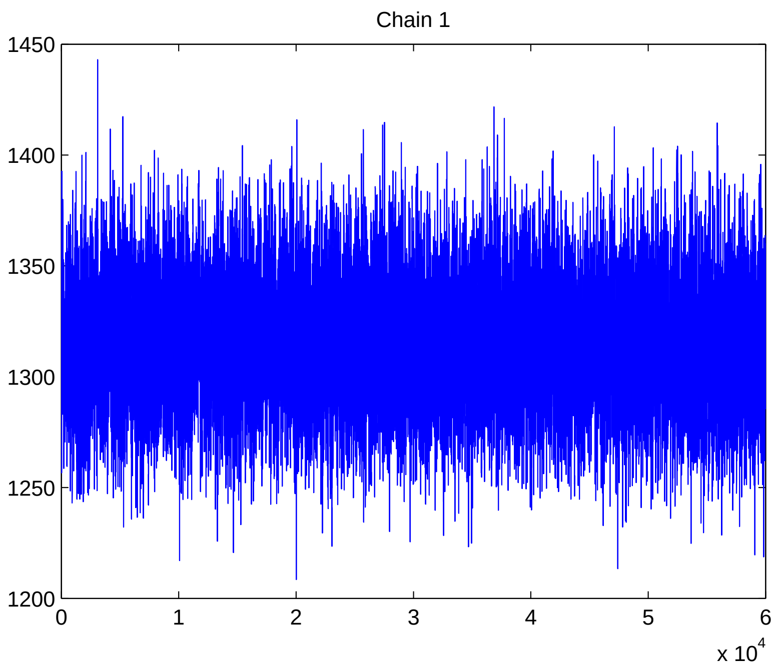

Figure 9: *T*heta 1.

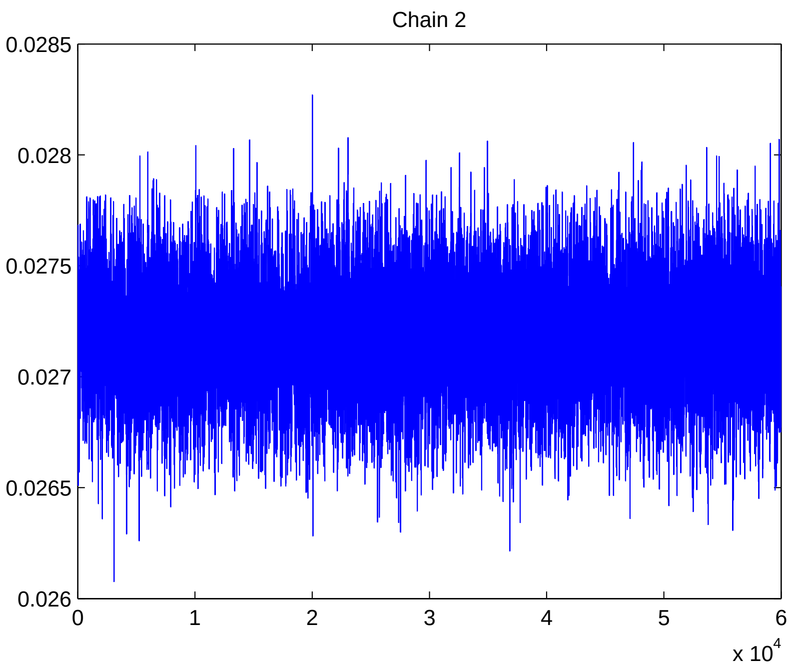

Figure 10: *T*heta 2.

