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THE BIFURCATIONAL BEHAVIOUR OF THE SPATIALLY DISTRIBUTED RAYLEIGH EQUATION

Examiners: Professor Heikki Haario
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ABSTRACT

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The bifurcational behaviour of the spatially distributed Rayleigh equation

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At the present work the bifurcational behaviour of the solutions of Rayleigh equation and corresponding spatially distributed system is being analysed. The conditions of oscillatory and monotonic loss of stability are obtained. In the case of oscillatory loss of stability, the analysis of linear spectral problem is being performed. For nonlinear problem, recurrent formulas for the general term of the asymptotic approximation of the self-oscillations are found, the stability of the periodic mode is analysed. Lyapunov-Schmidt method is being used for asymptotic approximation. The correlation between periodic solutions of ODE and PDE is being investigated. The influence of the diffusion on the frequency of self-oscillations is being analysed. Several numerical experiments are being performed in order to support theoretical findings.

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Alexey Kazarnikov

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List of Symbols and Abbreviations

ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
c.j.	Complex conjugation
R-D	Reaction-Diffusion

1 INTRODUCTION

At the present time, a lot of mathematical papers are devoted to the analysis of bifurcations in nonlinear ODEs and partial differential equations. A special attention is given to reaction-diffusion systems. A lot of phenomena in different branches of science are being described by R-D systems. These systems have a wide range of application in biology (population dynamics, predator-prey models, the modelling of microorganism colonies), chemistry, physiology (nerve impulse propagation), social sciences and others. Therefore, the analysis of the solutions of R-D systems is actual nowadays.

Nonlinear ordinary differential equation of the second order can be transformed to a spatially distributed reaction-diffusion system by addition of diffusive terms. If a limit cycle exists in ODE, then a self-oscillatory periodic mode can appear in corresponding spatially distributed system. It should be pointed out that boundary conditions can influence the behaviour of periodic mode. At the present work, these phenomena are being analysed in a single ODE with a cubic nonlinearity.

1.1 Background

1.1.1 Rayleigh and Van-der-Pol equations. The relation between them.

Let us consider Rayleigh equation:

$$\ddot{x} - (\mu - \dot{x}^2)\dot{x} + x = 0 \tag{1}$$

If we introduce a new variable change $x = (1/\sqrt{3})y$ in (1), differentiate it by t and take $\dot{y}$ as new unknown variable z, we will arrive at Van-der-Pol equation:

$$\ddot{z} - (\mu - z^2)\dot{z} + z = 0 \tag{2}$$

Equations (1) and (2) are widely used in the analysis of periodic self-oscillations ([9], [1]). Here μ is dimensionless control parameter. There is no significant difference between these equations, and by using simple transformations one equation could be transformed into another and vice versa. At the present work we will mostly focus on Rayleigh equation.

Equation (1) is often written as:

$$\ddot{y} - \mu(1 - \dot{y}^2)\dot{y} + y = 0 \tag{3}$$

By transformation $x = \sqrt{\mu}y$ we can turn from (1) to (3). For details see ([1]).

Rayleigh equation has an unique equilibrium point. It is known that when $\mu < 0$ it is asymptotically stable. When $\mu > 0$ it is unstable and when $0 < \mu < 2$ an stable limit cycle exists in the system. For details see ([19]).

Rayleigh equation could be written as a system:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -x_1 + \mu x_2 - {x_2}^3 \end{cases} \tag{4}$$

If we incorporate diffusion terms in (4), we will arrive at the system of PDEs:

$$\begin{cases} u_t = \nu_1 u_{xx} + v \\ v_t = \nu_2 v_{xx} - u + \mu v - v^3 \end{cases} \tag{5}$$

1.1.2 The physical meaning of Van-der-Pol oscillator. Its applications in electrotechnics

Van-der-Pol oscillator was initially introduced by Holland physicist and engineer Balthasar Van-der-Pol. Under certain assumptions, it simulates the behaviour of vacuum tube oscillator, an electrotechnical device, which is able to generate periodic electrical self-oscillations ([9]). The circuit of this device is presented in the picture:

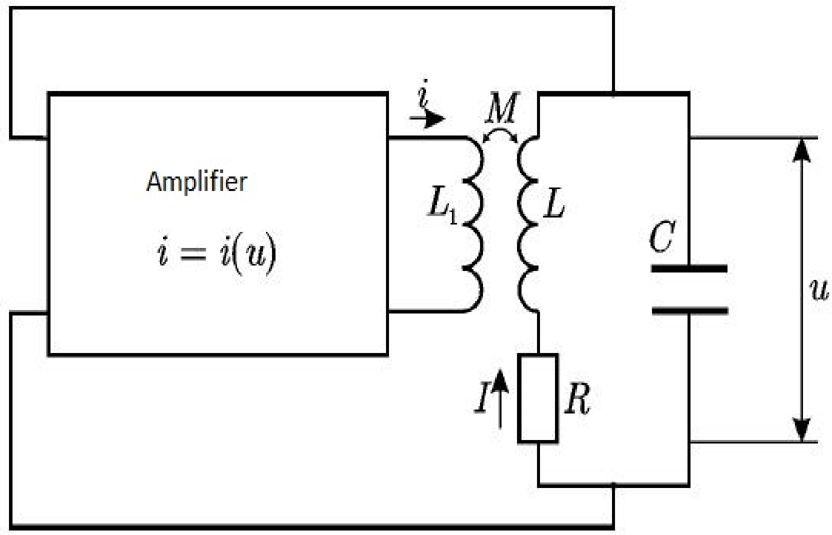

Figure 1: Electrotechnical generator of self-oscillations ([9], p. 162)

This circuit contains an oscillatory circuit and an amplifier. Oscillatory circuit consists of inductance L, resistance R and capacitor C. The voltage from oscillatory circuit comes to amplifier. The volt-ampere characteristics of the amplifier $i(u)$ is known. We will assume that it could be approximated by cubic polynomial, i.e. $i(u) = g_0 u - g_2 u^3$. Coefficients g_0, g_2 are positive. The output of the amplifier comes to inductance L_1, which is connected with inductance L because of the presence of self-induction.

We will briefly describe the process of ODE derivation, which describes the behaviour of this device. We denote as I an amperage on the resistance R, and as u the voltage on capacitor C. After applying the second law of Kirchhoff to an oscillatory circuit, we will get:

$$L\dot{I} + RI + \frac{1}{C}\int I dt = M\dot{i}(u), \qquad u = \frac{1}{C}\int I dt$$

From here, taking into account the expression for $i(u)$, we will get:

$$\ddot{u} - \omega_0^2(Mg_0 - RC - 3Mg_2u^2)\dot{u} + \omega_0^2 u = 0, \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

If we introduce a new variable $\tau = \omega_0 t$, $x = u\sqrt{3\omega_0 M g_2}$, we will get an ODE, which contains one dimensionless parameter:

$$\ddot{x} - (\lambda - x^2)\dot{x} + x = 0, \qquad \lambda = \frac{Mg_0 - RC}{\sqrt{LC}}$$

1.1.3 Other applications of the equations under study

Nerve impulse propagation. FitzHugh-Nagumo model At the second half of 20th century, mathematical models of electrotechnical devices have found a wide range of applications in biology and medicine. In 1952, Alan Lloyd Hodgkin and Andrew Huxley has introduced a mathematical model, describing the generation and propagation of nerve impuse in the giant axon of squid ([8]). This model consisted of four ODEs. In this model, cell memrbane was modelled as an electrical circuit ([7]). For the development of this model, authors were awarded by Nobel Prize in physiology and medicine in 1963.

In 1961 English physiologist Richard FitzHugh has developed an two-dimensional simplification of Hodgkin-Huxley model ([6]). In 1963, Japanese engineer Jin-Ichi Nagumo has constructed an electrical circuit, corresponding to this model. He also has studied the spatially distributed version of this system ([13]). An ODE system, which is called FitzHugh-Nagumo model is written below:

$$\begin{cases} \dot{u} = \varepsilon(v - \alpha u - \beta) \\ \dot{v} = -u + v - v^3 + z \end{cases} \tag{6}$$

Here v is the membrane potential and u is the recovery variable. Parameters α, β, ε are positive, z describes the stimulating impulse.

It should be pointed out that if in (6) set $\alpha = 0$, $\beta = 0$, $z = 0$, $\varepsilon = 1$, then we will arrive at Rayleigh equation, which is the special case of FitzHugh-Nagumo model.

Van-der-Pol oscillator and earthquake model Van-der-Pol oscillator is used in seismology as well. In paper ([5]) authors analyze the model of the interaction of two plates in a geological fault. This process is described by the following PDE:

$$\frac{\partial^2 \chi}{\partial t^2} = c^2 \frac{\partial^2 \chi}{\partial x^2} - (\chi - \nu t) - \gamma \phi(\frac{\partial \chi}{\partial t})$$

Here $\chi(x,t)$ is the time-dependent local longitudinal deformation of the surface of the upper plate in the static reference of the lower plate, $\phi(\partial\chi/\partial t) = (\partial\chi/\partial t)^3/3 - \partial\chi/\partial t$ is the friction function, γ is the magnitude of the friction, c defines the longitudinal speed of sound, ν is the pulling velocity of slip rate.

This equation is being transformed into a system

$$\begin{cases} \frac{\partial\psi}{\partial t} = \gamma(\eta - \phi(\psi)) \\ \frac{\partial\eta}{\partial t} = -\frac{1}{\gamma}(\psi - \nu - c^2\frac{\partial^2\psi}{\partial x^2}) \end{cases}$$

By searching for the solutions of the system in the form $\psi(x,t) = f(\overline{z})$, $\quad \overline{z} = x/v + t$ and introducing the variable change $z = \overline{z}/\sqrt{1 - c^2/v^2}$, we will arrive at Van-der-Pol equation:

$$\frac{d^2f}{dz^2} + \mu(f^2 - 1)\frac{df}{dz} + f = \nu$$

The propagation fronts are the periodic solutions of this equation.

1.1.4 The applications of spatially distributed systems. FitzHugh-Nagumo model with diffusion

If we incorporate diffusion terms in (6) we will arrive at spatially distributed FitzHugh-Nagumo model. It is often written in the form:

$$\begin{cases} u_t = \nu_1 u_{xx} + \varepsilon(v - \alpha u - \beta) \\ v_t = \nu_2 v_{xx} - u + v - v^3 + z \end{cases} \tag{7}$$

Let us note that spatially distributed Rayleigh equation is the special case of system (7) when $\alpha = 0$, $\beta = 0$, $z = 0$, $\varepsilon = 1$.

This model has the same physical meaning as (6). However, now $u = u(x,t)$, $v = v(x,t)$, which means that the functions now depend on space variable as well.Space variable describes the position of the point on the membrane. It follows from the physical meaning that variable u changes along the axis x very slowly, therefore diffusion coefficient ν_1 is often taken small, i.e. $\nu_1 << 1$. The coefficient is often taken $\nu_1 = 0$. For details see ([11], [17]).

System (7) has also applications in chemistry. It describes the reactions of type activator-inhibitor ([14]). Variables $u(x,t)$, $v(x,t)$ are treated as reagent concentrations, parameters α, β, ε govern the reaction process.

The system under study has the biological meaning as well. It is used as the model of interaction between two micro organism species ([11]). In that case, $u(x,t)$ and $v(x,t)$ are interpreted as amounts of populations, parameters α, β, ε govern the process of species interaction.

1.2 Research problem, objectives and delimitation

Let us consider Rayleigh equation

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -x_1 + \mu x_2 - {x_2}^3 \end{cases} \tag{8}$$

and the corresponding spatially distributed system

$$\begin{cases} u_t = \nu u_{xx} + v \\ v_t = \nu v_{xx} - u + \mu v - v^3 \end{cases} \tag{9}$$

where μ is the dimensionless real control parameter, $\nu > 0$ is the fixed viscosity parameter.

The main purpose of the present work is to investigate the bifurcational behaviour of the spatially distributed system in the case of different boundary conditions (Dirichlet, Neumann, etc) and to understand the influence of boundary conditions on the behaviour of the periodic modes. We will also study the relation between periodic solutions of ODE (8) and periodic modes in spatially distributed system (9), and perform several numerical experiments in order to support our theoretical findings.

1.3 Research methodology

A wide range of methods for analysis of periodic modes in ODEs and PDEs is nowadays available in literature. For example: Linstedt-Poincare method, Van-der-Pol method, central manifold method and etc. At the present work we will be using Lyapunov-Schmidt method in the form, developed by V.I.Yudovich ([22]). The method is applicable to ODEs and as well to PDEs, including Navier-Stokes equations ([21]). More detailed description of this method could be found in Appendix C. Nowadays, the method has a wide range of applications in fluid dynamics. For details see ([16], [12]).

We will use Maple and MATLAB software for numerical experiments.

1.4 Organization of the study

The present work is divided into several sections. The first section is devoted to the analysis of Rayleigh equation without diffusive terms. In this section we obtain first terms of the asymptotic approximation of the periodic solution and discuss the behaviour of the higher terms of the expansion. The second section is devoted to the investigation of spatially distributed system. We incorporate three different types of boundary conditions and analyse the influence of the type of boundary conditions on the self-oscillations. In the third section we consider our spatially distributed system in generalised domain and derive formulas for the asymptotic approximation of the periodic mode, suitable for a wide range of boundary conditions. And the fourth section is devoted to the numerical experiments.

All terms and definitions, which are necessary for the understanding of the material are presented in Appendix A. In Appendix B the process of numerical solution of PDE systems by using the MATLAB function pdepe() is described.

2 ODE analysis

We will start from the analysis of Rayleigh equation (1) without diffusion. It is known, that when parameter μ is small and positive, an stable limit cycle exists in the system. In this section we will find an asymptotic approximation for the periodic solution. For this purpose we will be using Lyapunov-Schmidt method. We will be looking for a periodic solution in the form of a series. We will explicitly derive first elements of the series, and discuss the behaviour of higher terms.

In order to apply the method, we first need to write down equation (1) in operator form.

2.1 Presenting ODE in operator form

Let us consider system (4). It is known that when $\mu = 0$ Hopf bifurcation takes place in the system. Therefore, $\mu = 0$ is a critical value of the parameter. By defining $\mu = 0 + \varepsilon^2$, we will get

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = \varepsilon^2 x_2 - {x_2}^3 - x_1 \end{cases} \tag{10}$$

This system can be written in vector form:

$$\dot{\vec{x}} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x} = \varepsilon^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \vec{x} - K(\vec{x}, \vec{x}, \vec{x}) \tag{11}$$

Here, $K(\vec{x}, \vec{x}, \vec{x}) = \begin{pmatrix} 0 \\ x_2 x_2 x_2 \end{pmatrix}$. Note, that $K(\vec{x}, \vec{x}, \vec{x})$ does not depends on the order of the operands.

Let us introduce new definitions:

$$\frac{d}{dt} = \omega \frac{d}{d\tau}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Then, system (11) could be rewritten as:

$$\omega \dot{\vec{x}} + B\vec{x} = \varepsilon^2 C\vec{x} - K(\vec{x}, \vec{x}, \vec{x}) \tag{12}$$

2.2 Eigenvalues and eigenvectors of the linearised system, corresponding to the critical value of parameter

Let us first find the eigenvalues of matrix B. We will consider

$$\lambda^2 + 1 = 0$$

From here we can find $\lambda_{1,2} = \pm i$. Let us consider eigenvalue $\lambda_2 = -i$ and find the corresponding eigenvector. In order to do this, we have to solve the linear system:

$$\begin{cases} i\phi_1 - \phi_2 = 0 \\ \phi_1 + i\phi_2 = 0 \end{cases} \tag{13}$$

We can set $\phi_1 = 1$, then $\phi_2 = i$. Therefore, vector $\vec{\phi} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is the needed eigenvector. We will denote as $\vec{\phi}^* = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ a vector, conjugated to vector $\vec{\phi}$.

By performing similar actions, we can find the eigenvalues of matrix B^*. It is clear that $\lambda_{1,2} = \pm i$. Now, Let us find the eigenvectors. To find the eigenvector, corresponding to $\lambda = i$, we have to consider a linear system:

$$\begin{cases} -i\psi_1 + \psi_2 = 0 \\ -\psi_1 - i\psi_2 = 0 \end{cases} \tag{14}$$

Let us set $\psi_1 = 1$, then $\psi_2 = i$. Therefore, vector $\vec{\psi} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector. Again, we will denote as $\vec{\psi}^* = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ a conjugated vector to vector $\vec{\psi}$. Note, that $(\vec{\phi}, \vec{\psi}) = 2$.

2.3 Preliminary actions

In the process of applying Lyapunov-Schmidt method, we will have to deal with the equations of the following type:

$$\dot{\vec{x}} + B\vec{x} = \begin{pmatrix} 0 \\ M \end{pmatrix} e^{ik\tau}, \qquad k \in \mathbb{Z}, k > 1, M \in \mathrm{C} \tag{15}$$

Let us find explicit formulas for their solutions. We will be looking for the solution in the form:

$$\vec{x}_p = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} e^{ik\tau} \tag{16}$$

By substituting (16) into (15), we will arrive at the linear system:

$$\begin{cases} (ik)\alpha - \beta = 0 \\ \alpha + (ik)\beta = M \end{cases}$$

From here we can find the expressions for the coefficients:

$$\begin{cases} \alpha = \frac{M}{1-k^2} \\ \beta = \frac{ikM}{1-k^2} \end{cases}$$

2.4 The analysis of nonlinear problem

We will be looking for $\vec{x}$ and ω in the form of a series:

$$\vec{x} = \sum_{i=1}^{\infty} \varepsilon^i \vec{x}_i \qquad \omega = \sum_{i=0}^{\infty} \varepsilon^i \omega_i, \quad \omega_0 = 1 \tag{17}$$

By substituting (17) into (12) and equating the coefficients of like powers of ε, we will arrive at the set of equations for obtaining $\vec{x_i}$, ω_i.

$$\varepsilon^1: \quad \dot{\vec{x}}_1 + B\vec{x}_1 = 0 \tag{18}$$

$$\varepsilon^2: \quad \dot{\vec{x}}_2 + B\vec{x}_2 = -\omega_1\dot{\vec{x}}_1 \tag{19}$$

$$\varepsilon^3: \quad \dot{\vec{x}}_3 + B\vec{x}_3 = C\vec{x}_1 - K(\vec{x}_1,\vec{x}_1,\vec{x}_1) - \omega_1\dot{\vec{x}}_2 - \omega_2\dot{\vec{x}}_1 \tag{20}$$

$$\varepsilon^4: \quad \dot{\vec{x}}_4 + B\vec{x}_4 = C\vec{x}_2 - K(\vec{x}_2,\vec{x}_1,\vec{x}_1) - K(\vec{x}_1,\vec{x}_2,\vec{x}_1) - K(\vec{x}_1,\vec{x}_1,\vec{x}_2) - \\ -\omega_1\dot{\vec{x}}_3 - \omega_2\dot{\vec{x}}_2 - \omega_3\dot{\vec{x}}_1 \tag{21}$$

$$\varepsilon^5: \quad \dot{\vec{x}}_5 + B\vec{x}_5 = C\vec{x}_3 - K(\vec{x}_3,\vec{x}_1,\vec{x}_1) - K(\vec{x}_2,\vec{x}_2,\vec{x}_1) - K(\vec{x}_2,\vec{x}_1,\vec{x}_2) - \\ -K(\vec{x}_1,\vec{x}_1,\vec{x}_3) - -K(\vec{x}_1,\vec{x}_3,\vec{x}_1) - K(\vec{x}_1,\vec{x}_2,\vec{x}_2) - \\ -\omega_1\dot{\vec{x}}_4 - \omega_2\dot{\vec{x}}_3 - \omega_3\dot{\vec{x}}_2 - \omega_4\dot{\vec{x}}_1 \tag{22}$$

The periodic solution of (18) has the form:

$$\vec{x}_1 = \alpha_1\vec{\phi}e^{i\tau} + \text{c.j.}, \qquad \alpha_1 > 0 \tag{23}$$

An inhomogeneous equation (19) will have an periodic solution if and only if the condition of solvability is satisfied:

$$\int_0^{2\pi} (-\omega_1\dot{\vec{x}}_1, \vec{\psi})e^{-i\tau}d\tau = 0 \tag{24}$$

The the meaning of the condition of solvability is explained in Appendix C.