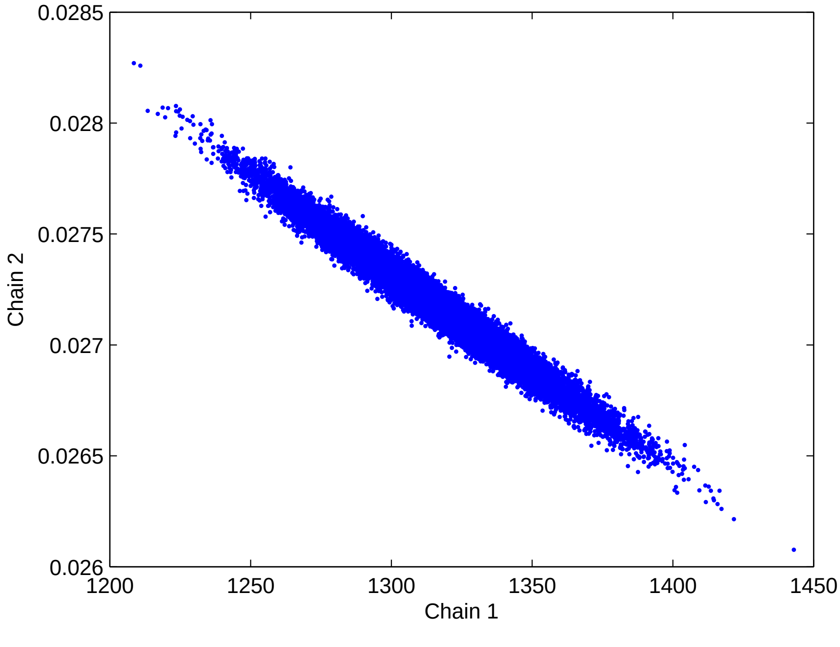

Figure 11: *T*he correlation between Theta 1 and Theta 2.

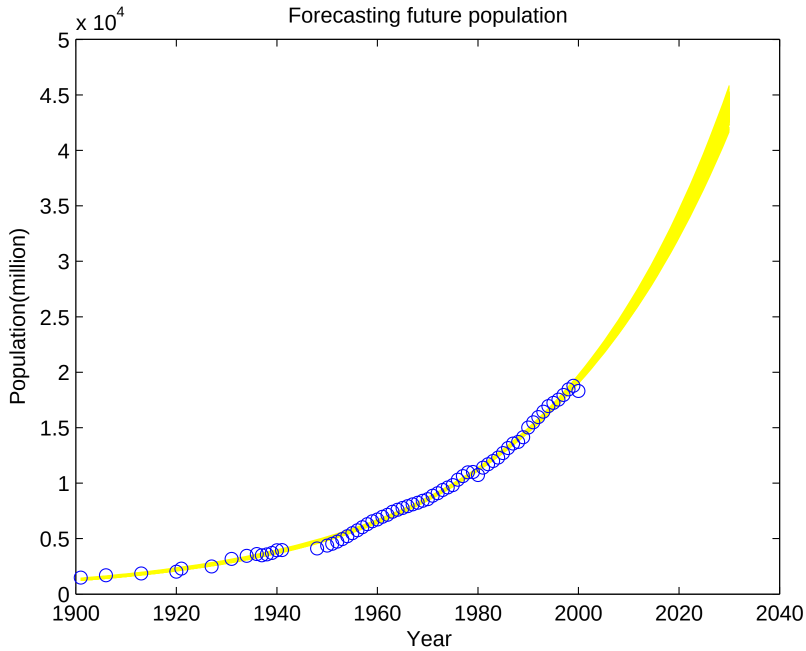

Figure 12: *Fitting, projection estimates under proposed model: Exponential function* .

7.2.2 Case 2

The following results are based on the model $P_i = \frac{\beta_1\beta_2}{\beta_1+(\beta_2-\beta_1)exp\left[-\beta_3\left(\frac{t_i-mean(t)}{std(t)}\right)\right]} + \beta_4$, the following values were taken into consideration, we equate *mean(t)* = 8.52 and *std(t)* = 4.8774 respectively. Table 1 provides the census figures of Ghana at the different interval years from 1901 to 2000 and estimated values from 2005 to 2030 respectively, which have been used to fit the model and to make the future projections. Using this census data and the logistic growth model described above, an adaptive metropolis program was developed to make a Bayesian analysis of the data and to provide projections and statistical reliability bounds for the projections of the population of Ghana. Adaptive metropolis codes for the model are given in the appendice 2. During the implementation of the program, we have taken three chains to run for each parameter. The results below show that this model is over-parameterized for this data.

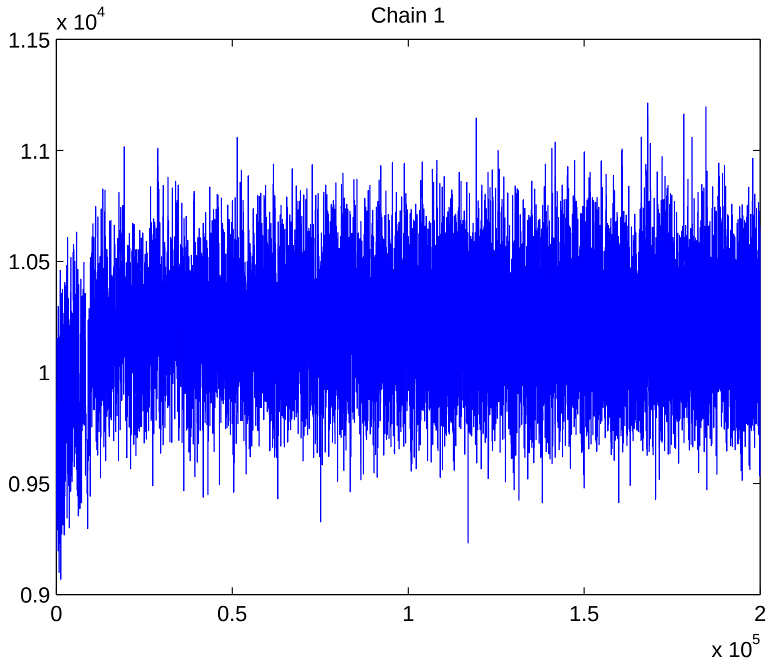

Figure 13: *B*eta 1.

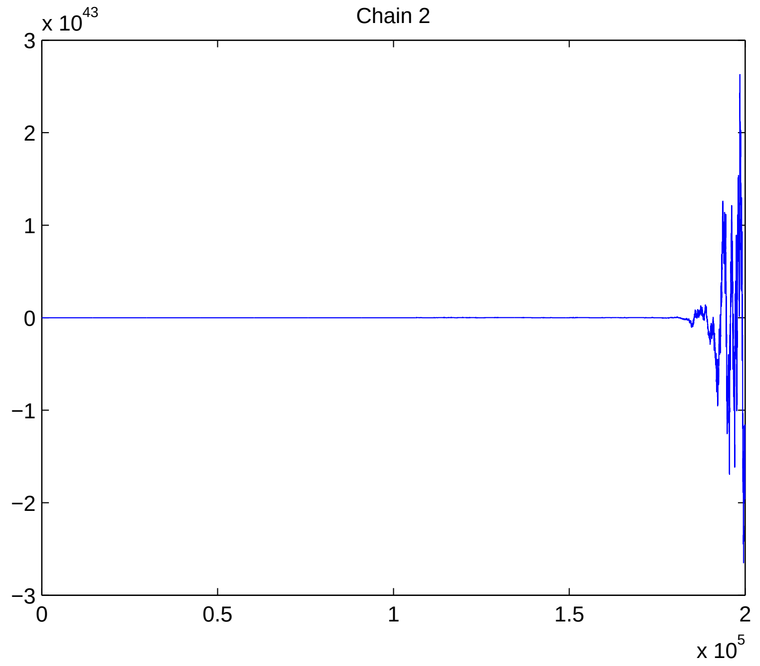

Figure 14: *B*eta 2.