By performing necessary computations and noting that $e^{i\tau}$ is 2π-periodic function, we will get:

$$\int_0^{2\pi} (-\omega_1\dot{\vec{x}}_1, \vec{\psi})d\tau = \int_0^{2\pi} (-\omega_1 i\alpha_1(\vec{\phi}e^{i\tau} + \vec{\phi}^*e^{-i\tau}), \vec{\psi})e^{-i\tau}d\tau = -2\omega_1 i\alpha_1 2\pi \tag{25}$$

From the condition $\alpha_1 > 0$ follows, that $\omega_1 = 0$. Then, the solution of (19) is:

$$\vec{x}_2 = \alpha_2\vec{\phi}e^{i\tau} + \text{c.j.} \tag{26}$$

Now, we will check the condition of solvability for (20):

$$\int_0^{2\pi} (C\vec{x}_1 - K(\vec{x}_1,\vec{x}_1,\vec{x}_1) - \omega_1\dot{\vec{x}}_2 - \omega_2\dot{\vec{x}}_1, \vec{\psi})d\tau = 0 \tag{27}$$

After performing the computations, we get:

$$\int_0^{2\pi} (\dots)d\tau = -2\pi(\alpha_1 + 3\alpha_1{}^3 + 2i\alpha_1\omega_2) = 0 \tag{28}$$

After splitting this expression into real and imaginary part, we will get:

$$\begin{cases} 3{\alpha_1}^2 - 1 = 0 \\ 2i\omega_2 = 0 \end{cases} \tag{29}$$

Therefore, the solution of the system is:

$$\alpha_1 = \frac{1}{\sqrt{3}}, \quad \omega_2 = 0 \tag{30}$$

Now, by deleting the terms, equal to zero, we can write (20) in the form:

$$\dot{\vec{x}}_3 + B\vec{x}_3 = \frac{1}{3\sqrt{3}} \begin{pmatrix} 0 \\ i \end{pmatrix} e^{3i\tau} + \text{c.j.} \tag{31}$$

By applying formulas, derived in previous subsection, we can write the solution:

$$\vec{x}_p = \begin{pmatrix} \frac{1}{24i\sqrt{3}} \\ \frac{1}{8\sqrt{3}} \end{pmatrix} e^{3i\tau} + \text{c.j.}$$

Finally, the solution of (20) is:

$$\vec{x}_3 = \alpha_3\vec{\phi}e^{i\tau} + \text{c.j.} + \begin{pmatrix} \frac{1}{24i\sqrt{3}} \\ \frac{3}{24\sqrt{3}} \end{pmatrix} e^{3i\tau} + \text{c.j.} = \vec{x}_{30} + \text{c.j.} + \vec{a}_3 e^{3i\tau} + \text{c.j.} \tag{32}$$

Now, we will satisfy the condition of solvability for (22):

$$\int_0^{2\pi} (C\vec{x}_2 - 3K(\vec{x}_2, \vec{x}_1, \vec{x}_1) - \omega_3\dot{\vec{x}}_1, \vec{\psi})e^{-i\tau} d\tau = 0 \tag{33}$$

By performing the computations, we can obtain:

$$i\omega_3(\vec{\phi}, \vec{\psi}) = \alpha_2(C\vec{\phi}, \vec{\psi}) - 9{\alpha_1}^2\alpha_2(K(\vec{\phi}, \vec{\phi}, \vec{\phi}^*), \vec{\psi}) \tag{34}$$

From here we can arrive at the equation:

$$2i\omega_3 = \alpha_2 - 9{\alpha_1}^2\alpha_2 \tag{35}$$

Therefore, the solution is:

$$\alpha_2 = 0, \qquad \omega_3 = 0 \tag{36}$$

And finally, the solution of (22) has the form:

$$\vec{x}_4 = \alpha_4 \vec{\phi} e^{i\tau} + \text{c.j.} \tag{37}$$

Let us satisfy the condition of solvability for (22):

$$\int_0^{2\pi} (C\vec{x}_3 - 3K(\vec{x}_3, \vec{x}_1, \vec{x}_1), \vec{\psi}) e^{-i\tau} d\tau = 0 \tag{38}$$

By performing the computations and noting that $K(\vec{x}_3, \vec{x}_1, \vec{x}_1) = K(\vec{x}_{30}, \vec{x}_1, \vec{x}_1) + K(\vec{a}_3 e^{3i\tau} + \text{c.j.}, \vec{x}_1, \vec{x}_1)$ we will get:

$$2i\alpha_1\omega_4 = -9{\alpha_1}^2\alpha_3 + \alpha_3 + \frac{9i}{24\sqrt{3}}{\alpha_1}^2 \tag{39}$$

From here we conclude that:

$$\omega_4 = \frac{3}{48}, \quad \alpha_3 = 0 \tag{40}$$

Now, we have found first two terms of the asymptotic approximation. The expressions for the first terms and cyclic frequency are:

$$x(t) = \frac{2}{\sqrt{3}}\varepsilon\cos(\omega t) + \frac{1}{12\sqrt{3}}\varepsilon^3 \sin(3\omega t) + O(\varepsilon^4), \qquad \omega = 1 + \frac{3}{48}\varepsilon^4 + O(\varepsilon^5) \tag{41}$$

On the figure below the comparison between analytical approximation and numerical solutions of the nonlinear equation (1) is presented.

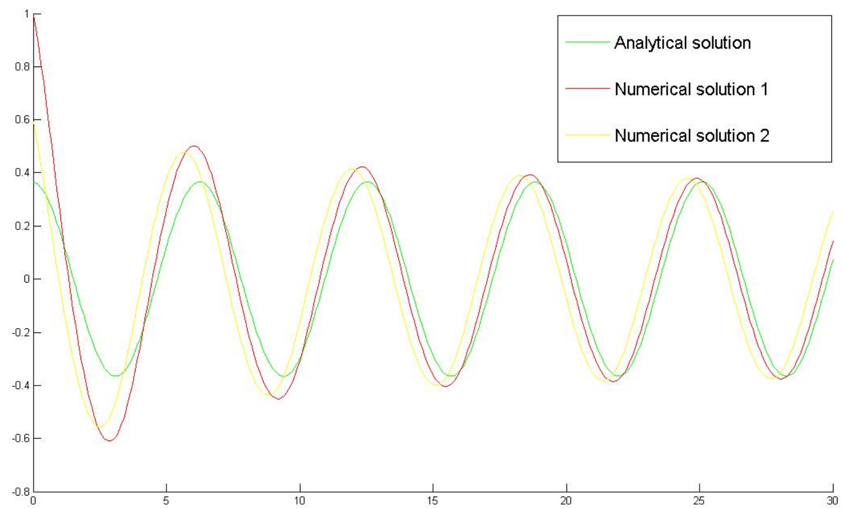

Figure 2: Analytical and numerical solutions when $\mu = 0.1$

2.5 Formulas for the consecutive terms of asymptotic

When $n \geq 3$, the expression around ε^n can be presented as:

$$\omega_0 \dot{\vec{x}}_n + B\vec{x}_n = C\vec{x}_{n-2} - \sum_{i=1}^{n-1} \omega_{n-i}\dot{\vec{x}}_i - \sum_{\substack{i_1+i_2+i_3=n \\ i_1 \leq i_2 \leq i_3}} 3K(\vec{x}_{i_1}, \vec{x}_{i_2}, \vec{x}_{i_3}) \tag{42}$$

In this section we will derive formulas for the terms of expansion around ε^n. We will start from the proof of a simple statement.

Statement 1. For all $k \in \mathbb{N}$ $\quad \vec{x}_{2k} = 0, \quad \alpha_{2k} = 0, \quad \omega_{2k-1} = 0$

Proof. We will apply mathematical induction. It was found that:

$$\alpha_2 = 0, \alpha_4 = 0$$
$$\omega_1 = 0, \omega_3 = 0$$
$$\vec{x}_2 = 0, \vec{x}_4 = 0$$

Let us suppose that for some even number p holds:

$$\alpha_k = 0, k = 2, 4, \ldots, p-4$$
$$\omega_k = 0, k = 1, 3, \ldots, p-3$$
$$\vec{x}_k = 0, k = 2, 4, \ldots, p-4$$

We will prove that the following is also valid for $p+1$. For that, Let us consider the expression around $p-2$ degree. It is clear that it will have the following form:

$$\omega_0 \dot{\vec{x}}_{p-2} + B\vec{x}_{p-2} = 0$$

Therefore, $\vec{x}_{p-2} = \alpha_{p-2}\vec{\varphi}e^{i\tau} + \text{ c.j.}$

Let us consider the expression around p-th degree:

$$\omega_0 \dot{\vec{x}}_p + B\vec{x}_p = C\vec{x}_{p-2} - \sum_{i=1}^{p-1} \omega_{p-i}\dot{\vec{x}}_i - \sum_{\substack{i_1+i_2+i_3=p \\ i_1 \leq i_2 \leq i_3}} 3K(\vec{x}_{i_1}, \vec{x}_{i_2}, \vec{x}_{i_3})$$

From the fact that p is even and even number cannot be the sum of three odd numbers we conclude that:

$$\sum_{\substack{i_1+i_2+i_3=p \\ i_1 \le i_2 \le i_3}} 3K(\vec{x}_{i_1}, \vec{x}_{i_2}, \vec{x}_{i_3}) = 3K(\vec{x}_1, \vec{x}_1, \vec{x}_{p-2})$$

By applying the prepositions of the statement, we get:

$$\sum_{i=1}^{p-1} \omega_{p-i} \dot{\vec{x}}_i = \omega_{p-1} \dot{\vec{x}}_1$$

Then, our equation will be in the following form:

$$\omega_0 \dot{\vec{x}}_p + B\vec{x}_p = C\vec{x}_{p-2} - 3K(\vec{x}_1, \vec{x}_1, \vec{x}_{p-2}) - \omega_{p-1} \dot{\vec{x}}_1$$

The condition of solvability for this equation can be presented in the form:

$$\alpha_{p-2}(C\vec{\varphi}, \vec{\psi}) - 9\alpha_1^2 \alpha_{p-2}(K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi}^*), \vec{\psi}) = i\omega_{p-1}(\vec{\varphi}, \vec{\psi})$$

After computing the inner products, we get:

$$-2\alpha_{p-2} = i\omega_{p-1}$$

Therefore, we conclude that $\omega_{p-1} = 0$, $\alpha_{p-2} = 0$, and $\vec{x}_{p-2} = 0$. The proof is finished.

We have found the expressions for even terms. Now, Let us describe the process of finding the formulas for odd terms.

For odd n, equation (42) can be written as:

$$\omega_0 \dot{\vec{x}}_n = A\vec{x}_n + C\vec{x}_{n-2} - i\omega_{n-1} \dot{\vec{x}}_1 - \sum_{\substack{i_1+i_2+i_3=n \\ i_1 \le i_2 \le i_3}}^{i_1+i_2+i_3=2k+1} 3K(\vec{x}_{i_1}, \vec{x}_{i_2}, \vec{x}_{i_3}) \tag{43}$$

Let us note that:

$$C\vec{x}_{n-2} = \alpha_{n-2} C\vec{\varphi} e^{i\tau} + \alpha_{n-2} C\vec{\varphi^*} e^{-i\tau} + C(\vec{x}^{\mathrm{p}}_{n-2})$$

We have denoted as $\vec{x}^{\mathrm{p}}_{n-2}$ the solution of inhomogeneous equation.

Now, Let us transform the sum:

$$\sum_{\substack{i_1+i_2+i_3=n\\ i_1\le i_2\le i_3}}^{i_1+i_2+i_3=2k+1} 3K(\vec{x}_{i_1},\vec{x}_{i_2},\vec{x}_{i_3}) = 3K(\vec{x}_1,\vec{x}_1,\vec{x}_{p-2}) +$$

$$+ \sum_{\substack{i_1+i_2+i_3=n\\ i_1\le i_2\le i_3}}^{(i_1,i_2,i_3)\neq(1,1,p-2)} 3K(\vec{x}_{i_1},\vec{x}_{i_2},\vec{x}_{i_3})$$

Let us note that:

$$K(\vec{x}_1,\vec{x}_1,\vec{x}_{p-2}) = K(\vec{x}_1,\vec{x}_1,\vec{x}^{\mathrm{hom}}_{p-2}) + K(\vec{x}_1,\vec{x}_1,\vec{x}^{\mathrm{p}}_{p-2})$$

We will denote the right part of the expression (43) as:

$$f_n = C(\vec{x}^{\mathrm{p}}_{n-2}) - 3K(\vec{x}_1,\vec{x}_1,\vec{x}^{\mathrm{p}}_{p-2}) - \sum_{\substack{i_1+i_2+i_3=n\\ i_1\le i_2\le i_3}}^{(i_1,i_2,i_3)\neq(1,1,p-2)} 3K(\vec{x}_{i_1},\vec{x}_{i_2},\vec{x}_{i_3}) \qquad (44)$$

Then, we can write the condition of solvability for this equation in the form:

$$\int_0^{2\pi} ((\alpha_{n-2}C\vec{\varphi}e^{i\tau} + \alpha_{n-2}C\vec{\varphi}^{*}e^{-i\tau},\vec{\psi})d\tau -$$

$$-\int_0^{2\pi} i\omega_{n-1}(\dot{\vec{x}}_1,\vec{\psi}) - 3(K(\vec{x}_1,\vec{x}_1,\vec{x}_{p-2},\vec{\psi}) + (f_n,\vec{\psi}))e^{-i\tau}d\tau = 0$$

After performing the computations, we get:

$$\alpha_{n-2} + i\alpha_1\omega_{n-1} = (f_n,\vec{\psi})$$

From here we conclude that

$$\alpha_{n-2} = Re((f_n, \vec{\psi})), \qquad \omega_{n-1} = \sqrt{3} Im((f_n, \vec{\psi}))$$

Now, Let us discuss the process of obtaining the solution of inhomogeneous equation (43). It is clear, that it can be written as:

$$\omega_0 \dot{\vec{x}}_n + B\vec{x}_n = \begin{pmatrix} M_{3n} \\ N_{3n} \end{pmatrix} e^{3i\tau} + \cdots + \begin{pmatrix} M_{mn} \\ N_{mn} \end{pmatrix} e^{im\tau} + \text{c.j.} \tag{45}$$

We will be looking for the solution in the following form:

$$\vec{x}_n = \begin{pmatrix} \alpha_{3n} \\ \beta_{3n} \end{pmatrix} e^{3i\tau} + \cdots + \begin{pmatrix} \alpha_{mn} \\ \beta_{mn} \end{pmatrix} e^{im\tau} + \text{c.j.} \tag{46}$$

By substituting (46) into (45) and grouping the expressions, we will arrive at the finite set of linear systems. Formulas for their solutions were derived earlier. By using these formulas, we can find unknown coefficients in expression (46).

3 The analysis of spatially distributed system

In this section we will be dealing with the periodic modes in spatially distributed system (5). We shall start from obtaining the asymptotic approximation for the periodic mode in the case of three different types of boundary conditions. Then, we will generalize our results for a wider range of boundary conditions.

For obtaining the asymptotic approximation, we will be using again Lyapunov-Schmidt method. It is applicable to PDEs as well, therefore the process will not significantly change. However, the computations would become more complicated.

As in the case of ODE, we should start from writing our system in operator form.

3.1 Presenting PDE in operator form

Let us consider operator A:

$$\mathrm{A}\vec{\phi} = \nu\vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + \mu\mathrm{C}\vec{\phi} : L_2[0,1] \times L_2[0,1] \to L_2[0,1] \times L_2[0,1]$$

Here $\vec{\phi} = \begin{pmatrix} u \\ v \end{pmatrix}$, $\mathrm{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathrm{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Let us consider trilinear operator $K(\vec{x}, \vec{x}, \vec{x})$, defined in the following way:

$$K(\vec{\phi}, \vec{\phi}, \vec{\phi}) = \begin{pmatrix} 0 \\ \phi_2\phi_2\phi_2 \end{pmatrix}, \quad K(\vec{x}, \vec{x}, \vec{x}) : (L_2[0,1])^6 \to (L_2[0,1])^6$$

Note, that again $K(\vec{x}, \vec{x}, \vec{x})$ does not depends on the order of operands.

Then, system (5) can be written in operator form:

$$\dot{\vec{\phi}} = \mathrm{A}\vec{\phi} - K(\vec{\phi}, \vec{\phi}, \vec{\phi}) \tag{47}$$

Operator A is defined on the domain $D(A)$, which belongs to $L_2[0,1] \times L_2[0,1]$. And operator K is defined on $D(A) \times D(A) \times D(A)$.

It should be pointed out, that by selecting $D(A)$ we incorporate boundary conditions.

3.2 Conjugate operator

Let us derive the expression for A^*, which is an operator, conjugated to A.

Inner product in $L_2[0,1] \times L_2[0,1]$ is defined as:

$$\forall u = (u_1, u_2), v = (v_1, v_2) : u, v \in L_2[0,1] \times L_2[0,1]$$
$$(u, v) = \int\limits_0^1 (u_1 v_1 + u_2 v_2) dx$$

Conjugate operator must satisfy the following condition:

$$(\mathrm{A}u, v) = (u, \mathrm{A}^* v) \tag{48}$$

Let us compute $(\mathrm{A}u, v)$:

$$(\mathrm{A}u,v) = \left(\begin{pmatrix} \nu(u_1)_{xx} + u_2 \\ \nu(u_2)_{xx} - u_1 + \mu u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \int_0^1 \nu(u_1)_{xx} v_1 dx + \\ + \int_0^1 \nu(u_2)_{xx} v_2 dx + \int_0^1 (u_2 v_1 - u_1 v_2 + \mu u_2 v_2) dx \tag{49}$$

By applying integration by parts to (49) and by grouping terms around u_1, u_2, we will get:

$$(Au, v) = \int_0^1 [(\nu(v_1)_{xx} - v_2)u_1 + (\nu(v_2)_{xx} + v_1 + \mu v_2)u_2]$$

From here, we can find the conjugate operator:

$$\mathrm{A}^* \vec{v} = \vec{v}_{xx} - B\vec{v} + \mu C \vec{v} \tag{50}$$

3.3 Dirichlet boundary conditions

Let us consider equation (47) and incorporate Dirichlet boundary conditions. Then, the domain of operator A will be:

$$D(A) = \{u(x,t), v(x,t) \in W_2^2 : u(0,t) = u(1,t) = 0, v(0,t) = v(1,t) = 0\}$$

In this domain, system $\{\sin(\pi k x)\}_{k=1}^{\infty}$ is the basis.

3.3.1 Eigenvalues of operator A

Let us consider an eigenvalue problem:

$$\mathrm{A}\vec{\phi} = \lambda \vec{\phi} \tag{51}$$

We can write basis decomposition of vector $\vec{\phi}$ in the form:

$$\vec{\phi} = \sum_{k=1}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \sin(\pi k x) \tag{52}$$

If we substitute (52) into (51) and group expressions around basis functions, we will get:

$$\begin{pmatrix} -\nu(\pi k)^2 & 1 \\ -1 & \mu - \nu(\pi k)^2 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \lambda \begin{pmatrix} a_k \\ b_k \end{pmatrix} \qquad k = 1, 2, \dots$$

The determinant of a matrix in left side of the expression can be written as:

$$\varphi_{A_k}(\lambda) = \lambda^2 + (2\nu(\pi k)^2 - \mu)\lambda + \nu^2(\pi k)^4 - \mu\nu(\pi k)^2 + 1$$

The eigenvalues of operator A are the roots of polynomial $\varphi_{A_k}(\lambda)$:

$$\lambda_{1,2}^k = \frac{-(2\nu(\pi k)^2 - \mu) \pm \sqrt{\mu^2 - 4}}{2}$$

Now, Let us find the critical value of the parameter μ, which corresponds to oscillatory loss of stability. Stability conditions are:

$$\begin{cases} \mu - 2\nu(\pi k)^2 < 0 \\ \nu^2(\pi k)^4 - \mu\nu(\pi k)^2 + 1 > 0 \end{cases} \qquad k = 1, 2, \dots \tag{53}$$

Or:

$$\begin{cases} \mu < 2\nu(\pi k)^2 \\ \mu < \frac{1}{\nu(\pi k)^2} + \nu(\pi k)^2 \end{cases} \qquad k = 1, 2, \dots \tag{54}$$

As $\nu > 0$, we can consider only first pair of inequalities (when k = 1):

$$\begin{cases} \mu < 2\nu(\pi)^2 \\ \mu < \frac{1}{\nu(\pi)^2} + \nu(\pi)^2 \end{cases} \tag{55}$$

It is known fact that $\forall a \in \mathbb{R} : a > 0$ holds: $a + \frac{1}{a} < 2a \iff a > 1$. Then, it is clear that in the pair of inequalities (55), second inequality will dominate when $\nu(\pi)^2 > 1$. From here we can find the critical value of parameter μ:

$$\mu_{\text{cr}} = \left[\begin{array}{l} \frac{1}{\nu\pi^2} + \nu\pi^2, \quad when \quad \nu > \frac{1}{\pi^2} \\ 2\nu\pi^2, \quad when \quad \nu < \frac{1}{\pi^2} \end{array} \right.$$

When $\nu > \frac{1}{\pi^2}$ monotonic instability takes place in the system. When $\nu < \frac{1}{\pi^2}$ - oscillatory instability.

The process of finding the critical value of the parameter can be better understood from the graphical interpretation. In (55) we have infinite number of inequalities. To find the critical value means to find out, at what value of the parameter μ at least one of the conditions will become unsatisfied.

Let us treat right sides of the inequalities (55) as continuous functions of argument k. We will denote

$$f_1(k) = 2\nu\pi^2 k^2, \qquad f_2(k) = \frac{1}{\nu\pi^2 k^2} + \nu\pi^2 k^2$$

and assume that $k > 0, k \in \mathbb{R}$.

We can see that $f_1(k)$ is a monotonous function, $f_2(k)$ has one minimum point at $k_0 = \frac{1}{\sqrt{\nu}\pi}$. It can be easily found out that $f_1(k)$ and $f_2(k)$ will intersect exactly at the point of minimum. This happens because the diffusion coefficients in system(5) are taken equal. Therefore, function $g(k) = \min(f_1(k), f_2(k))$ will be also monotonous function.

However, we are interested in integer values of the function $g(k)$. In other words, the critical value of the parameter μ will be the minimal value of the function $g(k)$, when $k = 1, 2, \ldots$. We have understood earlier that $g(k)$ is monotonous, therefore it is evident that $\mu_{cr} = g(1)$. However, this value depends on the position of the minimum point k_0. If $k_0 > 1$ then $g(1) = f_1(1)$, otherwise $g(1) = f_2(1)$. And this is exactly the same condition, which was obtained earlier.