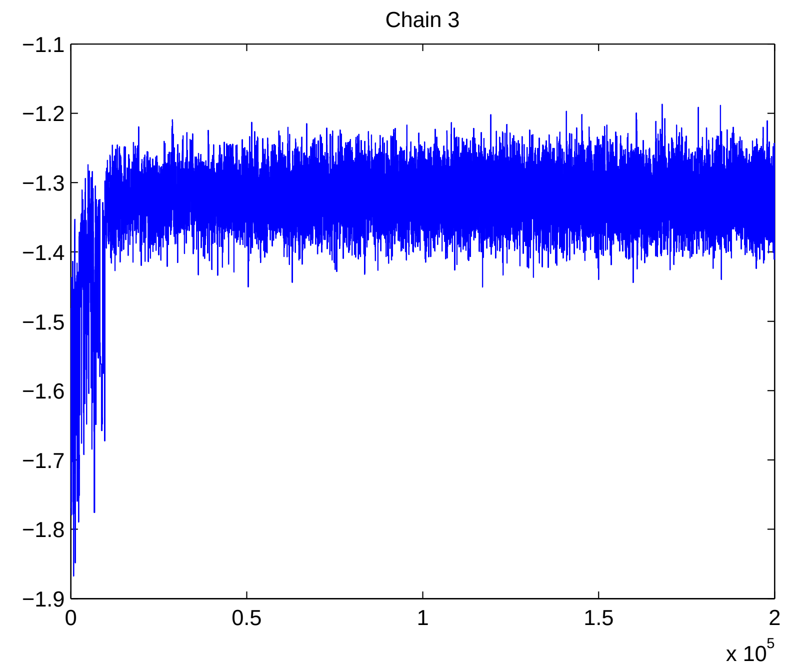

Figure 15: *B*eta 3.

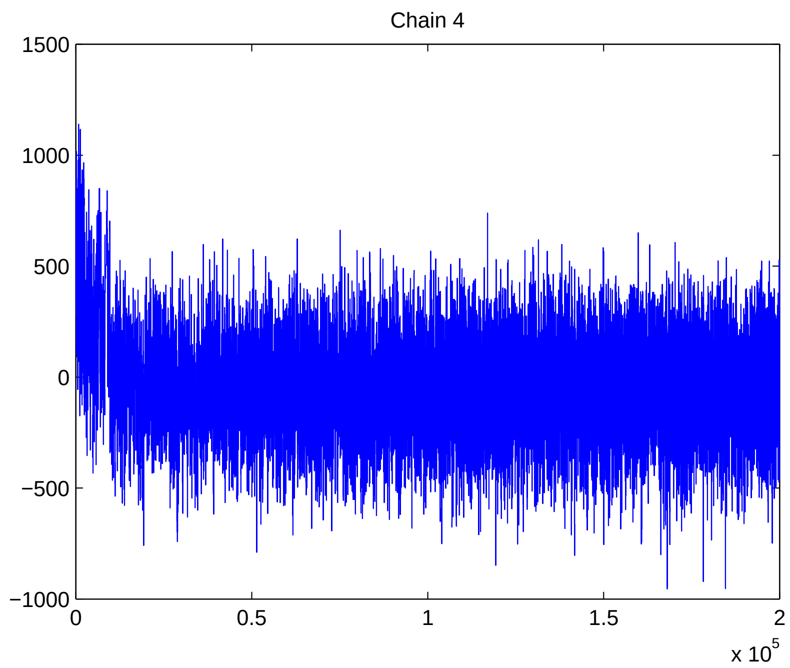

Figure 16: *B*eta 4.

We have obtained summary statistics for the estimates of the parameters of the model after discarding 60,000 initial updates. The number of iterations required to run after the convergence of the chains is assessed on the basis of Monte Carlo error for each parameter.

Simulation shall be continued from (103) until Monte Carlo error for each parameter of the sample standard deviation.

Table 1: Shows a collective census data from 1901 to 2000 and with earlier prediction from 2005 to 2030 (5 years interval) by statistical census data in Ghana.

Table 1: GHANA: historical demographical census data in million[18].

Year	Pop	Year	Pop	Year	Pop	Year	Pop	Year	Pop	Year	Pop
1901	1486	1939	3700	1957	6034	1968	8240	1979	11000	1990	15020
1906	1697	1940	3963	1958	6303	1969	8414	1980	10736	1991	15484
1913	1852	1941	3959	1959	6562	1970	8559	1981	11400	1992	15959
1920	2021	1948	4118	1960	6727	1971	8858	1982	11700	1993	16446
1921	2296	1950	4368	1961	6960	1972	9086	1983	12000	1994	16944
1927	2496	1951	4532	1962	7148	1973	9385	1984	12309	1995	17236
1931	3163	1952	4734	1963	7422	1974	9607	1985	12710	1996	17522
1934	3441	1953	4964	1964	7598	1975	9817	1986	13163	1997	17945
1936	3613	1954	5217	1965	7767	1976	10309	1987	13572	1998	18460
1937	3489	1955	5484	1966	7927	1977	10632	1988	13709	1999	18785
1938	3572	1956	5758	1967	8082	1978	10969	1989	14137	2000	18412
2005	23033	2010	26284	2015	29599	2020	32769	2025	35886	2030	38855

The following histogram diagrams show the reliability bounds of the future population of the nation from 2010 to 2025. The maximum bar may represents the most probable total number population within each histogram.

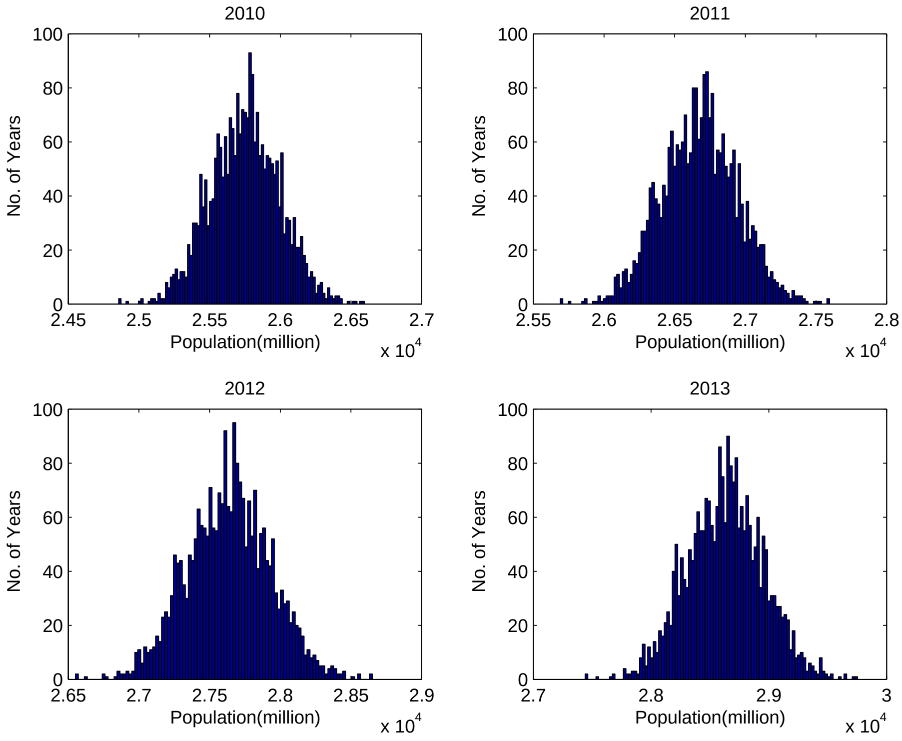

Figure 17: *Prediction from 2010 to 2013*.