On the figures below two different situations, mentioned above are presented:

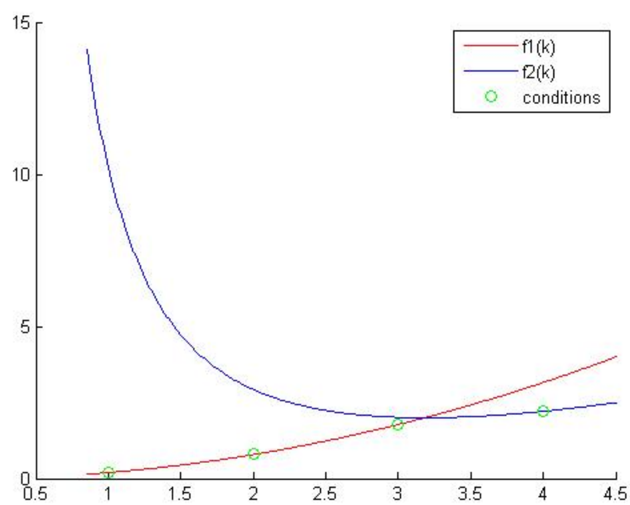

Figure 3: Oscillatory loss of stability. $\nu = 0.01$, $k_0 > 1$

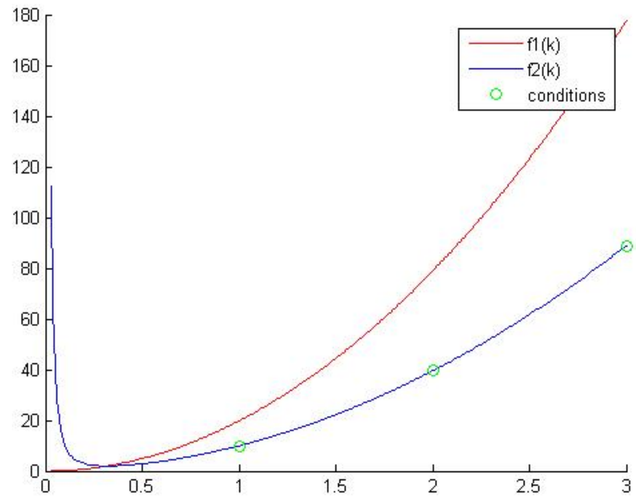

Figure 4: Monotonic loss of stability. $\nu = 1$, $k_0 < 1$

The situation, discussed above holds only when diffusion coefficients in (5) are taken equal. When $\nu_1 \neq \nu_2$ obtaining the critical value of the parameter μ becomes more complicated process. This happens because $g(k)$ will no longer be a monotonous function.

3.3.2 Eigenvectors of the operator A in the case of oscillatory instability

Let us consider an eigenvector problem:

$$\mathrm{A}\vec{\phi} = \lambda_{1,2}\vec{\phi}$$

Here $\mu = 2\pi^2\nu$, $\nu < \frac{1}{\pi^2}$, $\lambda_{1,2} = \pm i\sqrt{1 - \nu^2\pi^4}$.

Let us define $\omega_0 = \sqrt{1 - 4\nu^2\pi^4}$. Then, $\lambda_{1,2} = \pm i\omega_0$. Eigenvectors could be found from the system:

$$\begin{pmatrix} -\nu\pi^2 & 1 \\ -1 & \nu\pi^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm i\omega_0 \begin{pmatrix} a \\ b \end{pmatrix}$$

It can be rewritten as:

$$\begin{cases} (-\nu\pi^2 \mp i\omega_0)a + b = 0 \\ -a + (\nu\pi^2 \mp i\omega_0)b = 0 \end{cases}$$

If we take $a = 1$ and $b = \nu\pi^2 \pm i\omega_0$, we can find eigenvectors, corresponding to $\lambda_{1,2}$:

$$\vec{\phi}_{1,2} = \begin{pmatrix} 1 \\ \nu\pi^2 \pm i\omega_0 \end{pmatrix} \sin(\pi x)$$

3.3.3 Eigenvectors of the conjugate operator A^*

Let us find eigenvectors of conjugate operator A^*, which correspond to $\lambda_{1,2}$ when $\mu = \mu_{??}$ in the case of oscillatory instability. We should consider a system:

$$\begin{pmatrix} -\nu\pi^2 \mp i\omega_0 & -1 \\ 1 & \nu\pi^2 \mp i\omega_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From here we can find eigenvectors:

$$\vec{\psi}_{1,2} = \begin{pmatrix} 1 \\ \nu\pi^2 \mp i\omega_0 \end{pmatrix} \sin(\pi x)$$

3.3.4 Preliminary actions

In the process of applying Lyapunov-Schmidt method, we will have to deal with the equations of the following type:

$$(A - ik_1\omega_0 I)\vec{v} = \begin{pmatrix} 0 \\ (\gamma + i\delta) \end{pmatrix} \sin(\pi k_2 x), \qquad k_1 \in \mathbb{Z}, k_2 \in \mathbb{N} \tag{56}$$

Let us describe the process of obtaining the solutions of these equations. We will be looking for them in the form:

$$\vec{v} = \begin{pmatrix} M \\ N \end{pmatrix} \sin(\pi k_2 x) \tag{57}$$

If we substitute (57) into (56), we will get:

$$\left[B + aC - (\frac{a{k_2}^2}{2} + ik_1\omega_0)I\right] \vec{v} = \begin{pmatrix} 0 \\ \gamma + i\delta \end{pmatrix} \sin(\pi k_2 x)$$

By grouping terms around basis functions, we will get:

$$\begin{pmatrix} -(\frac{ak_2{}^2}{2} + ik_1\omega_0) & 1 \\ -1 & a - (\frac{ak_2{}^2}{2} + ik_1\omega_0) \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma + i\delta \end{pmatrix}$$

The solution of this system could be presented as:

$$\begin{cases} M = (\gamma + i\delta)/((a - (\frac{ak_2^2}{2} + ik_1\omega_0))(\frac{ak_2^2}{2} + ik_1\omega_0) - 1) \\ N = (\frac{ak_2^2}{2} + ik_1\omega_0)M \end{cases}$$

Let us introduce new notations:

$$\begin{cases} P_{k1}^{k2} = \frac{ak_2^2}{2} + ik_1\omega_0 \\ F_{k1}^{k2} = (a - P_{k1}^{k2})P_{k1}^{k2} - 1 \\ R_{k1}^{k2} = Re(P_{k1}^{k2})Re(F_{k1}^{k2}) - Im(P_{k1}^{k2})Im(F_{k1}^{k2}) \\ I_{k1}^{k2} = Re(P_{k1}^{k2})Im(F_{k1}^{k2}) - Im(P_{k1}^{k2})Re(F_{k1}^{k2}) \end{cases}$$

Then, the expressions for the coefficients could be written in the following form:

$$M = \frac{(\gamma + i\delta)(F_{k1}^{k2})^*}{|F_{k1}^{k2}|^2}, \quad N = \frac{(\gamma + i\delta)(R_{k1}^{k2} + iI_{k1}^{k2})}{|F_{k1}^{k2}|^2}$$

Therefore, the solution of (56) is:

$$\vec{v} = \begin{pmatrix} M \\ N \end{pmatrix} \sin(\pi k_2 x)$$

3.3.5 The analysis of nonlinear problem

Let us consider (47) in the case of oscillatory instability, i.e. when the condition $\nu < \frac{1}{\pi^2}$ holds. In that case, the critical value of parameter μ is $\mu_{\text{cr}} = 2\nu\pi^2$.

We will introduce new notation: $\mu = a + \delta$, $a = \mu_{\text{cr}}$, $\delta \ll 1$. Then, equation (47) could be rewritten as:

$$\dot{\vec{\phi}} = \nu\vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + a\mathrm{C}\vec{\phi} + \delta\mathrm{C}\vec{\phi} - K(\vec{\phi}, \vec{\phi}, \vec{\phi}) \tag{58}$$

Now, we will introduce a time variable change: $\tau = \omega t, \delta = \varepsilon^2$. By dot we will define the derivation by τ. Then, equation (58) can be rewritten as:

$$\omega\dot{\vec{\phi}} = \nu\vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + a\mathrm{C}\vec{\phi} + \varepsilon^2\mathrm{C}\vec{\phi} - K(\vec{\phi},\vec{\phi},\vec{\phi}) \tag{59}$$

We will be looking for $\vec{\phi}$ and ω in the form of series:

$$\vec{\phi} = \sum_{i=1}^{\infty}\varepsilon^i\vec{\phi}_i, \qquad \omega = \sum_{i=0}^{\infty}\varepsilon^i\omega_i \tag{60}$$

It is evident that $\omega_0 = \sqrt{1-\nu^2\pi^4}$. If we substitute (60) into (59) and equate the coefficients of like powers of ε, we will get:

$$\varepsilon^1: \quad \omega_0\dot{\vec{\phi}}_1 = \nu\frac{\partial^2\vec{\phi}_1}{\partial x^2} + \mathrm{B}\vec{\phi}_1 + a\mathrm{C}\vec{\phi}_1 \tag{61}$$

$$\varepsilon^2: \quad \omega_0\dot{\vec{\phi}}_2 = \nu\frac{\partial^2\vec{\phi}_2}{\partial x^2} + \mathrm{B}\vec{\phi}_2 + a\mathrm{C}\vec{\phi}_2 - \omega_1\dot{\vec{\phi}}_1 \tag{62}$$

$$\varepsilon^3: \quad \omega_0\dot{\vec{\phi}}_3 = \nu\frac{\partial^2\vec{\phi}_3}{\partial x^2} + \mathrm{B}\vec{\phi}_3 + a\mathrm{C}\vec{\phi}_3 - \omega_1\dot{\vec{\phi}}_2 - \omega_2\dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_1 - K(\vec{\phi}_1,\vec{\phi}_1,\vec{\phi}_1) \tag{63}$$

$$\varepsilon^4: \quad \omega_0\dot{\vec{\phi}}_4 = \nu\frac{\partial^2\vec{\phi}_4}{\partial x^2} + \mathrm{B}\vec{\phi}_4 + a\mathrm{C}\vec{\phi}_4 - \omega_1\dot{\vec{\phi}}_3 - \omega_2\dot{\vec{\phi}}_2 - \omega_3\dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_2 - -3K(\vec{\phi}_1,\vec{\phi}_1,\vec{\phi}_2) \tag{64}$$

$$\varepsilon^5: \quad \omega_0\dot{\vec{\phi}}_5 = \nu\frac{\partial^2\vec{\phi}_5}{\partial x^2} + \mathrm{B}\vec{\phi}_5 + a\mathrm{C}\vec{\phi}_5 - \omega_1\dot{\vec{\phi}}_4 - \omega_2\dot{\vec{\phi}}_3 - \omega_3\dot{\vec{\phi}}_2 - \omega_4\dot{\vec{\phi}}_1 + +\mathrm{C}\vec{\phi}_3 - 3K(\vec{\phi}_1,\vec{\phi}_1,\vec{\phi}_3) - 3K(\vec{\phi}_1,\vec{\phi}_2,\vec{\phi}_2) \tag{65}$$

Let us define: $\vec{\varphi} = \vec{\phi}_1$, $\vec{\psi} = \vec{\psi}_2$.

Now, we calculate $(\vec{\varphi},\vec{\psi})$:

$$(\vec{\varphi},\vec{\psi}) = \int_0^1 (1 + (\frac{a}{2} + i\omega_0)(\frac{a}{2} - i\omega_0))\sin^2(\pi x)dx = 2\int_0^1 \sin^2(\pi x)dx = 1$$

Let us start solving equations. Expression (61) is a linear homogeneous equation. Then, its periodic solution can be written as:

$$\vec{\phi}_1 = \alpha_1\vec{\varphi}e^{i\tau} + \text{c.j.}, \qquad \alpha_1 > 0$$

Inhomogeneous equation (62) has the periodic solution if and only if the condition of solvability is satisfied:

$$\int_0^{2\pi}(-\omega_1\dot{\vec{\phi}}_1,\vec{\psi})e^{-i\tau}d\tau = 0$$

By performing computations, we get:

$$\int_0^{2\pi}(-\omega_1\dot{\vec{\phi}}_1,\vec{\psi})e^{-i\tau}d\tau = -2\pi i\alpha_1\omega_1$$

From the condition $\alpha_1 > 0$, we conclude that $\omega_1 = 0$.

Then, the solution of (62) can be written as:

$$\vec{\phi}_2 = \alpha_2 \vec{\varphi} e^{i\tau} + \text{c.j.}$$

Now, Let us satisfy the condition of solvability for the equation (63):

$$\int_0^{2\pi} (-\omega_2 \dot{\vec{\phi}}_1 + K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_1) + \text{C}\vec{\phi}_1, \vec{\psi}) e^{-i\tau} d\tau = 0$$

After performing the computations, we get:

$$\int_0^{2\pi} (-\omega_2 \dot{\vec{\phi}}_1 - K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_1) + \text{C}\vec{\phi}_1, \vec{\psi}) e^{-i\tau} d\tau =$$
$$= -i\omega_2 \alpha_1 (\vec{\varphi}, \vec{\psi}) - 3\alpha_1^3 (K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi}^*), \vec{\psi}) + \alpha_1 (\text{C}\vec{\varphi}, \vec{\psi})$$

If we compute the inner products, we will get:

$$(K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi}^*), \vec{\psi}) = (\frac{a}{2} + i\omega_0)^2 (\frac{a}{2} - i\omega_0)^2 \int_0^1 \sin^4(\pi x) dx = \frac{3}{8}$$

$$(C\vec{\varphi}, \vec{\psi}) = (\frac{a^2}{4} + \omega_0^2) \int_0^1 \sin^2(\pi x) dx = \frac{1}{2}$$

Then, the condition of solvability will be:

$$-i\omega_2 \alpha_1 - \frac{9\alpha_1^3}{8} + \frac{\alpha_1}{2} = 0$$

By splitting this expression into real and imaginary parts, we get:

$$\alpha_1 = \frac{2}{3}, \omega_2 = 0$$

From here we conclude that periodic mode will be in the system when δ is positive.

Now, Let us solve (63). We will be looking for solution in the form:

$$\vec{\phi}_3 = {\alpha_1}^3 \vec{\phi}_{33} e^{3i\tau} + \alpha_1 \vec{\phi}_{31} e^{i\tau} + \text{c.j.} \tag{66}$$

If we substitute (66) into (63) and group the terms, we will arrive at two equations:

$$(\mathrm{A} - i\omega_0 \mathrm{I})\vec{\phi}_{31} = -\mathrm{C}\vec{\varphi} + 3\alpha_1^2 K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi^*}) \tag{67}$$

$$(\mathrm{A} - (3i\omega_0)\mathrm{I})\vec{\phi}_{33} = K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi}) \tag{68}$$

Let us solve (67). The right side of the equation could be written as:

$$-\mathrm{C}\vec{\varphi} + 3\alpha_1^2 K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi^*}) = -\frac{1}{3}\begin{pmatrix} 0 \\ \frac{a}{2} + i\omega_0 \end{pmatrix}\sin(3\pi x)$$

Therefore, we will be looking for solution in the form:

$$\vec{\phi}_{31} = \begin{pmatrix} M \\ N \end{pmatrix}\sin(3\pi x)$$

By using derived formulas for coefficients, we get:

$$\vec{\phi}_{31} = -\frac{1}{3|F_1^3|^2}\begin{pmatrix} (F_1^3)^*(\frac{a}{2} + i\omega_0) \\ (R_1^3 + iI_1^3)(\frac{a}{2} + i\omega_0) \end{pmatrix}\sin(3\pi x) \tag{69}$$

Now, we will solve (68). The right side of the equation can be transformed to:

$$K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi}) = \frac{3}{4}\begin{pmatrix} 0 \\ \frac{a}{2} + i\omega_0 \end{pmatrix}\sin(\pi x) - \frac{1}{4}\begin{pmatrix} 0 \\ \frac{a}{2} + i\omega_0 \end{pmatrix}\sin(3\pi x)$$

We will be looking for solution in the form:

$$\vec{\phi}_{33} = \begin{pmatrix} M \\ N \end{pmatrix}\sin(\pi x) + \begin{pmatrix} P \\ Q \end{pmatrix}\sin(3\pi x)$$

Then, we get:

$$\vec{\phi}_{33} = \frac{3(\frac{a}{2} + i\omega_0)^3}{4|F_3^1|^2}\begin{pmatrix} (F_3^1)^* \\ (R_3^1 + iI_3^1) \end{pmatrix}\sin(\pi x) - \\ -\frac{(\frac{a}{2} + i\omega_0)^3}{4|F_3^3|^2}\begin{pmatrix} (F_3^3)^* \\ (R_3^3 + iI_3^3) \end{pmatrix}\sin(3\pi x) \tag{70}$$

Finally, the solution of (63) is:

$$\vec{\phi}_3 = \alpha_3 e^{i\tau}\vec{\varphi} + \alpha_1\vec{\phi}_{31}e^{i\tau} + {\alpha_1}^3\vec{\phi}_{33}e^{3i\tau} + \text{c.j.}$$

Let us satisfy the condition of solvability for equation (64).

$$\int_0^{2\pi}(-\omega_3\dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_2 + 3K(\vec{\phi}_1,\vec{\phi}_1,\vec{\phi}_2),\vec{\psi})e^{-i\tau}d\tau = 0 \tag{71}$$

If we transform integral (71), we will get:

$$\frac{\alpha_2}{2} - \frac{3\alpha_2}{2} - \frac{2i\omega_3}{3} = 0$$

From here we conclude that:

$$\alpha_2 = 0, \quad \omega_3 = 0$$

Therefore, the solution of (64) is:

$$\vec{\phi}_4 = \alpha_4 e^{i\tau}\vec{\varphi} + \text{c.j.}$$

Now, Let us satisfy the condition of solvability for (65):

$$\int_0^{2\pi}(-\omega_4\dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_3 - 3K(\vec{\phi}_1,\vec{\phi}_1,\vec{\phi}_3),\vec{\psi})e^{-i\tau}d\tau = 0$$

After performing the computations, we will get:

$$-i\alpha_1\omega_4(\vec{\varphi},\vec{\psi}) + \alpha_3(C\vec{\varphi},\vec{\psi}) + \alpha_1(C\vec{\phi}_{31},\vec{\psi}) - 9\alpha_1^2\alpha_3(K(\vec{\varphi},\vec{\varphi},\vec{\varphi}^*),\vec{\psi}) -$$
$$-6\alpha_1^3(K(\vec{\phi}_{31},\vec{\varphi},\vec{\varphi}^*),\vec{\psi}) - 3\alpha_1^3(K(\vec{\phi}_{31}^*,\vec{\varphi},\vec{\varphi}),\vec{\psi}) -$$
$$-3\alpha_1^5(K(\vec{\phi}_{33},\vec{\varphi}^*,\vec{\varphi}^*),\vec{\psi}) = 0$$

Let us transform this equation. We will note that:

$$(C\vec{\phi}_{31},\vec{\psi}) = 0, \quad \text{because} \int_0^1 \sin(\pi x)\sin(3\pi x)dx = 0$$

Then, we get:

$$-i\alpha_1\omega_4 + \tfrac{1}{2}\alpha_3 - \alpha_3\tfrac{3}{2} = 6\alpha_1^3(K(\vec{\phi}_{31}, \vec{\varphi}, \vec{\varphi}^*), \vec{\psi}) + 3\alpha_1^3(K(\vec{\phi}^*_{31}, \vec{\varphi}, \vec{\varphi}), \vec{\psi}) +$$
$$+3\alpha_1^5(K(\vec{\phi}_{33}, \vec{\varphi}^*, \vec{\varphi}^*), \vec{\psi})$$

After performing the computations, we arrive at:

$$-i\alpha_1\omega_4 - \alpha_3 = \frac{2}{27}\frac{R_1^3 + iI_1^3}{|F_1^3|^2} + \frac{1}{27}\frac{R_1^3 - iI_1^3}{|F_3^1|^2} + \frac{1}{9}\frac{R_3^1 + iI_3^1}{|F_3^1|^2} + \frac{1}{81}\frac{R_3^3 + iI_3^3}{|F_3^3|^2}$$

Therefore:

$$\begin{cases} \alpha_3 = -\frac{3}{27}\frac{R_1^3}{|F_1^3|^2} - \frac{1}{9}\frac{R_3^1}{|F_3^1|^2} - \frac{1}{81}\frac{R_3^3}{|F_3^3|^2} \\ \omega_4 = -\frac{3}{2}\left(\frac{1}{27}\frac{I_1^3}{|F_1^3|^2} - \frac{1}{9}\frac{I_3^1}{|F_3^1|^2} - \frac{1}{81}\frac{I_3^3}{|F_3^3|^2}\right) \end{cases} \quad (72)$$

Finally, the expression for two first terms in asymptotic is:

$$\begin{cases} \vec{\phi} = \varepsilon\frac{2}{3}(e^{i\tau}\vec{\varphi}) + \varepsilon^3(\alpha_3 e^{i\tau}\vec{\varphi} + \alpha_1\vec{\phi}_{31}e^{i\tau} + {\alpha_1}^3\vec{\phi}_{33}e^{3i\tau}) + \text{c.j.} + O(\varepsilon^4) \\ \omega = \sqrt{1 - \nu^2\pi^4} + \varepsilon^4\omega_4 + O(\varepsilon^5) \end{cases} \quad (73)$$

Here, $\vec{\phi}_{31}$, $\vec{\phi}_{33}$, α_3, ω_4 are given by formulas (69), (70), (72) respectively.

On the figures below the visualisation of asymptotic is presented.

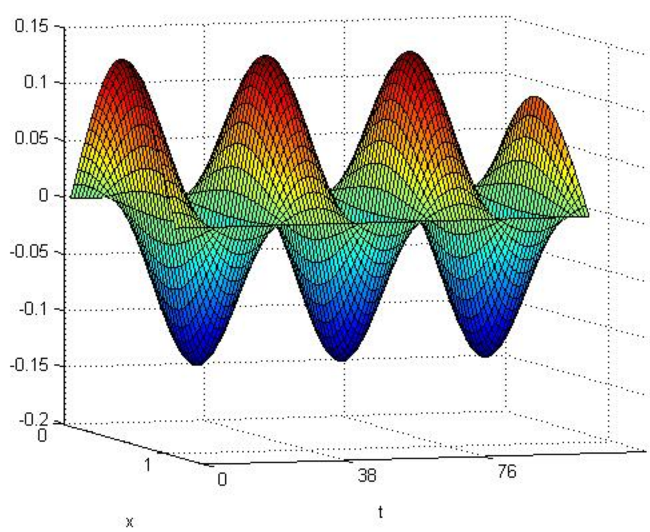

Figure 5: The asymptotic of u(x,t) when $\nu = 0.1$ and $\mu = a + 0.01$

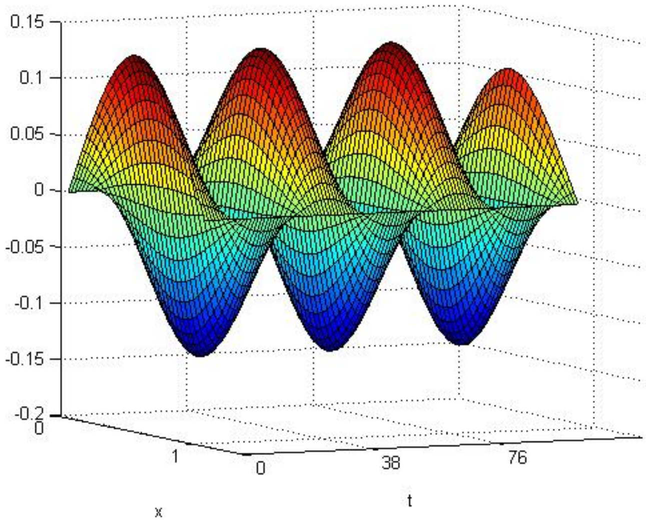

Figure 6: The asymptotic of v(x,t) when $\nu = 0.1$ and $\mu = a + 0.01$

On the next figures we plot numerical solution. Initial conditions were taken from asymptotic.