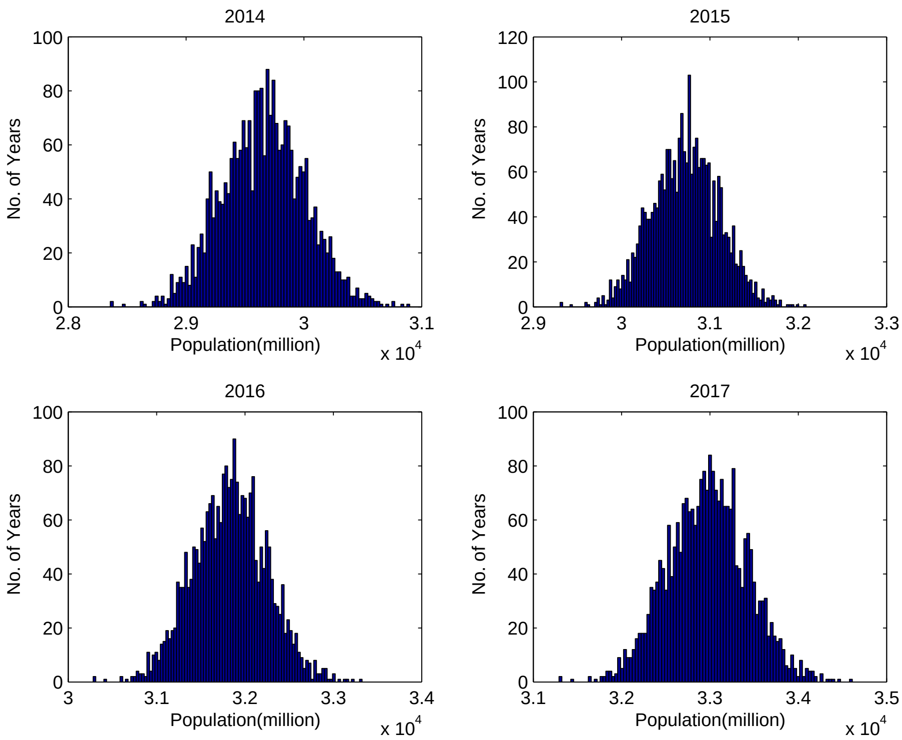

Figure 18: *Prediction from 2014 to 2017*.

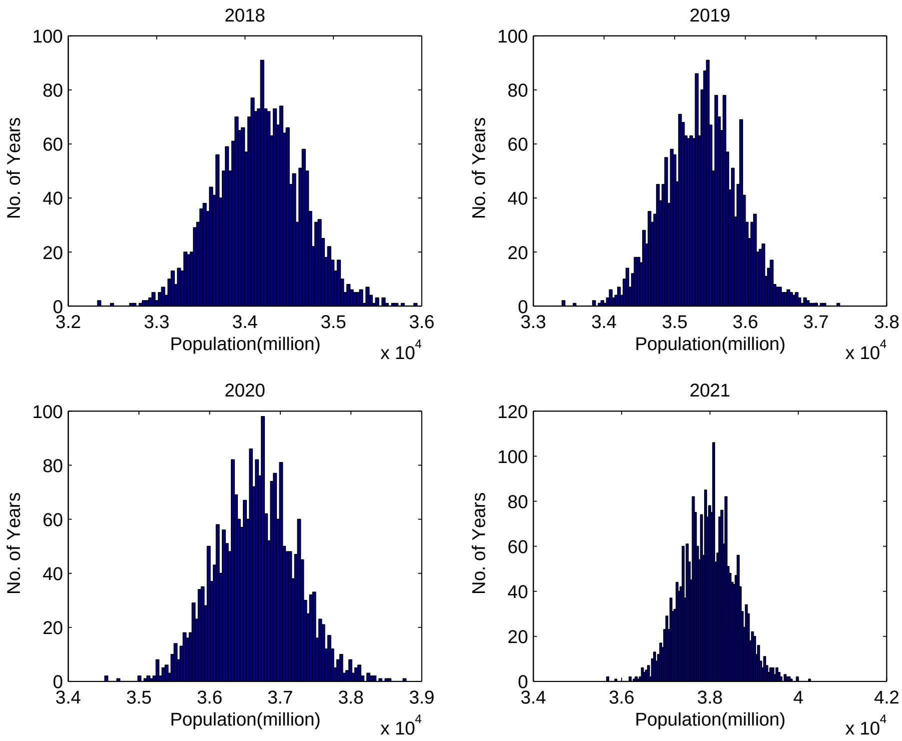

Figure 19: *Prediction from 2018 to 2021*.