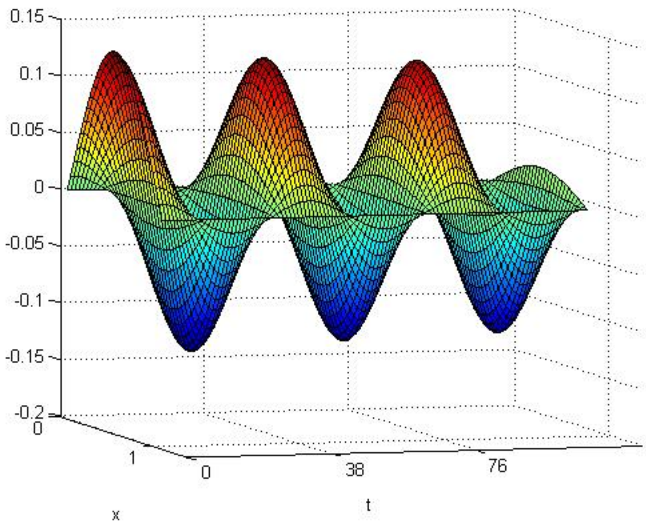

Figure 7: Numerical solution u(x,t) when $\nu = 0.1$ and $\mu = a + 0.01$

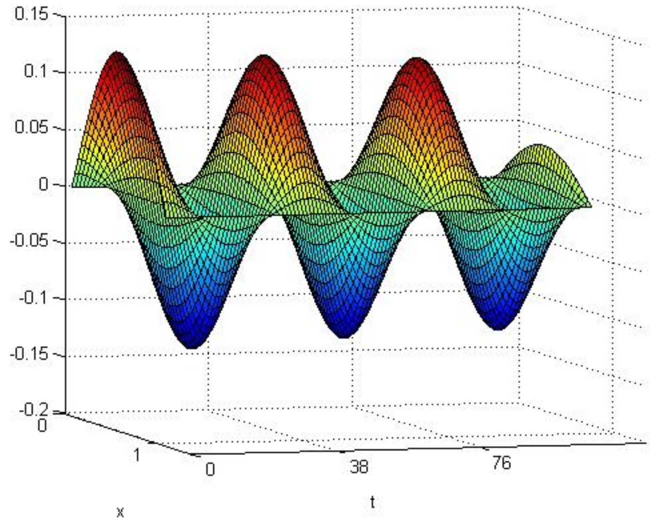

Figure 8: Numerical solution v(x,t) when $\nu = 0.1$ and $\mu = a + 0.01$

3.3.6 Formulas for the consecutive terms of asymptotic

When $n \geq 3$, expression around ε^n can be written in the form:

$$\omega_0 \dot{\vec{\phi}}_n = A\vec{\phi}_n + C\vec{\phi}_{n-2} - \sum_{i=1}^{n-1} \omega_{n-i}\dot{\vec{\phi}}_i - \sum_{\substack{i_1+i_2+i_3=n \\ i_1 \leq i_2 \leq i_3}} 3K(\vec{\phi}_{i_1}, \vec{\phi}_{i_2}, \vec{\phi}_{i_3}) \qquad (74)$$

In this section we will derive formulas for higher terms of the asymptotic. Again, we will prove a simple statement.

Statement 2. For any $k \in \mathbb{N}$ $\quad \vec{\phi}_{2k} = 0, \quad \alpha_{2k} = 0, \quad \omega_{2k-1} = 0$

Proof. We will use mathematical induction. From computed first terms of the expansion we can conclude that:

$$\alpha_2 = 0, \alpha_4 = 0$$
$$\omega_1 = 0, \omega_3 = 0$$
$$\vec{\phi}_2 = 0, \vec{\phi}_4 = 0$$

Let us assume that for some even number p holds:

$$\alpha_k = 0, k = 2, 4, \ldots, p-4$$
$$\omega_k = 0, k = 1, 3, \ldots, p-3$$
$$\vec{\phi}_k = 0, k = 2, 4, \ldots, p-4$$

We will prove that this holds for $p+1$ as well. To do this, Let us first consider the equation for $p-2$ term. By incorporating our assumptions we conclude that:

$$\omega_0 \dot{\vec{\phi}}_{p-2} = A\vec{\phi}_{p-2}$$

From here we conclude that $\vec{\phi}_{p-2} = \alpha_{p-2}\vec{\varphi}e^{i\tau} + $ c.j.

Now, Let us consider the equation around p term:

$$\omega_0 \dot{\vec{\phi}}_p = A\vec{\phi}_p + C\vec{\phi}_{p-2} - \sum_{i=1}^{p-1} \omega_{p-i}\dot{\vec{\phi}}_i - \sum_{\substack{i_1+i_2+i_3=p \\ i_1 \le i_2 \le i_3}} 3K(\vec{\phi}_{i_1}, \vec{\phi}_{i_2}, \vec{\phi}_{i_3})$$

From the fact that p is even number, and that even number cannot be the sum of three odd numbers, and incorporating the assumptions, we conclude that:

$$\sum_{\substack{i_1+i_2+i_3=p \\ i_1 \le i_2 \le i_3}} 3K(\vec{\phi}_{i_1}, \vec{\phi}_{i_2}, \vec{\phi}_{i_3}) = 3K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_{p-2})$$

Therefore:

$$\sum_{i=1}^{p-1} \omega_{p-i}\dot{\vec{\phi}}_i = \omega_{p-1}\dot{\vec{\phi}}_1$$

Then, our equation could be presented as:

$$\omega_0 \dot{\vec{\phi}}_p = A\vec{\phi}_p + C\vec{\phi}_{p-2} - 3K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_{p-2}) - \omega_{p-1}\dot{\vec{\phi}}_1$$

The condition of solvability for this equation could be written as:

$$\alpha_{p-2}(C\vec{\varphi}, \vec{\psi}) - 9\alpha_1^2\alpha_{p-2}(K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi}^*, \vec{\psi}) = i\omega_{p-1}(\vec{\varphi}, \vec{\psi})$$

After computing inner products, we will get:

$$-\frac{1}{2}\alpha_{p-2} = i\omega_{p-1}$$

From here we conclude that $\omega_{p-1} = 0$ and $\alpha_{p-2} = 0$, then $\vec{\phi}_{p-2} = 0$. The statement is proved.

We have obtained the formulas for even terms. Now, Let us get the expressions for odd terms.

For odd n equation (74) could be written as:

$$\omega_0 \dot{\vec{\phi}}_n = A\vec{\phi}_n + C\vec{\phi}_{n-2} - i\omega_{n-1}\dot{\vec{\phi}}_1 - \sum_{\substack{i_1+i_2+i_3=n \\ i_1 \le i_2 \le i_3}}^{i_1+i_2+i_3=2k+1} 3K(\vec{\phi}_{i_1}, \vec{\phi}_{i_2}, \vec{\phi}_{i_3}) \tag{75}$$

We will note that:

$$C\vec{\phi}_{n-2} = \alpha_{n-2}C\vec{\varphi}e^{i\tau} + \alpha_{n-2}C\vec{\varphi}^{*}e^{-i\tau} + C(\vec{\phi}^{\text{p}}_{n-2})$$

Here as $\vec{\phi}^{\text{p}}_{n-2}$ is defined the partial solution of inhomogeneous equation.

Now, Let us transform the sum:

$$\sum_{\substack{i_1+i_2+i_3=n \\ i_1 \le i_2 \le i_3}}^{i_1+i_2+i_3=2k+1} 3K(\vec{\phi}_{i_1}, \vec{\phi}_{i_2}, \vec{\phi}_{i_3}) = 3K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_{p-2}) + \\ + \sum_{\substack{i_1+i_2+i_3=n \\ i_1 \le i_2 \le i_3}}^{(i_1,i_2,i_3)\neq(1,1,p-2)} 3K(\vec{\phi}_{i_1}, \vec{\phi}_{i_2}, \vec{\phi}_{i_3})$$

It should be pointed out that:

$$K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_{p-2}) = K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}^{\text{hom}}_{p-2}) + K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}^{\text{p}}_{p-2})$$

We will introduce new notation:

$$f_n = C(\vec{\phi}^{\text{p}}_{n-2}) - 3K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}^{\text{p}}_{p-2}) - \sum_{\substack{i_1+i_2+i_3=n \\ i_1 \le i_2 \le i_3}}^{(i_1,i_2,i_3)\neq(1,1,p-2)} 3K(\vec{\phi}_{i_1}, \vec{\phi}_{i_2}, \vec{\phi}_{i_3}) \tag{76}$$

Then, the condition of solvability could be written as:

$$\int_0^{2\pi} ((\alpha_{n-2} C\vec{\varphi} e^{i\tau} + \alpha_{n-2} C\vec{\varphi}^* e^{-i\tau}, \vec{\psi}) - i\omega_{n-1}(\dot{\vec{\phi}}_1, \vec{\psi}) d\tau -$$
$$- \int_0^{2\pi} 3(K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_{p-2}, \vec{\psi}) + (f_n, \vec{\psi})) e^{-i\tau} d\tau = 0$$

After performing the computations, we get:

$$\alpha_{n-2} + i\alpha_1 \omega_{n-1} = (f_n, \vec{\psi})$$

From here we conclude that

$$\alpha_{n-2} = Re((f_n, \vec{\psi})), \qquad \omega_{n-1} = \frac{3}{2} Im((f_n, \vec{\psi}))$$

Now we will derive formulas for the solution of inhomogeneous equation (75). This equation could be written in the form:

$$\begin{aligned}
\omega_0 \dot{\vec{\phi}}_n = A\vec{\phi}_n - e^{i\tau}(& \begin{pmatrix} 0 \\ (a_{13} + ib_{13})(\frac{a}{2} + i\omega_0) \end{pmatrix} \sin(3\pi x) + \dots) - \\
-e^{3i\tau}(& \begin{pmatrix} 0 \\ (a_{31} + ib_{31})(\frac{a}{2} + i\omega_0)^3 \end{pmatrix} \sin(\pi x) + \dots) - \dots \\
-e^{in\tau}(& \begin{pmatrix} 0 \\ (a_{n1} + ib_{n1})(\frac{a}{2} + i\omega_0)^n \end{pmatrix} \sin(\pi x) + \dots)
\end{aligned} \tag{77}$$

We will be looking for the solution in the form:

$$\vec{\phi}_n = e^{i\tau}(\begin{pmatrix} M_{13} \\ N_{13} \end{pmatrix} \sin(3\pi x) + \dots) + \dots + e^{in\tau}(\begin{pmatrix} M_{n1} \\ N_{n1} \end{pmatrix} \sin(\pi x) + \dots) \tag{78}$$

If we substitute (78) into (77) and group the terms, we will arrive at a finite set of linear systems. Formulas for their solutions were derived earlier.Therefore, we can compute unknown coefficients in the solution of (78) by using these formulas.

Finally, we have got the following results:

$$\begin{cases} \vec{\phi}_{2k} = 0, \quad \alpha_{2k} = 0, \quad \omega_{2k+1} = 0 \\ \vec{\phi}_{2k+1} = e^{i\tau}(\begin{pmatrix} M_{13} \\ N_{13} \end{pmatrix} \sin(3\pi x) + \dots) + \dots + e^{in\tau}(\begin{pmatrix} M_{n1} \\ N_{n1} \end{pmatrix} \sin(\pi x) + \dots) \\ \alpha_{2k-1} = Re((f_{2k+1}, \vec{\psi})), \quad \omega_{2k} = \frac{3}{2} Im((f_{2k+1}, \vec{\psi})) \end{cases}$$

Here f_{2k+1} is given by formula (76). Formulas for coefficients M_{ij}, N_{ij} were derived earlier.

3.4 Neumann boundary conditions

Now, we will consider system (47) in the case of Neumann boundary conditions. In that case, the domain of operator A will be:

$$D(A) = \{u(x,t), v(x,t) \in W_2^2 : u_x(0,t) = u_x(1,t) = 0, v_x(0,t) = v_x(1,t) = 0\}$$

In $D(A)$ the system $\{\cos(\pi kx)\}_{k=0}^{\infty}$ is the basis.

3.4.1 Eigenvalues of operator A

We will consider an eigenvalue problem:

$$\mathrm{A}\vec{\phi} = \lambda\vec{\phi} \tag{79}$$

We can write decomposition $\vec{\phi}$ in the form:

$$\vec{\phi} = \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \cos(\pi kx) \tag{80}$$

If we substitute (80) into (79) and group the terms around basis functions, we will get:

$$\begin{pmatrix} -\nu(\pi k)^2 & 1 \\ -1 & \mu - \nu(\pi k)^2 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \lambda \begin{pmatrix} a_k \\ b_k \end{pmatrix} \qquad k = 0, 1, 2, \dots \tag{81}$$

The determinant of matrix in the left side could be written as:

$$\varphi_{A_k}(\lambda) = \lambda^2 + (2\nu(\pi k)^2 - \mu)\lambda + \nu^2(\pi k)^4 - \mu\nu(\pi k)^2 + 1$$

We can find eigenvalues of operator A as the roots of $\varphi_{A_k}(\lambda)$:

$$\lambda_{1,2}^k = \frac{-(2\nu(\pi k)^2 - \mu) \pm \sqrt{\mu^2 - 4}}{2}$$

Now, we can find the critical value of parameter μ. Stability conditions has the form:

$$\begin{cases} \mu - 2\nu(\pi k)^2 < 0 \\ \nu^2(\pi k)^4 - \mu\nu(\pi k)^2 + 1 > 0 \end{cases} \qquad k = 0, 1, 2, \ldots$$

From here we can conclude that when $\mu < 0$ $\lambda_{1,2}^k \in C^- \quad \forall k$.

Therefore, when $\mu = 0$, operator A has a pair or conjugated eigenvalues $\lambda_{1,2} = \pm i$. When $\mu > 0$ $Re(\lambda_{1,2}) > 0$.

3.4.2 Eigenvectors of the operator A

Let us consider an eigenvector problem:

$$\mathrm{A}\vec{\phi} = \lambda_{1,2}\vec{\phi}$$

Here $\mu = 0$, $\lambda_{1,2} = \pm i$.

Eigenvectors could be obtained from the system:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm i \begin{pmatrix} a \\ b \end{pmatrix}$$

This system could be rewritten as:

$$\begin{cases} \mp ia + b = 0 \\ -a + \mp ib = 0 \end{cases}$$

If we take $a = 1$ and $b = \pm i$, we arrive at the expressions for eigenvectors:

$$\vec{\phi}_{1,2} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

3.4.3 Eigenvectors of the conjugated operator A^*

We need to find eigenvectors of the conjugate operator A^*, corresponding to $\lambda_{1,2}$ when $\mu = 0$. They could be found from the system:

$$\begin{pmatrix} \mp i & -1 \\ 1 & \pm i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From here we can find eigenvectors:

$$\vec{\psi}_{1,2} = \begin{pmatrix} 1 \\ \mp i \end{pmatrix}$$

3.4.4 The analysis of nonlinear problem

We will consider equation (47) in the case of Neumann boundary conditions. In that case, critical value of the parameter μ is $\mu_{cr} = 0$.

Let us introduce new notation: $\mu = 0 + \delta$, $\delta \ll 1$. Then, equation (47) can be rewritten as:

$$\dot{\vec{\phi}} = \nu \vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + \delta \mathrm{C}\vec{\phi} - K(\vec{\phi}, \vec{\phi}, \vec{\phi}) \tag{82}$$

We will introduce new variable change: $\tau = \omega t, \delta = \varepsilon^2$. We will define as dot differentiating by τ. Then (82) could be presented in the form:

$$\omega \dot{\vec{\phi}} = \nu \vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + \varepsilon^2 \mathrm{C}\vec{\phi} - K(\vec{\phi}, \vec{\phi}, \vec{\phi}) \tag{83}$$

We will be looking for $\vec{\phi}$ and ω in the form of series:

$$\vec{\phi} = \sum_{i=1}^{\infty} \varepsilon^i \vec{\phi}_i, \qquad \omega = \sum_{i=0}^{\infty} \varepsilon^i \omega_i \tag{84}$$

It is evident that $\omega_0 = 1$. By substituting (84) into (83) equating the coefficients of like powers of ε, we will arrive at the set of equations:

$$\varepsilon^1: \quad \omega_0 \dot{\vec{\phi}}_1 = \nu \frac{\partial^2 \vec{\phi}_1}{\partial x^2} + \mathrm{B}\vec{\phi}_1 + a\mathrm{C}\vec{\phi}_1 \tag{85}$$

$$\varepsilon^2: \quad \omega_0 \dot{\vec{\phi}}_2 = \nu \frac{\partial^2 \vec{\phi}_2}{\partial x^2} + \mathrm{B}\vec{\phi}_2 + a\mathrm{C}\vec{\phi}_2 - \omega_1 \dot{\vec{\phi}}_1 \tag{86}$$

$$\varepsilon^3: \quad \omega_0 \dot{\vec{\phi}}_3 = \nu \frac{\partial^2 \vec{\phi}_3}{\partial x^2} + \mathrm{B}\vec{\phi}_3 + a\mathrm{C}\vec{\phi}_3 - \omega_1 \dot{\vec{\phi}}_2 - \omega_2 \dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_1 - K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_1) \tag{87}$$

$$\varepsilon^4: \quad \omega_0 \dot{\vec{\phi}}_4 = \nu \frac{\partial^2 \vec{\phi}_4}{\partial x^2} + \mathrm{B}\vec{\phi}_4 + a\mathrm{C}\vec{\phi}_4 - \omega_1 \dot{\vec{\phi}}_3 - \omega_2 \dot{\vec{\phi}}_2 - \omega_3 \dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_2 - \\ -3K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_2) \tag{88}$$

$$\varepsilon^5: \quad \omega_0 \dot{\vec{\phi}}_5 = \nu \frac{\partial^2 \vec{\phi}_5}{\partial x^2} + \mathrm{B}\vec{\phi}_5 + a\mathrm{C}\vec{\phi}_5 - \omega_1 \dot{\vec{\phi}}_4 - \omega_2 \dot{\vec{\phi}}_3 - \omega_3 \dot{\vec{\phi}}_2 - \omega_4 \dot{\vec{\phi}}_1 + \\ +\mathrm{C}\vec{\phi}_3 - 3K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_3) - 3K(\vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_2) \tag{89}$$

It should be pointed out that in the expressions for eigenvectors there is no dependence on x. Then, periodic solution of the system under study will not depend on x as well. It can be easily concluded from the equations above that the solutions of these equations will be exactly the same as the solutions, obtained in the case of ODE. It means that the periodic solution, found for ODE will be the solution of spatially distributed system as well, however in this particular case of boundary conditions. Therefore, in the case of Neumann boundary conditions, a spatially homogeneous stable periodic mode exists in the system. First elements of its asymptotic approximation are given by the following expression:

$$\begin{cases} \vec{\phi} = \frac{2}{\sqrt{3}}\varepsilon \begin{pmatrix} \cos(\omega t) \\ -\omega\sin(\omega t) \end{pmatrix} + \frac{1}{12\sqrt{3}}\varepsilon^3 \begin{pmatrix} \sin(3\omega t) \\ 3\omega\cos(3\omega t) \end{pmatrix} + O(\varepsilon^4) \\ \omega = 1 - \frac{9}{48\sqrt{3}}\varepsilon^4 + O(\varepsilon^5) \end{cases}$$

On the figures below the visualisation of asymptotic is presented.

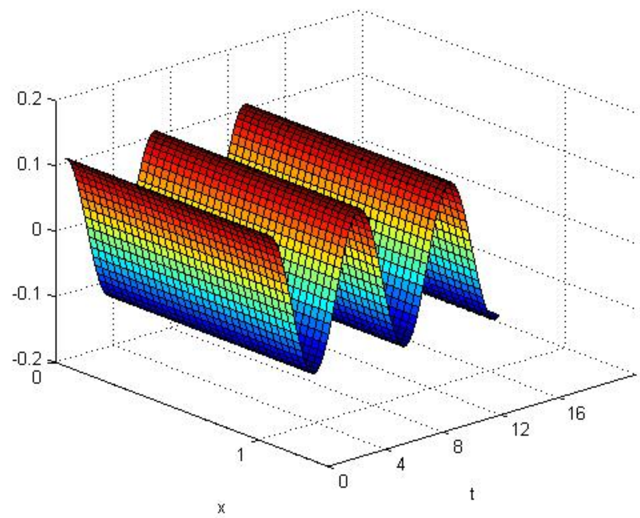

Figure 9: The asymptotic of u(x,t) when $\nu = 0.1$ and $\mu = 0.01$

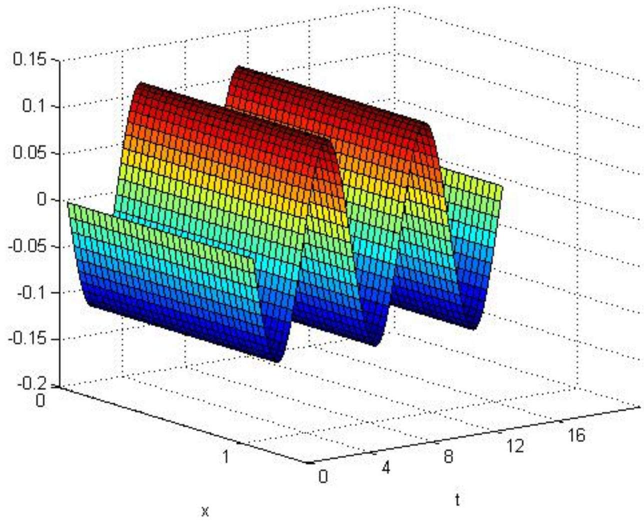

Figure 10: The asymptotic of v(x,t) when $\nu = 0.1$ and $\mu = 0.01$

On the next figures we plot numerical solution. Initial conditions were taken from asymptotic.

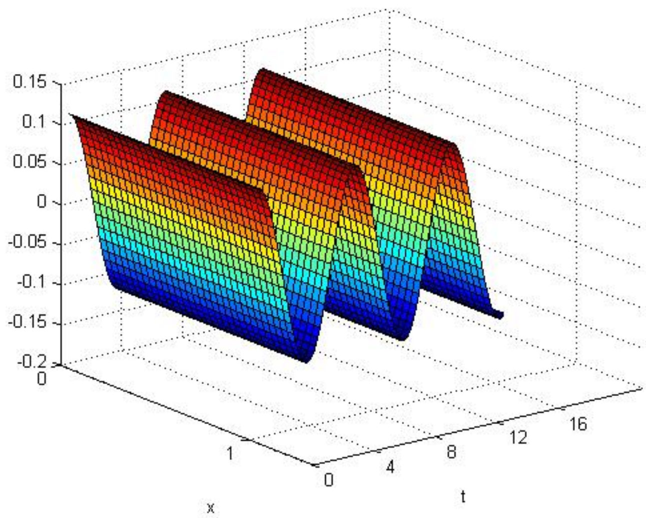

Figure 11: Numerical solution u(x,t) when $\nu = 0.1$ and $\mu = 0.01$

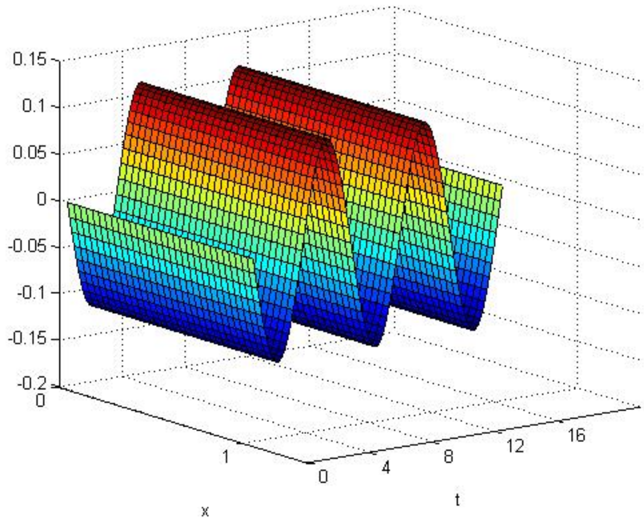

Figure 12: Numerical solution v(x,t) when $\nu = 0.1$ and $\mu = 0.01$

3.5 Neumann boundary conditions with additional requirement of zero average

In this section we will again consider equation (47) in the case of Neumann boundary conditions However, we will also add additional requirement of zero average.

The domain of A will be:

$$D(A) = \{(u, v) : u, v \in W_2^2, u_x(0, t) = u_x(1, t) = 0, v_x(0, t) = v_x(1, t) = 0, \int_0^1 u dx = 0, \int_0^1 v dx = 0\}$$

In $D(A)$ the system $\{\cos(\pi kx)\}_{k=1}^{\infty}$ is the basis.

3.5.1 Eigenvalues of operator A

The process of finding the eigenvalues is almost the same as was performed in two previous cases. Because of that we will write only the expressions for the eigenvalues of operator A:

$$\lambda_{1,2}^k = \frac{-(2\nu(\pi k)^2 - \mu) \pm \sqrt{\mu^2 - 4}}{2}, \qquad k = 1, 2, \ldots$$

The critical value of parameter μ will be:

$$\mu_{\text{cr}} = \left[\begin{array}{c} \frac{1}{\nu\pi^2} + \nu\pi^2, \quad when \quad \nu > \frac{1}{\pi^2} \\ 2\nu\pi^2, \quad when \quad \nu < \frac{1}{\pi^2} \end{array} \right.$$

When $\nu > \frac{1}{\pi^2}$ monotonic instability take place in the system. When $\nu < \frac{1}{\pi^2}$ - oscillatory instability.

3.5.2 Eigenvectors of the operator A in the case of oscillatory instability

After performing simple computations, we can find that the expressions for eigenvectors are:

$$\vec{\phi}_{1,2} = \begin{pmatrix} 1 \\ \nu\pi^2 \pm i\omega_0 \end{pmatrix} \cos(\pi x)$$

The eigenvectors of conjugate operator will be:

$$\vec{\psi}_{1,2} = \begin{pmatrix} 1 \\ \nu\pi^2 \mp i\omega_0 \end{pmatrix} \cos(\pi x)$$

3.5.3 Preliminary actions

As in previous cases, in the process of finding the asymptotic approximation we would deal with the following equations:

$$(A - ik_1\omega_0 I)\vec{v} = \begin{pmatrix} 0 \\ (\gamma + i\delta) \end{pmatrix} \cos(\pi k_2 x), \qquad k_1 \in \mathbb{Z}, k_2 \in \mathbb{N} \tag{90}$$

Again, we will be looking for the solution in the form:

$$\vec{v} = \begin{pmatrix} M \\ N \end{pmatrix} \cos(\pi k_2 x) \tag{91}$$

When we substitute (91) into (90), we will get:

$$\left[B + aC - (\frac{ak_2{}^2}{2} + ik_1\omega_0)I \right] \vec{v} = \begin{pmatrix} 0 \\ \gamma + i\delta \end{pmatrix} \cos(\pi k_2 x)$$

If we group the terms around basis functions, we will arrive at linear system:

$$\begin{pmatrix} -(\frac{ak_2{}^2}{2} + ik_1\omega_0) & 1 \\ -1 & a - (\frac{ak_2{}^2}{2} + ik_1\omega_0) \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma + i\delta \end{pmatrix}$$

The solution of the system could be written as:

$$\begin{cases} M = (\gamma + i\delta)/((a - (\frac{ak_2^2}{2} + ik_1\omega_0))(\frac{ak_2^2}{2} + ik_1\omega_0) - 1) \\ N = (\frac{ak_2^2}{2} + ik_1\omega_0)M \end{cases}$$

Let us introduce new notation:

$$\begin{cases} P_{k1}^{k2} = \frac{ak_2^2}{2} + ik_1\omega_0 \\ F_{k1}^{k2} = (a - P_{k1}^{k2})P_{k1}^{k2} - 1 \\ R_{k1}^{k2} = Re(P_{k1}^{k2})Re(F_{k1}^{k2}) - Im(P_{k1}^{k2})Im(F_{k1}^{k2}) \\ I_{k1}^{k2} = Re(P_{k1}^{k2})Im(F_{k1}^{k2}) - Im(P_{k1}^{k2})Re(F_{k1}^{k2}) \end{cases}$$

Then, the coefficients could be written as:

$$M = \frac{(\gamma + i\delta)(F_{k1}^{k2})^*}{|F_{k1}^{k2}|^2}, \quad N = \frac{(\gamma + i\delta)(R_{k1}^{k2} + iI_{k1}^{k2})}{|F_{k1}^{k2}|^2}$$

Then, the solution of (90) will be:

$$\vec{v} = \begin{pmatrix} M \\ N \end{pmatrix} \cos(\pi k_2 x)$$

3.5.4 Asymptotic approximation of periodic solution

We are considering equation (47) in the case of Neumann boundary conditions with additional requirement of zero average. The critical value of the parameter μ is $\mu_{\mathrm{cr}} = 2\nu\pi^2$. By performing similar actions as before, we arrive at the equation:

$$\omega\dot{\vec{\phi}} = \nu\vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + a\mathrm{C}\vec{\phi} + \varepsilon^2\mathrm{C}\vec{\phi} - K(\vec{\phi},\vec{\phi},\vec{\phi}) \tag{92}$$

We will be looking for periodic solution in the form of series:

$$\vec{\phi} = \sum_{i=1}^{\infty}\varepsilon^i\vec{\phi}_i, \qquad \omega = \sum_{i=0}^{\infty}\varepsilon^i\omega_i \tag{93}$$

Here $\omega_0 = \sqrt{1-\nu^2\pi^4}$

By substituting (93) into (92) and equating the coefficients of like powers of ε, we will get:

$$\varepsilon^1: \quad \omega_0\dot{\vec{\phi}}_1 = \nu\frac{\partial^2\vec{\phi}_1}{\partial x^2} + \mathrm{B}\vec{\phi}_1 + a\mathrm{C}\vec{\phi}_1 \tag{94}$$

$$\varepsilon^2: \quad \omega_0\dot{\vec{\phi}}_2 = \nu\frac{\partial^2\vec{\phi}_2}{\partial x^2} + \mathrm{B}\vec{\phi}_2 + a\mathrm{C}\vec{\phi}_2 - \omega_1\dot{\vec{\phi}}_1 \tag{95}$$

$$\varepsilon^3: \quad \omega_0\dot{\vec{\phi}}_3 = \nu\frac{\partial^2\vec{\phi}_3}{\partial x^2} + \mathrm{B}\vec{\phi}_3 + a\mathrm{C}\vec{\phi}_3 - \omega_1\dot{\vec{\phi}}_2 - \omega_2\dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_1 - K(\vec{\phi}_1,\vec{\phi}_1,\vec{\phi}_1) \tag{96}$$

$$\begin{aligned}\varepsilon^4: \quad \omega_0\dot{\vec{\phi}}_4 = \nu\frac{\partial^2\vec{\phi}_4}{\partial x^2} + \mathrm{B}\vec{\phi}_4 + a\mathrm{C}\vec{\phi}_4 - \omega_1\dot{\vec{\phi}}_3 - \omega_2\dot{\vec{\phi}}_2 - \omega_3\dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_2 - \\ -3K(\vec{\phi}_1,\vec{\phi}_1,\vec{\phi}_2)\end{aligned} \tag{97}$$

$$\begin{aligned}\varepsilon^5: \quad \omega_0\dot{\vec{\phi}}_5 = \nu\frac{\partial^2\vec{\phi}_5}{\partial x^2} + \mathrm{B}\vec{\phi}_5 + a\mathrm{C}\vec{\phi}_5 - \omega_1\dot{\vec{\phi}}_4 - \omega_2\dot{\vec{\phi}}_3 - \omega_3\dot{\vec{\phi}}_2 - \omega_4\dot{\vec{\phi}}_1 + \\ +\mathrm{C}\vec{\phi}_3 - 3K(\vec{\phi}_1,\vec{\phi}_1,\vec{\phi}_3) - 3K(\vec{\phi}_1,\vec{\phi}_2,\vec{\phi}_2)\end{aligned} \tag{98}$$

Let us introduce new notation: $\vec{\varphi} = \vec{\phi}_1$, $\vec{\psi} = \vec{\psi}_2$.

Let us compute $(\vec{\varphi},\vec{\psi})$:

$$(\vec{\varphi},\vec{\psi}) = \int_0^1 (1 + (\frac{a}{2} + i\omega_0)(\frac{a}{2} - i\omega_0))\cos^2(\pi x)dx = 2\int_0^1 \cos^2(\pi x)dx = 1$$

Let us start solving the equations.

The periodic solution of (94) will be:

$$\vec{\phi}_1 = \alpha_1\vec{\varphi}e^{i\tau} + \mathrm{c.j.}$$

Inhomogeneous equation (95) has the periodic solution if and only if the condition of solvability is satisfied:

$$\int_0^{2\pi} (-\omega_1 \dot{\vec{\phi}}_1, \vec{\psi}) e^{-i\tau} d\tau = 0$$

After performing the computations we get:

$$\int_0^{2\pi} (-\omega_1 \dot{\vec{\phi}}_1, \vec{\psi}) e^{-i\tau} d\tau = -2\pi i \alpha_1 \omega_1$$

From $\alpha_1 > 0$, we conclude that $\omega_1 = 0$.

Then, the solution of (95) will be:

$$\vec{\phi}_2 = \alpha_2 \vec{\varphi} e^{i\tau} + \text{c.j.}$$

Now, we will satisfy the condition of solvability for equation (96):

$$\int_0^{2\pi} (-\omega_2 \dot{\vec{\phi}}_1 + K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_1) + \mathrm{C}\vec{\phi}_1, \vec{\psi}) e^{-i\tau} d\tau = 0$$

After the computations we arrive at:

$$\int_0^{2\pi} (-\omega_2 \dot{\vec{\phi}}_1 - K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_1) + \mathrm{C}\vec{\phi}_1, \vec{\psi}) e^{-i\tau} d\tau =$$
$$= -i\omega_2 \alpha_1 (\vec{\varphi}, \vec{\psi}) - 3\alpha_1^3 (K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi}^*), \vec{\psi}) + \alpha_1 (\mathrm{C}\vec{\varphi}, \vec{\psi})$$

When we compute inner products, we get:

$$(K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi}^*), \vec{\psi}) = (\frac{a}{2} + i\omega_0)^2 (\frac{a}{2} - i\omega_0)^2 \int_0^1 \cos^4(\pi x) dx = \frac{3}{8}$$

$$(C\vec{\varphi}, \vec{\psi}) = (\frac{a^2}{4} + \omega_0^2) \int_0^1 \cos^2(\pi x) dx = \frac{1}{2}$$

And then we arrive at the equation:

$$-i\omega_2 \alpha_1 - \frac{9\alpha_1^3}{8} + \frac{\alpha_1}{2} = 0$$

From here we can find the solution: $\alpha_1 = \frac{2}{3}, \omega_2 = 0$. Therefore, the periodic mode will be presented in the system when δ is positive.

Let us solve the equation (96). We will be looking for solution in the form:

$$\vec{\phi}_3 = {\alpha_1}^3 \vec{\phi}_{33} e^{3i\tau} + \alpha_1 \vec{\phi}_{31} e^{i\tau} + \text{c.j.} \tag{99}$$

By substituting (99) into (96) we will arrive at two equations:

$$(\mathrm{A} - i\omega_0 \mathrm{I})\vec{\phi}_{31} = -\mathrm{C}\vec{\varphi} + 3\alpha_1^2 K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi^*}) \tag{100}$$

$$(\mathrm{A} - (3i\omega_0)\mathrm{I})\vec{\phi}_{33} = K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi}) \tag{101}$$

Let us solve (100). The right side of the equation could be rewritten as:

$$-\mathrm{C}\vec{\varphi} + 3\alpha_1^2 K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi^*}) = \frac{1}{3} \begin{pmatrix} 0 \\ \frac{a}{2} + i\omega_0 \end{pmatrix} \cos(3\pi x)$$

Then, we will be looking for the solution in the form:

$$\vec{\phi}_{31} = \begin{pmatrix} M \\ N \end{pmatrix} \cos(3\pi x)$$

The solution will be:

$$\vec{\phi}_{31} = \frac{1}{3|F_1^3|^2} \begin{pmatrix} (F_1^3)^* (\frac{a}{2} + i\omega_0) \\ (R_1^3 + iI_1^3)(\frac{a}{2} + i\omega_0) \end{pmatrix} \cos(3\pi x) \tag{102}$$

Now, Let us turn to the equation (101). We can rewrite its right side as:

$$K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi}) = \frac{3}{4} \begin{pmatrix} 0 \\ \frac{a}{2} + i\omega_0 \end{pmatrix} \cos(\pi x) + \frac{1}{4} \begin{pmatrix} 0 \\ \frac{a}{2} + i\omega_0 \end{pmatrix} \cos(3\pi x)$$

We will be looking for solution in the form:

$$\vec{\phi}_{33} = \begin{pmatrix} M \\ N \end{pmatrix} \cos(\pi x) + \begin{pmatrix} P \\ Q \end{pmatrix} \cos(3\pi x)$$

Then:

$$\begin{aligned} \vec{\phi}_{33} = \tfrac{3(\frac{a}{2} + i\omega_0)^3}{4|F_3^1|^2} \begin{pmatrix} (F_3^1)^* \\ (R_3^1 + iI_3^1) \end{pmatrix} \cos(\pi x) + \\ + \tfrac{(\frac{a}{2} + i\omega_0)^3}{4|F_3^3|^2} \begin{pmatrix} (F_3^3)^* \\ (R_3^3 + iI_3^3) \end{pmatrix} \cos(3\pi x) \end{aligned} \tag{103}$$

Therefore, the solution of (96) will be:

$$\vec{\phi}_3 = \alpha_3 e^{i\tau}\vec{\varphi} + \alpha_1\vec{\phi}_{31}e^{i\tau} + {\alpha_1}^3\vec{\phi}_{33}e^{3i\tau} + \text{c.j.}$$

Let us check the condition of solvability for (97).

$$\int_0^{2\pi} (-\omega_3\dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_2 + 3K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_2), \vec{\psi})e^{-i\tau}d\tau = 0 \tag{104}$$

After computing the integral (104), we will get:

$$\frac{\alpha_2}{2} - \frac{3\alpha_2}{2} - \frac{2i\omega_3}{3} = 0$$

If we split this expression into real and imaginary part, we will find the solution:

$$\alpha_2 = 0, \quad \omega_3 = 0$$

Therefore, the solution of (97) is:

$$\vec{\phi}_4 = \alpha_4 e^{i\tau}\vec{\varphi} + \text{c.j.}$$

Next, we will satisfy the condition of solvability for equation (98):

$$\int_0^{2\pi} (-\omega_4\dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_3 - 3K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_3), \vec{\psi})e^{-i\tau}d\tau = 0$$

After performing the computations, we will get:

$$-i\omega_4(\vec{\varphi}, \vec{\psi}) + \alpha_3(C\vec{\varphi}, \vec{\psi}) + \alpha_1(C\vec{\phi}_{31}, \vec{\psi}) - 9\alpha_1^2\alpha_3(K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi}^*), \vec{\psi}) -$$
$$-6\alpha_1^3(K(\vec{\phi}_{31}, \vec{\varphi}, \vec{\varphi}^*), \vec{\psi}) - 3\alpha_1^3(K(\vec{\phi}_{31}^*, \vec{\varphi}, \vec{\varphi}), \vec{\psi}) - 3\alpha_1^5(K(\vec{\phi}_{33}, \vec{\varphi}^*, \vec{\varphi}^*), \vec{\psi}) = 0$$

Let us transform this expression. We will note that:

$$(C\vec{\phi}_{31}, \vec{\psi}) = 0, \quad \text{because} \int_0^1 \sin(\pi x)\sin(3\pi x)dx = 0$$

Then, we will get:

$$-i\alpha_1\omega_4 - \alpha_3 9\alpha_1^2\alpha_3\tfrac{3}{8} = 6\alpha_1^3(K(\vec{\phi}_{31}, \vec{\varphi}, \vec{\varphi}^*), \vec{\psi}) + 3\alpha_1^3(K(\vec{\phi}_{31}^*, \vec{\varphi}, \vec{\varphi}), \vec{\psi}) +$$
$$+3\alpha_1^5(K(\vec{\phi}_{33}, \vec{\varphi}^*, \vec{\varphi}^*), \vec{\psi})$$

After performing the computations, we will get:

$$-i\alpha_1\omega_4 - \alpha_3 = \frac{2}{27}\frac{R_1^3 + iI_1^3}{|F_1^3|^2} + \frac{1}{27}\frac{R_1^3 - iI_1^3}{|F_3^1|^2} + \frac{1}{9}\frac{R_3^1 + iI_3^1}{|F_3^1|^2} + \frac{1}{81}\frac{R_3^3 + iI_3^3}{|F_3^3|^2}$$

From here we conclude that:

$$\begin{cases} \alpha_3 = -\frac{3}{27}\frac{R_1^3}{|F_1^3|^2} - \frac{1}{9}\frac{R_3^1}{|F_3^1|^2} - \frac{1}{81}\frac{R_3^3}{|F_3^3|^2} \\ \omega_4 = -\frac{3}{2}\left(\frac{1}{27}\frac{I_1^3}{|F_1^3|^2} - \frac{1}{9}\frac{I_3^1}{|F_3^1|^2} - \frac{1}{81}\frac{I_3^3}{|F_3^3|^2}\right) \end{cases} \quad (105)$$

Finally, the first terms of asymptotic approximation will be:

$$\begin{cases} \vec{\phi} = \varepsilon\frac{2}{3}(e^{i\tau}\vec{\varphi}) + \varepsilon^3(\alpha_3 e^{i\tau}\vec{\varphi} + \alpha_1\vec{\phi}_{31}e^{i\tau} + {\alpha_1}^3\vec{\phi}_{33}e^{3i\tau}) + \text{c.j.} + O(\varepsilon^4) \\ \omega = \sqrt{1-\nu^2\pi^4} + \varepsilon^4\omega_4 + O(\varepsilon^5) \end{cases} \quad (106)$$

Here, $\vec{\phi}_{31}$, $\vec{\phi}_{33}$, α_3, ω_4 are given by formulas (102), (103), (105) respectively.

On the figures below the visualisation of asymptotic is presented.