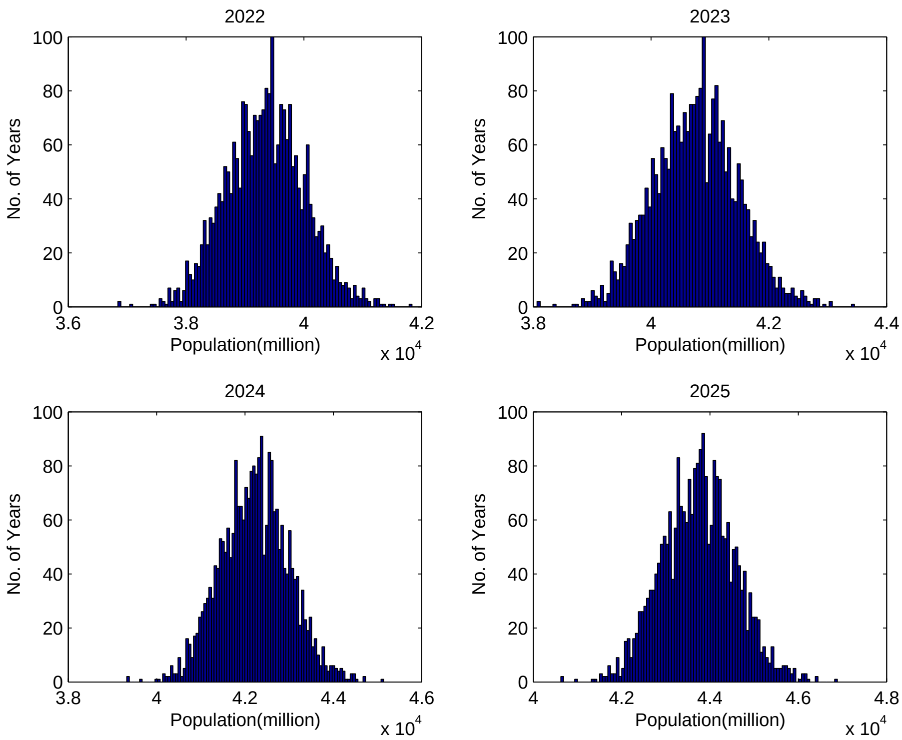

Figure 20: *Prediction from 2022 to 2025*.

Table 2: Population projections (millions) of Ghana (2010-2030).

Year	Predicted Population	Year	Predicted Population
2010	25.500	2021	35.228
2011	26.409	2022	35.499
2012	27.318	2023	36.137
2013	28.227	2024	37.046
2014	29.136	2025	37.955
2015	30.045	2026	38.864
2016	30.954	2027	39.773
2017	31.863	2028	40.682
2018	32.772	2029	41.591
2019	33.681	2030	42.500
2020	34.590		

The fitted values and the projections for the future using the logistic model are given in the Table 2. If we look critically from Table 1 the estimated population values computed by [18] and that Table 2 we find that differences values the years 2010, 2015, 2020, 2025 and 2030 were 784, 446, 1821, 2069 and 3645 respectively.

Table 3: The comparison on the two prediction of Ghana (2010-2030).

Earlier Prediction		Prediction by the formulated Model	
Year	Population	Year	Population
2010	26.284	2010	25.500
2015	29.599	2015	30.045
2020	32.769	2020	34.590
2025	35.886	2025	37.955
2030	38.855	2030	42.500

Figure 11 and 21 provide graphical presentation of the fitting of the model. It looks from the graph that the model provides a close fit to the census data. Looking at the future projections, we see that the yellow colour approached to 2030 at around 45 millions. The value of the carrying capacity (i.e $\beta_2 + \beta_4$) or the upper asymptote and the lower asymptote can be estimated as on the condition we the mean value of (4 and 4.5) million = 42.5 millions.