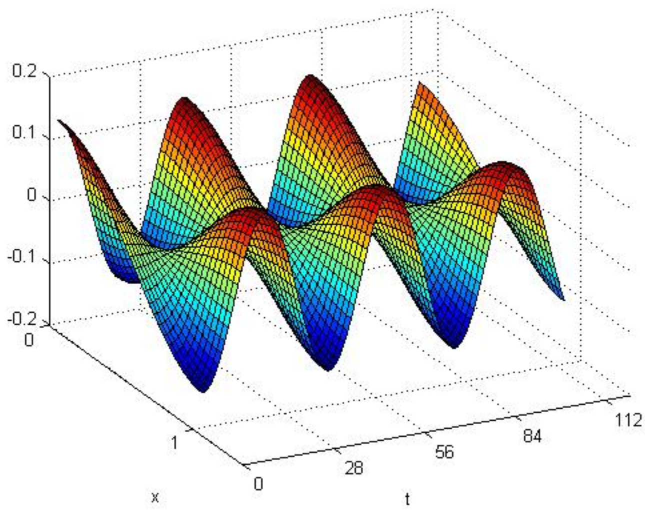

Figure 13: The asymptotic of u(x,t) when $\nu = 0.1$ and $\mu = a + 0.01$

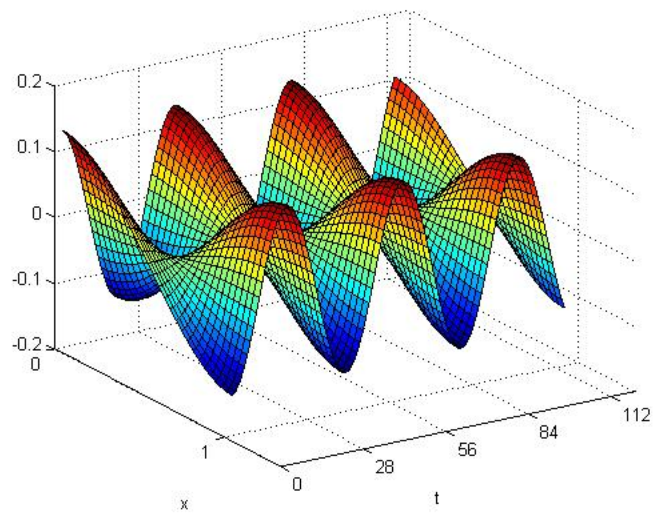

Figure 14: The asymptotic of v(x,t) when $\nu = 0.1$ and $\mu = a + 0.01$

3.5.5 Formulas for the consecutive terms of asymptotic

In order to get the expressions for the higher terms of the expression, the same actions are required, as were performed in the case of Dirichlet boundary conditions. We will not repeat them here.

4 Abstract scheme

In previous sections we have found asymptotic approximations for periodic mode in spatially distributed system (5) in the case of three different types of boundary conditions. However, it can be observed that our computations does not strongly differ in all examined situations. Therefore, we can try to generalize our results to wider range of boundary conditions. It means, in the language of functional analysis, that we need to generalize the domain $D(A)$, because boundary conditions of the system are incorporated by the selection of domain. In this section we will be again looking for asymptotic approximation of periodic mode in system (5), however we will try to generalize $D(A)$ as much as it possible.

4.1 Presenting PDE in operator form

Again, we will write system (5) in operator form

$$\dot{\vec{\phi}} = \mathrm{A}\vec{\phi} - K(\vec{\phi}, \vec{\phi}, \vec{\phi})$$

Here

$$\mathrm{A}\vec{\phi} = \nu\vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + \mu\mathrm{C}\vec{\phi} : L_2[0,1] \times L_2[0,1] \to L_2[0,1] \times L_2[0,1]$$

$$K(\vec{\phi}, \vec{\phi}, \vec{\phi}) = \begin{pmatrix} 0 \\ \phi_2\phi_2\phi_2 \end{pmatrix}, \quad K(\vec{x}, \vec{x}, \vec{x}) : (L_2[0,1])^6 \to (L_2[0,1])^6$$

And

$$\vec{\phi} = \begin{pmatrix} u \\ v \end{pmatrix}, \mathrm{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathrm{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

However, we will not use concrete domain $D(A)$. Instead, we will be assuming only that

$$D(A) = D \times D, D \subset L_2[0,1]$$

In that case, the domain of $K(\vec{\phi}, \vec{\phi}, \vec{\phi})$ will be

$$D(K) = D(A) \times D(A) \times D(A)$$

It is known, that in $L_2[0,1]$ exists the basis, which consists of the eigenfunctions of Laplace operator. Therefore, we can assume that the system

$$\{\psi_i\}_{i=1}^{+\infty}, \qquad -\{\psi_i\}_{xx} = \lambda_i\psi_i, \qquad \forall i \quad \lambda_i \geq 0, \qquad \forall i < j \quad \lambda_i \leq \lambda_j$$

is the basis in domain D. It is clear, that every element from $D(A)$ can be presented in the form of series in the following way:

$$\forall (u(x), v(x)) \in D(A) \qquad \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \sum_{i=1}^{+\infty} \begin{pmatrix} a_i \\ b_i \end{pmatrix} \psi_i,$$
$$a_i = (u, \psi_i)_{L_2}, \quad b_i = (v, \psi_i)_{L_2}$$

It should be pointed out that:

1. $\forall i, j \quad (\psi_i, \psi_j)_{L_2} = \delta_{ij}, \quad (\psi_i, \psi_i) = 1$

2. $\forall i \quad \lambda_1 \leq \lambda_i, \qquad \lambda_1 \geq 0$

4.2 Eigenvalues of operator A

Let us consider an eigenvalue problem:

$$\mathrm{A}\vec{\phi} = \theta\vec{\phi} \tag{107}$$

We can write vector $\vec{\phi}$ in the form of series:

$$\vec{\phi} = \sum_{k=1}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \psi_k \tag{108}$$

By substituting (108) into (107) and grouping the terms, we get:

$$\begin{pmatrix} -\nu\lambda_k & 1 \\ -1 & \mu - \nu\lambda_k \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \theta \begin{pmatrix} a_k \\ b_k \end{pmatrix} \qquad k = 1, 2, \ldots$$

The determinant of the matrix in the left side can be written as:

$$\varphi_{A_k}(\theta) = \theta^2 + (2\nu\lambda_k - \mu)\theta + \nu^2{\lambda_k}^2 - \mu\nu\lambda_k + 1$$

Eigenvalues of operator A could be found from equation $\varphi_{A_k}(\lambda) = 0$. The expressions for eigenvalues are:

$$\theta_{1,2}^k = \frac{-(2\nu\lambda_k - \mu) \pm \sqrt{\mu^2 - 4}}{2}$$

It should be pointed out that conjugate operator A^* will have the same eigenvalues.

Now, we will find the critical value of μ, in the case of oscillatory instability. Stability conditions are:

$$\begin{cases} \mu - 2\nu\lambda_k < 0 \\ \nu^2{\lambda_k}^2 - \mu\nu\lambda_k + 1 > 0 \end{cases} \qquad k = 1, 2, \ldots$$

Or, if we assume that $\lambda_1 > 0$:

$$\begin{cases} \mu < 2\nu\lambda_k \\ \mu < \frac{1}{\nu\lambda_k} + \nu\lambda_k \end{cases} \qquad k = 1, 2, \ldots$$

From the fact that $\nu > 0$, and that the sequence $\{\lambda_i\}_{i=1}^{+\infty}$ does not decrease, we can conclude that we can consider only the first pair of inequalities(when k = 1):

$$\begin{cases} \mu < 2\nu\lambda_1 \\ \mu < \frac{1}{\nu\lambda_1} + \nu\lambda_1 \end{cases} \tag{109}$$

It is known that $\forall a \in \mathbb{R} : a > 0$ holds: $a + \frac{1}{a} < 2a \Longleftrightarrow a > 1$. Then, it is clear that when $\nu\lambda_1 > 1$ the second inequality in (109) will be dominating. From here we can find the critical value of the parameter μ:

$$\mu_{\text{cr}} = \left[\begin{array}{c} \frac{1}{\nu\lambda_1} + \nu\lambda_1, \quad when \quad \nu > \frac{1}{\lambda_1} \\ 2\nu\lambda_1, \quad when \quad \nu < \frac{1}{\lambda_1} \end{array} \right.$$

Therefore, if $\lambda_1 > 0$, then in the case when $\nu \geq \frac{1}{\lambda_1}$, monotonic instability takes place in the system. In the case, when $\nu < \frac{1}{\lambda_1}$ - oscillatory instability. In that case, operator A has a pair of conjugated eigenvalues:

$$\theta_{1,2} = \pm i\sqrt{1 - \nu^2{\lambda_1}^2} = \pm i\omega_0$$

If $\lambda_1 = 0$, then the critical value of the parameter μ is equal to zero. In that case, oscillatory instability always takes place in the system and operator A has a pair of conjugated eigenvalues:

$$\theta_{1,2} = \pm i$$

4.3 Eigenvectors of the operator A in the case of oscillatory instability

Let us fix the value of parameter ν, which corresponds to oscillatory instability. Let us fix the value of μ equal to critical value, i.e. $\mu = 2\nu\lambda_1$.In that case all eigenvalue of operator A has negative real part, except one conjugated pair $\theta_{1,2} = \pm i\omega_0$. Let us find eigenvectors, which corresponds to these eigenvalues.

We will consider an eigenvalue problem

$$A\vec{\phi} = \pm i\omega_0\vec{\phi} \tag{110}$$

We will be looking for $\vec{\phi}$ in the form

$$\vec{\phi} = \sum_{i=1}^{+\infty} \begin{pmatrix} a_i \\ b_i \end{pmatrix} \psi_i \tag{111}$$

After substituting (111) into (110) and grouping the terms, we will get

$$\begin{cases} (-\nu\lambda_i \mp i\omega_0)a_i + b_i = 0 \\ -a_i + (-\nu\lambda_i + 2\nu\lambda_1 \mp i\omega_0)b_i = 0 \end{cases} \qquad i = 1, 2, \dots \tag{112}$$

It is clear that any of these systems will have non trivial solution only when $\lambda_i = \lambda_1$. In general case, we will have m these systems. We can assume that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_m, \qquad \lambda_{m+1} > \lambda_m$$

Therefore, only first m systems in (112) will have non trivial solutions. The solutions could be written as

$$\begin{cases} (-\nu\lambda_1 \mp i\omega_0)a_i + b_i = 0 \\ -a_i + (\nu\lambda_1 \mp i\omega_0)b_i = 0 \end{cases} \qquad i = 1, 2, \dots, m$$

From here we can find coefficients a_i, b_i, by setting, for example:

$$a_i = 1, \qquad b_i = \nu\lambda_1 \pm i\omega_0, \quad i = 1, 2, \dots, m$$

Therefore, the eigenvalues of A, corresponding to $\theta_{1,2} = \pm i\omega_0$ has the form

$$\vec{\phi}_{1,2} = \sum_{i=1}^{m} \begin{pmatrix} 1 \\ \nu\lambda_1 \pm i\omega_0 \end{pmatrix} \psi_i$$

We should not that the eigenvectors of conjugate operator A^*, corresponding to $\theta_{1,2} = \pm i\omega_0$ will have the form

$$\vec{\psi}_{1,2} = \sum_{i=1}^{m} \begin{pmatrix} 1 \\ \nu\lambda_1 \mp i\omega_0 \end{pmatrix} \psi_i$$

Therefore, it can be concluded that eigenvectors will consist of linear combinations of first m basis functions.

4.4 Preliminary actions

In the process of obtaining the periodic solution, we will have to deal with generalized versions of linear equations:

$$(A - ik_1\omega_0 I)\vec{v} = \begin{pmatrix} 0 \\ (\gamma + i\delta) \end{pmatrix} \psi_{k_2}(x), \qquad k_1 \in \mathbb{Z}, k_2 \in \mathbb{N} \tag{113}$$

We will be looking for solution in the form:

$$\vec{v} = \begin{pmatrix} M \\ N \end{pmatrix} \psi_{k_2}(x) \tag{114}$$

By substituting (114) into (113), we will get:

$$[B + aC - (\nu\lambda_n + ik_1\omega_0)I]\,\vec{v} = \begin{pmatrix} 0 \\ \gamma + i\delta \end{pmatrix} \psi_{k_2}(x)$$

From here we arrive at a linear system:

$$\begin{pmatrix} -(\nu\lambda_n + ik_1\omega_0) & 1 \\ -1 & a - (\nu\lambda_n + ik_1\omega_0) \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma + i\delta \end{pmatrix}$$

The solution of this system can be written as:

$$\begin{cases} M = (\gamma + i\delta)/((a - (\nu\lambda_n + ik_1\omega_0))(\nu\lambda_n + ik_1\omega_0) - 1) \\ N = (\nu\lambda_n + ik_1\omega_0)M \end{cases}$$

We will introduce new notation:

$$\begin{cases} P_{k1}^{k2} = \nu\lambda_{k2} + ik_1\omega_0 \\ F_{k1}^{k2} = (a - P_{k1}^{k2})P_{k1}^{k2} - 1 \\ R_{k1}^{k2} = \nu\lambda_{k_2} + ik_1\omega_0 \end{cases}$$

The coefficients could be rewritten as:

$$M = \frac{(\gamma + i\delta)(F_{k1}^{k2})^*}{|F_{k1}^{k2}|^2}, \quad N = \frac{(\gamma + i\delta)((F_{k1}^{k2})^* R_{k1}^{k2})}{|F_{k1}^{k2}|^2}$$

Therefore, the solution of (113) will be:

$$\vec{v} = \begin{pmatrix} M \\ N \end{pmatrix} \psi_{k_2}(x)$$

4.5 The analysis of nonlinear problem

We will consider the case when operator A has the pair of simple conjugated eigenvalues $\theta_{1,2} = \pm i\omega_0$, when parameter μ is equal to critical value. This means that $\lambda_1 < \lambda_2$ and eigenvectors of operator A, corresponding to $\theta_{1,2}$ have the following form:

$$\vec{\phi}_{1,2} = \begin{pmatrix} 1 \\ \nu\lambda_1 \pm i\omega_0 \end{pmatrix} \psi_1$$

And the eigenvectors of conjugate operator:

$$\vec{\psi}_{1,2} = \begin{pmatrix} 1 \\ \nu\lambda_1 \mp i\omega_0 \end{pmatrix} \psi_1$$

We will introduce new notation. Let us define:

$$\vec{\varphi} = \frac{1}{\sqrt{2}}\vec{\phi}_1, \qquad \vec{\psi} = \frac{1}{\sqrt{2}}\vec{\psi}_2$$

We did this in order to get $(\vec{\phi}, \vec{\psi}) = 1$.

As in previous sections, we will consider system (5) in the case of oscillatory instability. Then, the critical value of the parameter μ is equal to $2\nu{\lambda_1}^2$. If we denote $\mu = a + \delta$, $a = \mu_{\text{cr}}$, $\delta \ll 1$, we will get the system

$$\dot{\vec{\phi}} = \nu\vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + a\mathrm{C}\vec{\phi} + \delta\mathrm{C}\vec{\phi} - K(\vec{\phi}, \vec{\phi}, \vec{\phi}) \tag{115}$$

Let' introduce the variable change $\tau = \omega t, \delta = \varepsilon^2$. As dot, we will define differentiating by τ. From here we get the system:

$$\omega\dot{\vec{\phi}} = \nu\vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + a\mathrm{C}\vec{\phi} + \varepsilon^2\mathrm{C}\vec{\phi} - K(\vec{\phi}, \vec{\phi}, \vec{\phi}) \tag{116}$$

We will be looking for periodic solution $\vec{\phi}$ and unknown cyclic frequency ω in the form of series:

$$\vec{\phi} = \sum_{i=1}^{\infty} \varepsilon^i \vec{\phi}_i, \qquad \omega = \sum_{i=0}^{\infty} \varepsilon^i \omega_i \tag{117}$$

It is clear that $\omega_0 = \sqrt{1 - \nu^2 \lambda_1{}^2}$. By substituting (117) into (116) and equating the coefficients of like powers of ε, we will arrive at:

$$\varepsilon^1 : \quad \omega_0 \dot{\vec{\phi}}_1 = \nu \frac{\partial^2 \vec{\phi}_1}{\partial x^2} + \mathrm{B}\vec{\phi}_1 + a\mathrm{C}\vec{\phi}_1 \tag{118}$$

$$\varepsilon^2 : \quad \omega_0 \dot{\vec{\phi}}_2 = \nu \frac{\partial^2 \vec{\phi}_2}{\partial x^2} + \mathrm{B}\vec{\phi}_2 + a\mathrm{C}\vec{\phi}_2 - \omega_1 \dot{\vec{\phi}}_1 \tag{119}$$

$$\varepsilon^3 : \quad \omega_0 \dot{\vec{\phi}}_3 = \nu \frac{\partial^2 \vec{\phi}_3}{\partial x^2} + \mathrm{B}\vec{\phi}_3 + a\mathrm{C}\vec{\phi}_3 - \omega_1 \dot{\vec{\phi}}_2 - \omega_2 \dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_1 - -K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_1) \tag{120}$$

$$\varepsilon^4 : \quad \omega_0 \dot{\vec{\phi}}_4 = \nu \frac{\partial^2 \vec{\phi}_4}{\partial x^2} + \mathrm{B}\vec{\phi}_4 + a\mathrm{C}\vec{\phi}_4 - \omega_1 \dot{\vec{\phi}}_3 - \omega_2 \dot{\vec{\phi}}_2 - \omega_3 \dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_2 - -3K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_2) \tag{121}$$

$$\varepsilon^5 : \quad \omega_0 \dot{\vec{\phi}}_5 = \nu \frac{\partial^2 \vec{\phi}_5}{\partial x^2} + \mathrm{B}\vec{\phi}_5 + a\mathrm{C}\vec{\phi}_5 - \omega_1 \dot{\vec{\phi}}_4 - \omega_2 \dot{\vec{\phi}}_3 - \omega_3 \dot{\vec{\phi}}_2 - \omega_4 \dot{\vec{\phi}}_1 + +\mathrm{C}\vec{\phi}_3 - 3K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_3) - 3K(\vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_2) \tag{122}$$

Let us start to solve the equations. The periodic solution of (118) has the form:

$$\vec{\phi}_1 = \alpha_1 \vec{\varphi} e^{i\tau} + \text{c.j.}, \qquad \alpha_1 > 0$$

Inhomogeneous equation (119) has the periodic solution if and only if the condition of solvability is satisfied:

$$\int_0^{2\pi} (-\omega_1 \dot{\vec{\phi}}_1, \vec{\psi}) e^{-i\tau} d\tau = 0$$

By performing the computations, we get:

$$\int_0^{2\pi} (-\omega_1 \dot{\vec{\phi}}_1, \vec{\psi}) e^{-i\tau} d\tau = -4\pi i \alpha_1 \omega_1 = 0$$

From the fact that $\alpha_1 > 0$, we conclude that $\omega_1 = 0$.

Therefore, the solution of (119) will be:

$$\vec{\phi}_2 = \alpha_2 \vec{\varphi} e^{i\tau} + \text{c.j.}$$

Let us satisfy the condition of solvability for (120):

$$\int_0^{2\pi} (-\omega_2 \dot{\vec{\phi}}_1 - K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_1) + \mathrm{C}\vec{\phi}_1, \vec{\psi})e^{-i\tau} d\tau = 0$$

After the computations, we will get:

$$\int_0^{2\pi} (-\omega_2 \dot{\vec{\phi}}_1 - K(\vec{\phi}_1, \vec{\phi}_1, \vec{\phi}_1) + \mathrm{C}\vec{\phi}_1, \vec{\psi})e^{-i\tau} d\tau =$$
$$= -i\omega_2\alpha_1(\vec{\varphi}, \vec{\psi}) - 3\alpha_1^3(K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi}^*), \vec{\psi}) + \alpha_1(\mathrm{C}\vec{\varphi}, \vec{\psi})$$

From here we arrive at the equation for obtaining α_1 and ω_2

$$-i\omega_2\alpha_1 + \frac{1}{2}\alpha_1 - \frac{3}{4}{\alpha_1}^3({\psi_1}^3, \psi_1) = 0$$

If we split this expression into real and imaginary parts, we will get the expressions for α_1 and ω_2

$$\alpha_1 = \frac{\sqrt{2}}{\sqrt{3({\psi_1}^3, \psi_1)}}, \quad \omega_2 = 0 \tag{123}$$

From here we conclude, that the periodic mode will be in the system when δ is positive.

Now, we will find the periodic solution of inhomogeneous equation (120). This equation could be rewritten as

$$\omega_0\dot{\vec{\phi}}_3 = A\vec{\phi}_3 + [\alpha_1 C\vec{\varphi} - 3{\alpha_1}^3 K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi}^*)]e^{i\tau} - \alpha_1^3 K(\vec{\varphi}, \vec{\varphi}, \vec{\varphi})e^{3i\tau} + \text{c.j.}$$

By writing inhomogeneous part in vector form, we get

$$\omega_0\dot{\vec{\phi}}_3 = A\vec{\phi}_3 + (\frac{a}{2} + i\omega_0)\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}\alpha_1\psi_1 - 3\frac{1}{2^{\frac{3}{2}}}\alpha_1^3\psi_1^3 \end{pmatrix} e^{i\tau} +$$
$$+(\frac{a}{2} + i\omega_0)^3 \begin{pmatrix} 0 \\ -\frac{1}{2^{\frac{3}{2}}}\alpha_1^3\psi_1^3 \end{pmatrix} e^{3i\tau} + \text{c.j.}$$

We can present the inhomogeneous part in the right side of the equation as Fourier series. We will arrive at infinite set of linear systems, which we can solve. It should be specially pointed out that due to the condition of solvability, the Fourier series of the expression around $e^{i\tau}$ will start from the second basis function ψ_2. Because of that, there will be no resonant terms in the solution. Then, the solution of the inhomogeneous equation (120) could be written in the form:

$$\begin{aligned}\vec{\phi_3}^{\,\mathrm{p}} &= \frac{3}{2^{\frac{3}{2}}}\alpha_1^3(\tfrac{a}{2}+i\omega_0)\sum_{j=2}^{+\infty}\left(\begin{pmatrix}\frac{(F_1^j)^*}{|F_1^j|^2}\\ \frac{(F_1^j)^*R_1^j}{|F_1^j|^2}\end{pmatrix}(\psi_1^3,\psi_j)\psi_j\right)e^{i\tau}+\\ &+\frac{1}{2^{\frac{3}{2}}}\alpha_1^3(\tfrac{a}{2}+i\omega_0)^3\sum_{j=1}^{+\infty}\left((\psi_1^3,\psi_j)\psi_j\begin{pmatrix}\frac{(F_3^j)^*}{|F_3^j|^2}\\ \frac{(F_3^j)^*R_3^j}{|F_3^j|^2}\end{pmatrix}\right)e^{3i\tau}+\text{c.j.}=\\ &=\frac{3}{2^{\frac{3}{2}}}\alpha_1^3(\tfrac{a}{2}+i\omega_0)\begin{pmatrix}f_1(x)\\ g_1(x)\end{pmatrix}e^{i\tau}+\frac{1}{2^{\frac{3}{2}}}\alpha_1^3(\tfrac{a}{2}+i\omega_0)^3\begin{pmatrix}f_3(x)\\ g_3(x)\end{pmatrix}e^{3i\tau}+\\ &+\text{c.j.}\end{aligned} \tag{124}$$

Then, general periodic solution of (120) is

$$\vec{\phi_3}=\alpha_3(\vec{\varphi}e^{i\tau}+\vec{\varphi}^*e^{-i\tau})+\vec{\phi_3}^{\,\mathrm{p}}$$

Let us satisfy the condition of solvability for equation (121)

$$\int_0^{2\pi}(-\omega_3\dot{\vec{\phi_1}}+\mathrm{C}\vec{\phi_2}-3K(\vec{\phi_1},\vec{\phi_1},\vec{\phi_2}),\vec{\psi})e^{-i\tau}d\tau=0$$

After performing the computations, we will get

$$-i\omega_3+\frac{1}{2}\alpha_2-\frac{3}{4}{\alpha_1}^2\alpha_2({\psi_1}^3,\psi_1)=0$$

From here we conclude that

$$\alpha_2=0,\quad \omega_3=0$$

.

Then, the periodic solution of (121) will be:

$$\vec{\phi_4}=\alpha_4e^{i\tau}\vec{\varphi}+\text{c.j.}$$

Let us consider the condition of solvability for (122)

$$\int_0^{2\pi}(-\omega_4\dot{\vec{\phi_1}}+\mathrm{C}\vec{\phi_3}-3K(\vec{\phi_1},\vec{\phi_1},\vec{\phi_3}),\vec{\psi})e^{-i\tau}d\tau=0$$

By performing the computations and noting that $(Cf_1, \vec{\psi}) = 0$, we will get

$$\alpha_3 + i\alpha_1\omega_4 = \frac{3\alpha_1^5}{2^{\frac{5}{2}}}((g_3\psi_1^2, \psi_1) + 6(g_1\psi_1^2, \psi_1) + 3(g_1^*{\psi_1}^2, \psi_1))$$

From here we get the expressions for α_3 and ω_4

$$\begin{cases} \alpha_3 = Re(\frac{3\alpha_1^5}{2^{\frac{5}{2}}}((g_3\psi_1^2, \psi_1) + 6(g_1\psi_1^2, \psi_1) + 3(g_1^*{\psi_1}^2, \psi_1))) \\ \omega_4 = \frac{1}{\alpha_1} Im(\frac{3\alpha_1^5}{2^{\frac{5}{2}}}((g_3\psi_1^2, \psi_1) + 6(g_1\psi_1^2, \psi_1) + 3(g_1^*{\psi_1}^2, \psi_1))) \end{cases} \quad (125)$$

Finally, first terms of the asymptotic are given by formulas

$$\begin{cases} \vec{\phi} = \varepsilon\alpha_1(e^{i\omega t}\vec{\varphi} + e^{-i\omega t}\vec{\varphi^*}) + \varepsilon^3(\alpha_3(e^{i\omega t}\vec{\varphi} + e^{-i\omega t}\vec{\varphi^*}) + \vec{\phi}_3^{\mathrm{p}}) + O(\varepsilon^4) \\ \omega = \sqrt{1 - \nu^2\lambda_1^2} + \varepsilon^4\omega_4 + O(\varepsilon^5) \end{cases} \quad (126)$$

Here α_1, $\vec{\phi_3}^{\mathrm{p}}$, α_3, ω_4 are given by (123), (124) and (125) respectively.

4.6 Formulas for the consecutive terms of the asymptotic

When $n \geq 3$, the expression around ε^n can be written as:

$$\omega_0\dot{\vec{\phi}}_n = A\vec{\phi}_n + C\vec{\phi}_{n-2} - \sum_{i=1}^{n-1}\omega_{n-i}\dot{\vec{\phi}}_n - \sum_{\substack{i_1+i_2+i_3=n \\ i_1 \leq i_2 \leq i_3}} 3K(\vec{\phi}_{i_1}, \vec{\phi}_{i_2}, \vec{\phi}_{i_3}) \quad (127)$$

We will find the formulas for higher terms of the asymptotic in this section. Again, we will start from deriving formulas for even terms. As in previous cases, the following statement holds.

Statement 3. For every $k \in \mathbb{N}$ $\quad \vec{\phi}_{2k} = 0, \quad \alpha_{2k} = 0, \quad \omega_{2k-1} = 0$

Proof. The proof of this statement is the same as the proof, performed in the section, devoted to the analysis of Dirichlet boundary conditions. We will not repeat it here.

Now, Let us derive the formulas for odd terms in the expansion.

For odd n, equation (127) could be written as:

$$\omega_0\dot{\vec{\phi}}_n = A\vec{\phi}_n + C\vec{\phi}_{n-2} - i\omega_{n-1}\dot{\vec{\phi}}_1 - \sum_{\substack{i_1+i_2+i_3=n \\ i_1\le i_2\le i_3}}^{i_1+i_2+i_3=2k+1} 3K(\vec{\phi}_{i_1},\vec{\phi}_{i_2},\vec{\phi}_{i_3}) \qquad (128)$$

We will note that:

$$C\vec{\phi}_{n-2} = \alpha_{n-2}C\vec{\varphi}e^{i\tau} + \alpha_{n-2}C\vec{\varphi}^{*}e^{-i\tau} + C(\vec{\phi}^{\mathrm{p}}_{n-2})$$

Here, as $\vec{\phi}^{\mathrm{p}}_{n-2}$ is denoted the partial solution of inhomogeneous equation.

Now, we will transform the sum:

$$\sum_{\substack{i_1+i_2+i_3=n \\ i_1\le i_2\le i_3}}^{i_1+i_2+i_3=2k+1} 3K(\vec{\phi}_{i_1},\vec{\phi}_{i_2},\vec{\phi}_{i_3}) = 3K(\vec{\phi}_1,\vec{\phi}_1,\vec{\phi}_{n-2}) + \\ + \sum_{\substack{i_1+i_2+i_3=n \\ i_1\le i_2\le i_3}}^{(i_1,i_2,i_3)\neq(1,1,n-2)} 3K(\vec{\phi}_{i_1},\vec{\phi}_{i_2},\vec{\phi}_{i_3})$$

We should note that:

$$K(\vec{\phi}_1,\vec{\phi}_1,\vec{\phi}_{n-2}) = K(\vec{\phi}_1,\vec{\phi}_1,\vec{\phi}^{\mathrm{hom}}_{n-2}) + K(\vec{\phi}_1,\vec{\phi}_1,\vec{\phi}^{\mathrm{p}}_{n-2})$$

We will introduce new notation for inhomogeneous part:

$$f_n = C(\vec{\phi}^{\mathrm{p}}_{n-2}) - 3K(\vec{\phi}_1,\vec{\phi}_1,\vec{\phi}^{\mathrm{p}}_{n-2}) - \sum_{\substack{i_1+i_2+i_3=n \\ i_1\le i_2\le i_3}}^{(i_1,i_2,i_3)\neq(1,1,n-2)} 3K(\vec{\phi}_{i_1},\vec{\phi}_{i_2},\vec{\phi}_{i_3}) \qquad (129)$$

Then, the condition of solvability for this equation will be:

$$\int\limits_0^{2\pi}((\alpha_{n-2}C\vec{\varphi}e^{i\tau} + \alpha_{n-2}C\vec{\varphi}^{*}e^{-i\tau},\vec{\psi}) - i\omega_{n-1}(\dot{\vec{\phi}}_1,\vec{\psi})d\tau - \\ - \int\limits_0^{2\pi} 3(K(\vec{\phi}_1,\vec{\phi}_1,\vec{\phi}_{n-2},\vec{\psi}) + (f_n,\vec{\psi}))e^{-i\tau}d\tau = 0$$

After performing the computations, we will get:

$$2\alpha_{n-2} + 2i\alpha_1\omega_{n-1} = (f_n, \vec{\psi})$$

From here we conclude that

$$\alpha_{n-2} = \frac{1}{2}Re((f_n, \vec{\psi})), \qquad \omega_{n-1} = \frac{1}{2\alpha_1}Im((f_n, \vec{\psi}))$$

Now, Let us find the solution of (128). The equation could be rewritten as:

$$\begin{aligned} \omega_0\dot{\vec{\phi}}_n &= A\vec{\phi}_n + (\tfrac{a}{2} + i\omega_0)(\begin{pmatrix} 0 \\ g_1(x) \end{pmatrix} e^{i\tau} + \cdots + \\ &+(\tfrac{a}{2} + i\omega_0)^n(\begin{pmatrix} 0 \\ g_n(x) \end{pmatrix} e^{in\tau} + \text{c.j.} \end{aligned} \tag{130}$$

If we write $g_i(x)$ as Fourier series and solve the obtained linear systems, we will find the solution of inhomogeneous equation. The formulas for the solutions of these systems was derived earlier. We should point out that because of the condition of solvability, there will be no resonant terms in the solution.

Finally, we have got the following results

$$\begin{cases} \vec{\phi}_{2k} = 0, \quad \alpha_{2k} = 0, \quad \omega_{2k+1} = 0 \\ \vec{\phi}_{2k+1} = (\frac{a}{2} + i\omega_0)f_1(x)e^{i\tau} + \cdots + \cdots + (\frac{a}{2} + i\omega_0)^n f_n(x)e^{in\tau} + \text{c.j.} \\ \alpha_{2k-1} = \frac{1}{2}Re((f_{2k+1}, \vec{\psi})), \quad \omega_{2k} = \frac{1}{2\alpha_1}Im((f_{2k+1}, \vec{\psi})) \end{cases}$$

Here f_{2k+1} is given by (129). The expressions for $f_i(x)$ are

$$f_i(x) = \sum_{j=1}^{+\infty} \begin{pmatrix} \frac{(F_i^j)^*}{|F_i^j|^2} \\ \frac{(F_i^j)^* R_i^j}{|F_i^j|^2} \end{pmatrix} (g_i(x), \psi_j)\psi_j$$

4.7 The applications of abstract scheme

Formulas, derived in this sections allow us to calculate first terms of the asymptotic approximation of periodic mode. These formulas are applicable to a wide range of boundary conditions. In order to apply them, we have to fix boundary conditions, define the corresponding domain $D(A)$, select the sequence of basis functions $\{\psi_i\}_{i=1}^{+\infty}$ and perform the computations by using formula (126).

Here we will demonstrate the applications of generalized formulas (126) to different types of boundary conditions. It should be pointed out that the results, obtained for Dirichlet and Neumann boundary conditions could be obtained as well from formula (126). We will not repeat the computations here, we will just specify the domain and a system of basis functions.

- **Dirichlet boundary conditions.**

  $$D(A) = \{u(x,t), v(x,t) \in W_2^2 : u(0,t) = u(1,t) = 0,$$
  $$v(0,t) = v(1,t) = 0\}$$

  $\{\sqrt{2}\sin(\pi kx)\}_{k=1}^{\infty}$ - basis in $D(A)$.

  An spatially inhomogeneous periodic mode exists in the system. The diffusion slows down the frequency of self-oscillations.

- **Neumann boundary conditions**

  $$D(A) = \{u(x,t), v(x,t) \in W_2^2 : u_x(0,t) = u_x(1,t) = 0,$$
  $$v_x(0,t) = v_x(1,t) = 0\}$$

  $\{\sqrt{2}\cos(\pi kx)\}_{k=0}^{\infty}$ - basis in $D(A)$.

  An spatially homogeneous periodic mode exists in the system. The diffusion does not influence on the frequency of self-oscillations.

- **Neumann boundary conditions with additional requirement of zero average**

  $$D(A) = \{u(x,t), v(x,t) \in W_2^2 : u_x(0,t) = u_x(1,t) = 0,$$
  $$v_x(0,t) = v_x(1,t) = 0, \int_0^1 u dx = 0, \int_0^1 v dx = 0\}$$

  $\{\sqrt{2}\cos(\pi kx)\}_{k=1}^{\infty}$ - basis in $D(A)$.

  An spatially inhomogeneous periodic mode exists in the system. The diffusion slows down the frequency of self-oscillations.

- **Mixed boundary conditions**

$$D(A) = \{u(x,t), v(x,t) \in W_2^2 : u(0,t) = u_x(1,t) = 0,$$
$$v(0,t) = v_x(1,t) = 0\}$$

$\{\sqrt{2}\sin(\pi\frac{2k-1}{2}x)\}_{k=1}^{\infty}$ - basis in $D(A)$.

$$\alpha_1 = \frac{2}{3}, \qquad a = \frac{\nu\pi^2}{2}, \qquad \vec{\varphi} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \frac{a}{2} + i\omega_0 \end{pmatrix}\sin(\frac{\pi}{2}x)$$

$$f_1(x) = -\frac{1}{\sqrt{2}}\frac{(F_3^1)^*}{|F_3^1|^2}\sin(\frac{3\pi}{2}x), \qquad g_1(x) = -\frac{1}{\sqrt{2}}\frac{(F_3^1)^* R_1^3}{|F_3^1|^2}\sin(\frac{3\pi}{2}x)$$

$$f_3(x) = -\frac{3}{\sqrt{2}}\frac{(F_1^3)^*}{|F_1^3|^2}\sin(\frac{\pi}{2}x) - \frac{1}{\sqrt{2}}\frac{(F_3^3)^*}{|F_3^3|^2}\sin(\frac{3\pi}{2}x)$$

$$g_3(x) = -\frac{3}{\sqrt{2}}\frac{(F_1^3)^* R_1^3}{|F_1^3|^2}\sin(\frac{\pi}{2}x) - \frac{1}{\sqrt{2}}\frac{(F_3^3)^* R_3^3}{|F_3^3|^2}\sin(\frac{3\pi}{2}x)$$

$$\vec{\phi}_3^p = \frac{3}{2^{\frac{3}{2}}}\alpha_1^3(\frac{a}{2}+i\omega_0)\begin{pmatrix} f_1(x) \\ g_1(x) \end{pmatrix} e^{i\tau} + \frac{1}{2^{\frac{3}{2}}}\alpha_1^3(\frac{a}{2}+i\omega_0)^3\begin{pmatrix} f_3(x) \\ g_3(x) \end{pmatrix} e^{3i\tau} + \text{c.j.}$$

$$\alpha_3 = -Re(\frac{1}{9}(F_1^3)^* R_1^3 + \frac{1}{81}(F_3^3)^* R_3^3 + \frac{2}{27}(F_3^1)^* R_3^1 + \frac{1}{27}(F_3^1)(R_3^1)^*)$$

$$\omega_4 = -\frac{1}{\alpha_1}Im(\frac{1}{9}(F_1^3)^* R_1^3 + \frac{1}{81}(F_3^3)^* R_3^3 + \frac{2}{27}(F_3^1)^* R_3^1 + \frac{1}{27}(F_3^1)(R_3^1)^*)$$

An spatially inhomogeneous periodic mode exists in the system. The diffusion slows down the frequency of self-oscillations.

On the figures below we show the visualization of asymptotic approximation.

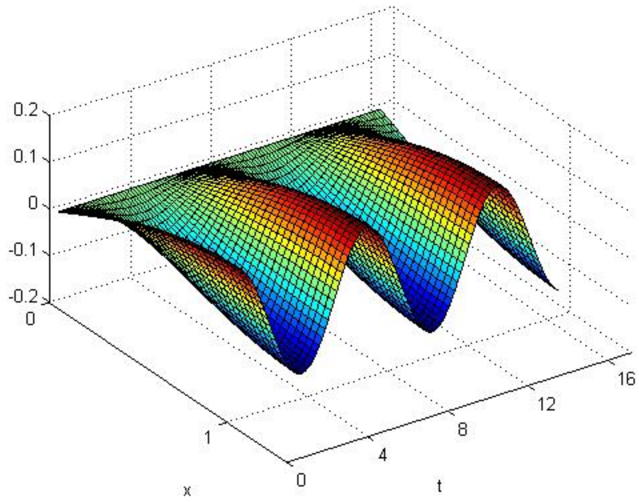

Figure 15: The asymptotic of u(x,t) when $\nu = 0.1$ and $\mu = a + 0.01$

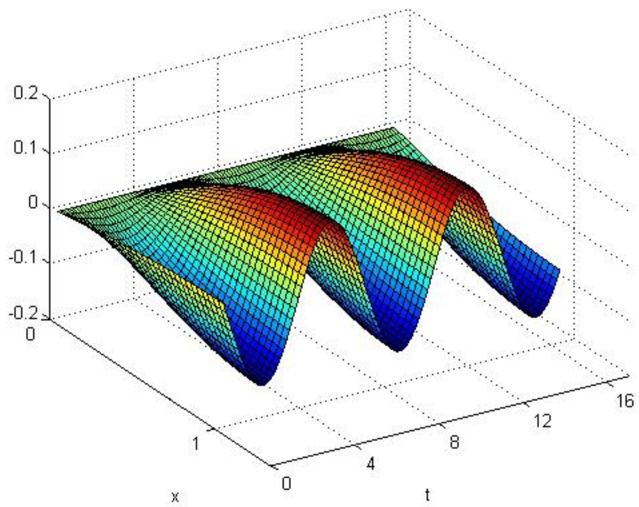

Figure 16: The asymptotic of v(x,t) when $\nu = 0.1$ and $\mu = a + 0.01$

On the next figures we plot numerical solution. Initial conditions were taken from asymptotic.

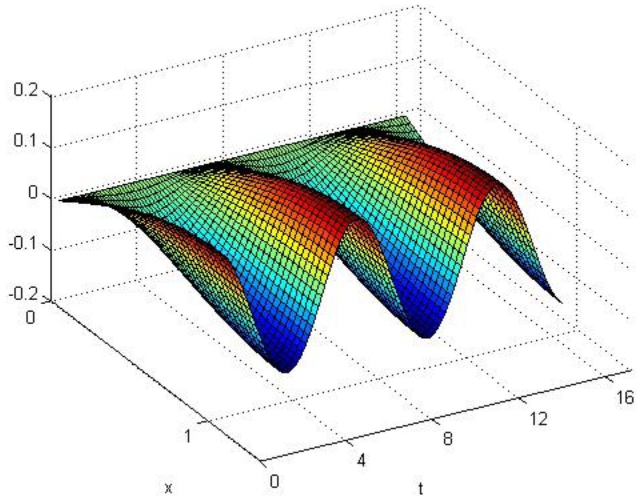

Figure 17: Numerical solution u(x,t) when $\nu = 0.1$ and $\mu = a + 0.01$

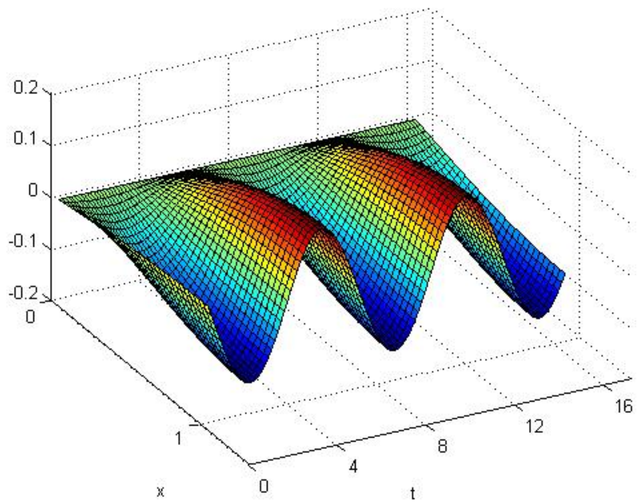

Figure 18: Numerical solution v(x,t) when $\nu = 0.1$ and $\mu = a + 0.01$

- **Periodic boundary conditions**

$$D(A) = \{u(x,t), v(x,t) \in W_2^2 : u(0,t) = u(1,t) = 0, u_x(0,t) = u_x(1,t),$$
$$v(0,t) = v(1,t) = 0, v_x(0,t) = v_x(1,t)\}$$

$\{1, \{\sqrt{2}\sin(\pi k x)\}_{k=1}^{\infty}, \{\sqrt{2}\cos(\pi k x)\}_{k=1}^{\infty}\}$ - basis in $D(A)$.

$$\alpha_1 = \frac{2}{\sqrt{3}}, \qquad \vec{\varphi} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$f_1(x) = 0, \qquad g_1(x) = 0$$

$$f_3(x) = \frac{(F_3^1)^*}{|F_3^1|^2}, \qquad g_3(x) = \frac{(F_3^1)^* R_3^1}{|F_3^1|^2}$$

$$\vec{\phi}_3^p = -\frac{i}{3\sqrt{3}}\begin{pmatrix} f_3(x) \\ g_3(x) \end{pmatrix} + \text{c.j.}$$

$$\alpha_3 = 0$$

$$\omega_4 = \frac{3}{48}$$

An spatially homogeneous periodic mode exists in the system. The diffusion does not influence on the frequency of self-oscillations. The formula of the asymptotic is again the same as in the case of ODE.

4.8 Remarks

Here we will discuss several important features of our abstract scheme. Firstly, we should point out that although we have derived our formulas in assumption that $x \in [0,1]$, nothing will change in the case when, for example $x \in [a,b] \times [c,d]$. This means that our scheme is applicable for higher dimensions as well. Moreover, if our domain and system are defined in non Cartesian coordinate system, for example, in polar coordinates, our formulas will be still applicable.

Another important observation should be noted as well. In section 2, when we were studying spatially distributed system in the case of fixed boundary conditions, the solutions of inhomogeneous equations consisted of finite number of terms. However, in our generalized formulas, derived in section 3 we can observe series, i.e. higher terms of the asymptotic approximation contain infinite number of elements. We have seen that during the application of derived formulas to the boundary conditions, considered at the present work, the series transform into finite sums. However, this does not holds for all types of boundary conditions. For example, if we apply our formulas to boundary conditions of the third type (Robin boundary conditions), the series will not disappear.