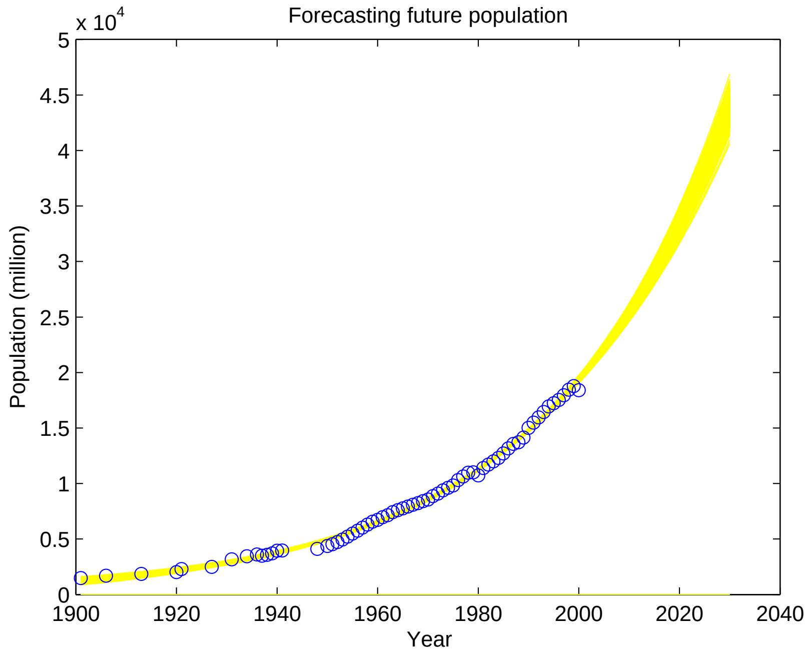

Figure 21: *Fitting, projection estimates under proposed model. Model fit: Ghana (1901-2030)* .

8 Conclusion and Discussion

The beginning of this work was to demonstrate how the Leslie matrix provides understanding of the mathematics behind the parameters in the matrix. The major outcome was that the matrix depends only on the fertility and survival rate. In future when we get statistical data trends on mortality, fertility and immigration in population growth, it is appropriate to apply these factors to the population of the country consecutive years in the future, starting with the population size and structure being put in place. Projection and its reliability bounds provides forecast for the future, which help with analysis and finally the understanding of current rates of the situation.

In this present study, efforts have show application and appropriateness of the MCMC tool in Bayesian Data analysis for fitting population census data and making predictions of the future population using the Logistic growth model. The predicted population values shown in the Table2 are the values of fits based on the past census data from Table1 and the projections for the future period of time depended on the proposed logistic model. As we can observe from Table3 earlier predicted values and our model are quite close to each other. Here, our interest is not to make comparison of different predicting methods, but to present the basics of the implementation of the Bayesian data analysis with a demonstration of the population prediction.

Moreover, present attempt appears to provide acceptable predictions for the Ghana. We will like to make further remark that the logistic growth model can still be used to fit the previous census data and predict the population, the model (103) shows exponential growth model in Fig 21 as well as Fig 12. In line with this, future work would require more information and data about the influence of demographic components (i.e. birth, death and migration). Leslie type model could be used to provide forecasts if sufficient data to estimate the model parameters.

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Appendices

Appendix 1: Codes for case 1

Exponential function growth

P = theta1*exp(theta2.*(t-1901); Parameters

theta1 = 1846

theta2 = 0.03

With the same data in appendix 2

Appendix 2: Codes for case 2

Four parameter logistic function

meant = 8.52; stdt = 4.8774;

P = beta1*beta2./(beta1+(beta2-beta1).*exp(beta3.*(t-meant)./stdt)) + beta4;

Parameters

beta1 = 6960.556;

beta2 = 18412;

theta3 = -3.503;

beta4 = 1485.856

```
years = [1901 1906 1913 1920 1921 1927 1931 1934 1936 1937 1938 ...
1939 1940 1941 1948 1950 1951 1952 1953 1954 1955 1956 ...
1957 1958 1959 1960 1961 1962 1963 1964 1965 1966 1967 ...
1968 1969 1970 1971 1972 1973 1974 1975 1976 1977 1978 ...
1979 1980 1981 1982 1983 1984 1985 1986 1987 1988 1989 ...
1990 1991 1992 1993 1994 1995 1996 1997 1998 1999 2000];

pop = [1486 1697 1852 2022 2296 2496 3164 3441 3614 3489 3572 ...
3700 3956 3963 4118 4368 4532 4734 4964 5217 5484 5758 ...
6034 6303 6562 6727 6960 7148 7422 7598 7767 7927 8082 ...
8240 8414 8559 8858 9086 9385 9607 9817 10309 10632 10969 ...
11000 10736 11400 11700 12000 12309 12710 13163 13572 13709 14137 ...
15020 15484 15959 16446 16944 17236 17522 17945 18460 18785 18412];

figure
```

```
plot(t,pop,'o',t,P,t,P,'r*-')
xlabel('Years since 1901')
ylabel('Population(million)')
title('Ghana Population Data')
```