5 Numerical experiments

In this section we will discuss the results of the numerical experiments, which were performed in order to support our theoretical results. Our main purpose is to visualise the results, obtained in the previous sections and to give graphical explanations of the phenomena, which appear in the system under study. MATLAB software is being used for numerical computations, while symbolic manipulation are performed in Maple package.

As we have understood in previous sections, there are two qualitative different behaviours of the self-oscillations in spatially distributed system (5). The periodic mode can be spatially homogeneous or inhomogeneous. Therefore, in numerical experiments we will consider our system in two situations: Dirichlet and Neumann boundary conditions.

We have used MATLAB procedure pdepe() to obtain numerical solutions. This procedure allows to solve initial-boundary value problems for systems of partial differential equations in the one space variable. The detailed information on the algorithm of obtaining numerical solution in given in Appendix B.

5.1 Dirichlet boundary conditions

5.1.1 The behaviour of the system when $\mu < \mu_{cr}$

As we have found out earlier, when system (5) is considered in the case of Dirichlet boundary conditions in the case when $\nu < \frac{1}{\pi}$, an oscillatory instability takes place in the system. The critical value of the parameter μ is $\mu_{cr} = a = 2\nu\pi^2$. We have proved that when $\mu = a + \delta, \delta << 1$, a stable periodic mode exists in this system. The periodic mode will be spatially inhomogeneous.

However, when $\mu < a$, there will be no self-oscillations in the system. The solutions of the system will decay to zero, when $t \to +\infty$.

In the following figures we show the results of numerical simulation. Here we have taken $\nu = 0.1$, $\mu < a$. Initial conditions for $u(x,t)$ and $v(x,t)$ are taken from asymptotic. We will show only the solution for $u(x,t)$, because $v(x,t)$ behaves in the similar way.

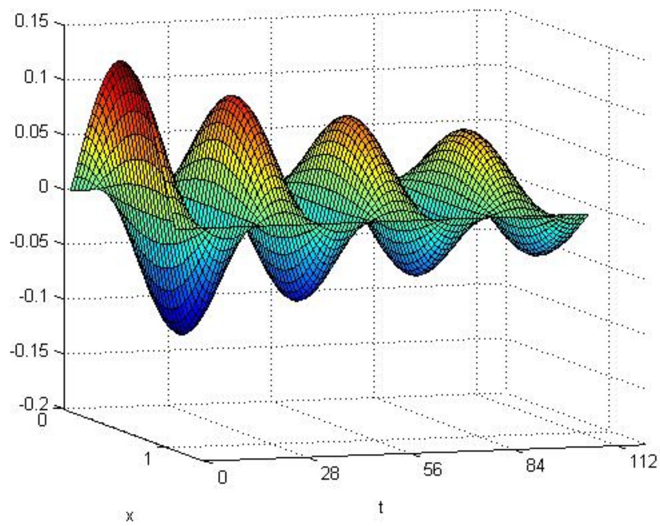

Figure 19: The numerical solution when $\mu = a - 0.01$

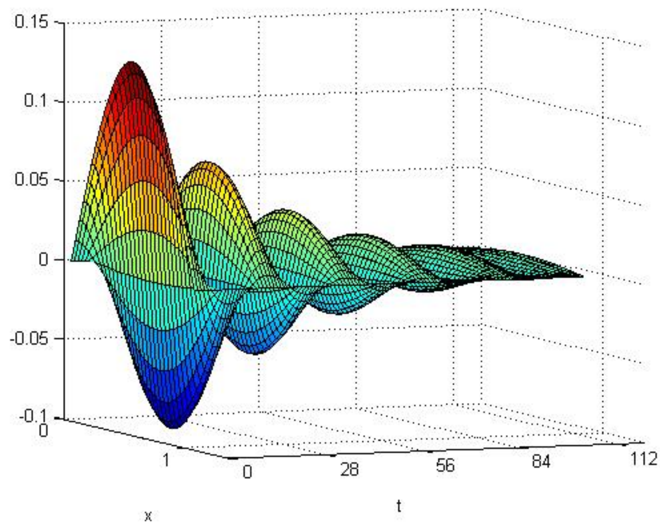

Figure 20: The numerical solution when $\mu = a - 0.05$

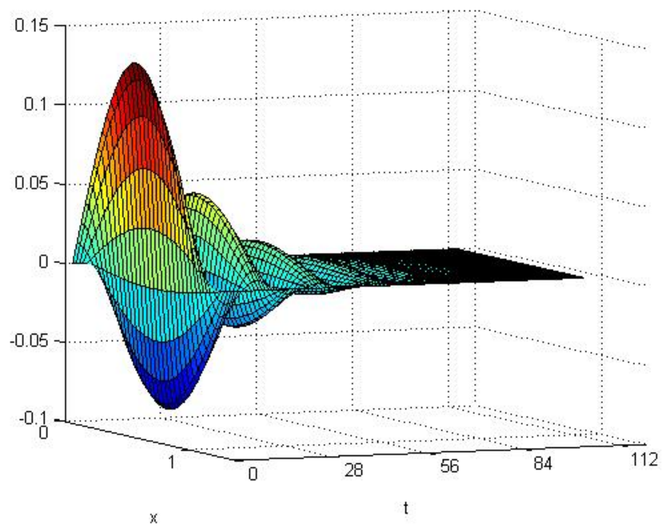

Figure 21: The numerical solution when $\mu = a - 0.1$

We can see from the figures, that the solution decays to zero and that the speed of decay is increasing when $\mu \searrow 0$.

5.1.2 The behaviour of the system when $\mu > \mu_{cr}$

Now we will perform several simulations in the case when $\mu > a$ and the self-oscillations are presented in the system. We will illustrate the stability of the periodic mode. We will fix $\nu = 0.1$, $\mu = a + 0.01$ and perform several numerical simulations with different initial conditions. On the following figures we plot the results.

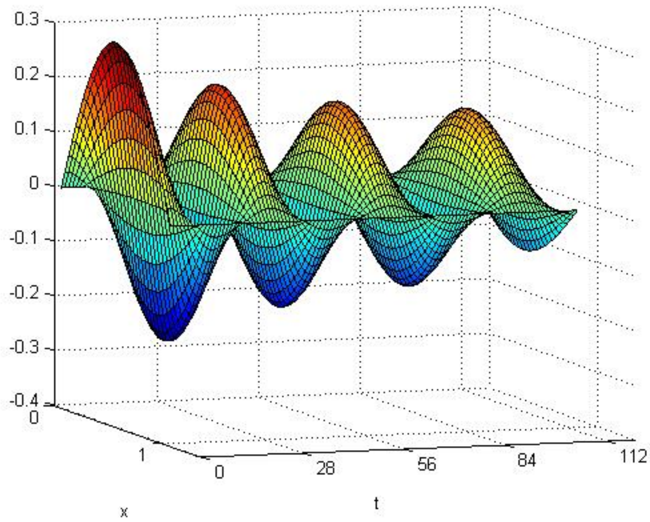

Figure 22: The numerical solution when $u_0(x) = v_0(x) = 0.2\sin(x)$

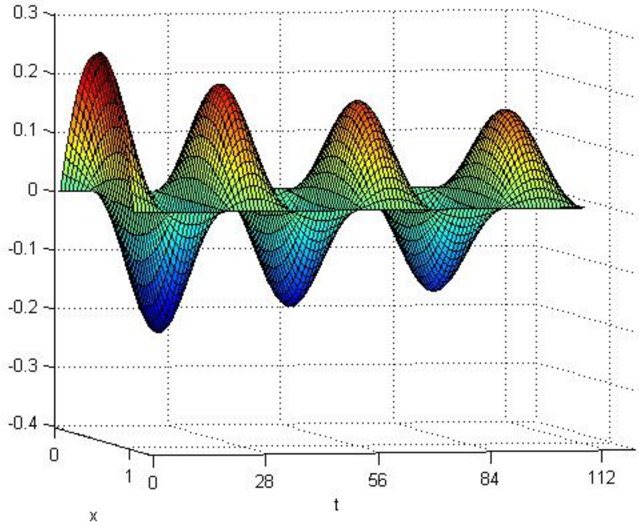

Figure 23: The numerical solution when $u_0(x) = v_0(x) = x(1-x)$

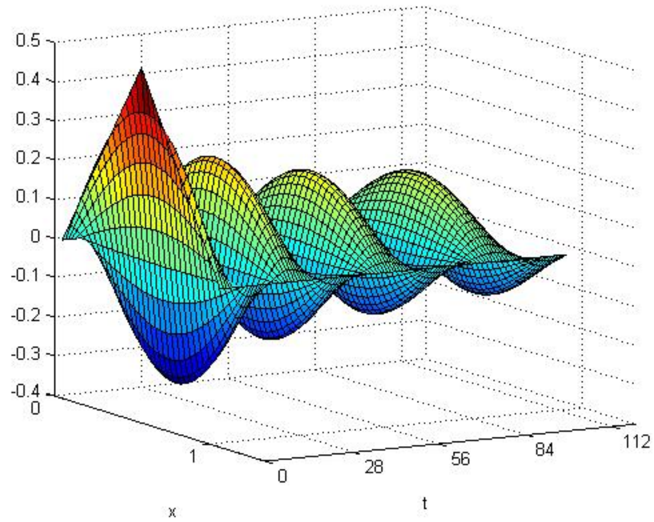

Figure 24: The numerical solution when $u_0(x) = v_0(x) = 0.5 - |x - 0.5|$

We observe from this figures, that the solutions with different initial conditions decay during the increase of time variable. However, if we perform the simulations with longer time interval, we will observe that the solutions tend to the stable periodic solution, approximated earlier. Therefore, we observe the situation, qualitatively different from the case when $\mu < a$.

5.2 Neumann boundary conditions

5.2.1 The behaviour of the system when $\mu < \mu_{cr}$

As we have found out before, in the case of Neumann boundary conditions, an oscillatory instability always takes place in the system. The critical value of the parameter μ is $\mu_{cr} = a = 0$. We have showed that when $\mu = \delta << 1$, a stable spatially homogeneous periodic mode exists in this system.

Again, we consider our system in the case when $\mu < 0$. In the following figures we show the results of numerical simulation. Here we have fixed $\nu = 0.1$. Initial conditions for $u(x, t)$ and $v(x, t)$ are taken from asymptotic. We will again show only the solution for $u(x, t)$.

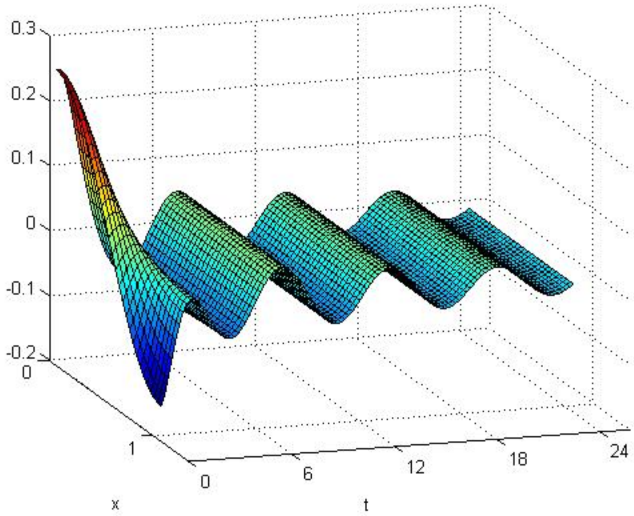

Figure 25: The numerical solution when $\mu = -0.1$

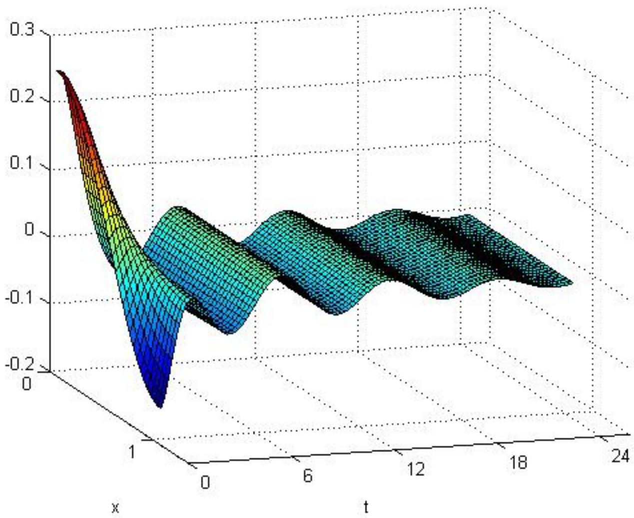

Figure 26: The numerical solution when $\mu = -0.2$

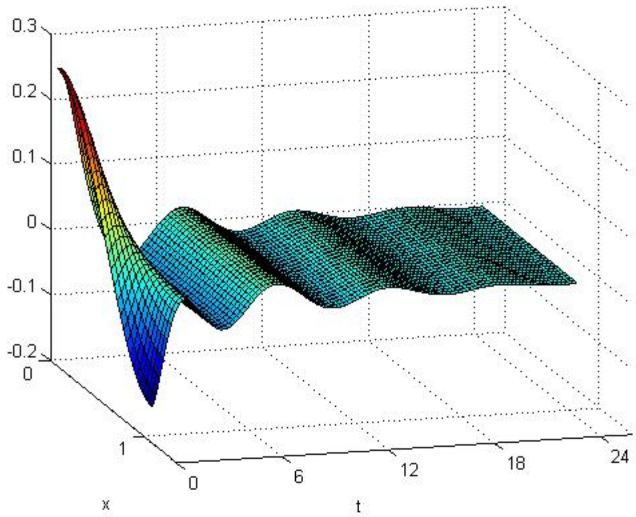

Figure 27: The numerical solution when $\mu = -0.3$

In this case we can see again that the solution tends to zero when $t \to +\infty$.

5.2.2 The behaviour of the system when $\mu > \mu_{cr}$

Here we will perform several numerical experiments in the case when $\mu > 0$. We will illustrate the stability of the periodic mode. Again we fix $\nu = 0.1$, $\mu = 0.01$ and perform several numerical simulations with different initial conditions. On the following figures the results are presented.

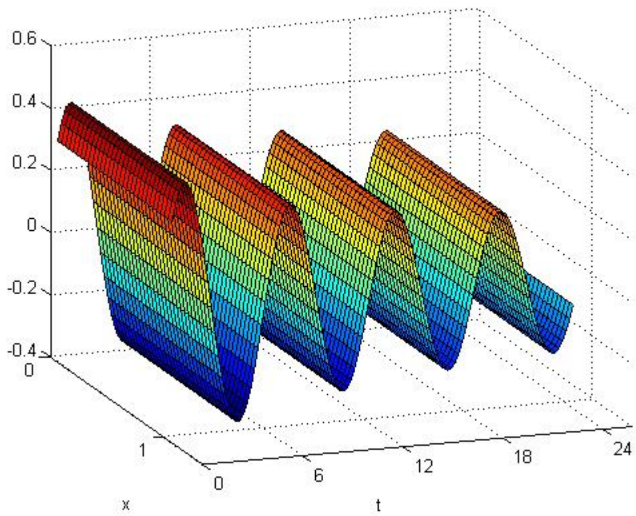

Figure 28: The numerical solution when $u_0(x) = v_0(x) = 0.2\sin(x)$

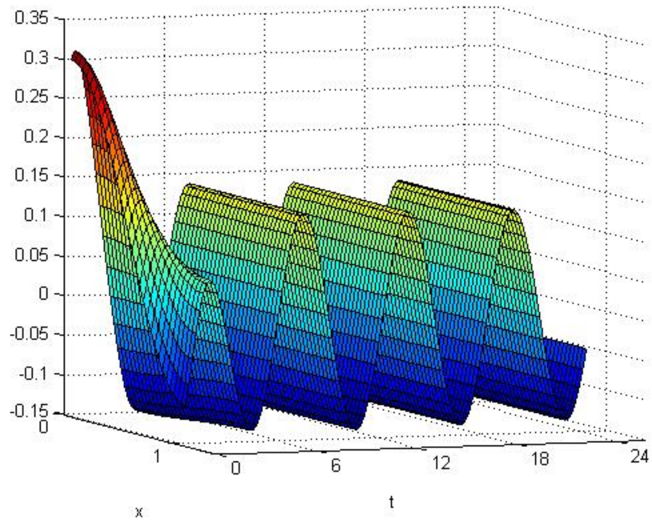

Figure 29: The numerical solution when $u_0(x) = v_0(x) = x(1-x)$

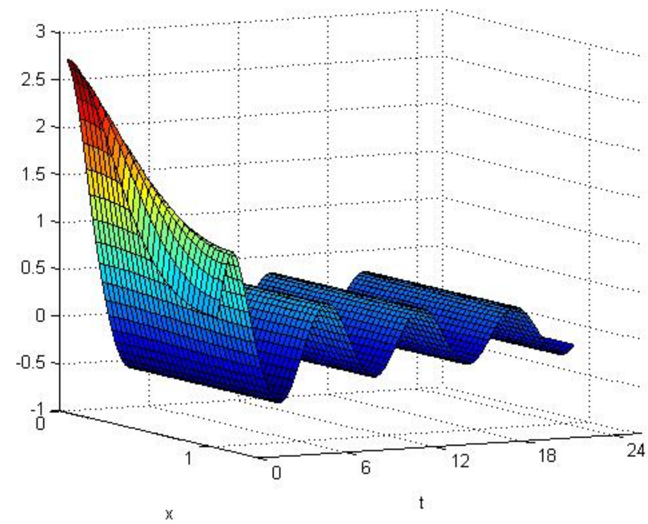

Figure 30: The numerical solution when $u_0(x) = v_0(x) = 0.5 - |x - 0.5|$

We observe that all considered solutions tends to the stable periodic solution when $t \to +\infty$. This phenomena could be better observed if we take longer time interval.

6 CONCLUSIONS

In the course of the present work we have performed the analysis of periodic modes in Rayleigh equation and the corresponding spatially distributed system. We have seen that self-oscillations does not disappear after the diffusion is added and that the form of the periodic mode depends on the type of the boundary conditions. We have observed two qualitative different forms of the periodic solution: spatially homogeneous and inhomogeneous self-oscillations. By using the methodology of functional analysis we have understood the conditions, leading to the presence of each type of self-oscillations. We have understood that the diffusion slows down the frequency of self-oscillations in the systems.

We have found out that secondary self-oscillatory mode will be stable, which means that soft loss of stability takes place in the system. The formulas for general term of the asymptotic was found as well. We have seen that in general, only odd terms will be presented in asymptotic approximation.

In our investigation, we have used Lyapunov-Schmidt method for theoretical analysis. Due to the selection of the method, we were able to use the same methodology while working with ODE and PDE. This allowed us to obtain the results, which can be easily compared. In addition, we were able to derive formulas for the asymptotic approximation of the periodic mode, suitable for almost any type of boundary conditions.

However, there are still special cases of boundary conditions when our derived formulas are not applicable. For example, periodic boundary conditions with additional requirement of zero average conservation. In that case, we cannot use our results, because the eigenvalues $\theta_{1,2}$, corresponding to critical value of the parameter μ will not be simple, i.e. each of them will have two corresponding independent eigenvectors. What happens in this situation is not understood yet, however this special case could be investigated in future.

References

[1] Akmerov R.R., Sadovsky B.N. Notes on the theory of ODEs (in Russian).

[2] Alzabut J.O., Tunc C. Existence of periodic solutions for Rayleigh equations with state-dependent delay. Electronic Journal of Differential Equations, vol. 2012, No 77, 2012.

[3] Andersen C.M., Geer J.M.. Power series expansions for the frequency and period of the limit cycle of the Van-der-Pol equation. SIAM Journal of Applied Mathematics, vol. 42, No 3, 1982.

[4] Benguria R.D. ,Depassier M. C. Variational principle for the Limit Cycle of Rayleigh's Equation. Differential Equations and Mathematical Physics. Proceedings of an International Conference held at University of Alabama at Birmingam, 1999.

[5] Cartwright J., Eguiluz V., Hernandez-Garcia E., Piro O. Dynamics of elastic excitable media. International Journal of Bifururcations and Chaos, vol. 9, 1999.

[6] Fitzhugh R. Impulses and physiological states in theoretical models of nerve membrane. Biophysical journal, vol. 1, 1961.

[7] Gerstner W., Kistler W.M. Spiking Neuron Models. Single neurons, populations, plasticity. Cambridge University Press, 2002.

[8] Hodgkin A.L., Huxley A.F. A quantitative description of membrane potential and its application to conduction and excitation in nerve. Journal of physiology, vol. 117, 1952.

[9] Kuznetsov A.P, Kuznetsov S.P., Ryskin N.M. Nonlinear oscillations (in Russian). M: Fizmatlit, 2002.

[10] Kuznetsov Y.A. Elements of Applied Bifurcation Theory, Second Edition. Springer, 1998.

[11] Kyoung L.J. Wave pattern selection in an excitable system. Physical Review Letters, vol. 79, 1997.

[12] Melekhov A.P., Revina S.V. Onset of Self-Oscillations upon the Loss of Stability of Spatially Periodic Two-Dimensional Viscous

Fluid Flows Relative to Long-Wave Perturbations. Fluid Dynamics, vol. 13, 2008.

[13] Nagumo J., Arimoto S., Yoshizawa S. An active pulse transmission line simulating nerve axon. Proceedings of the IRE, 1962.

[14] Petrich D.M., Goldstein R.E. Nonlocal contour dynamics model for chemical front motion. Physical Review Letters, vol. 72, 1994.

[15] Popescu M.N., Nistor G. Asymptotic study of the Rayleigh equation. Atti dell' Accademia Peloritana dei Pericolanti. Classe di Scienze Fisiche, Matematishe e Naturali. Vol. LXXXV, 2007.

[16] Revina S.V., Yudovich V.I. Initiation of self-oscillations at loss of stability of spatially-periodic, three-dimensional viscous flows with respect to long-wave perturbations. Fluid Dynamics, 2001.

[17] Rinzel J., Keener J.P. Hopf bifurcation to repetetive activity in nerve. SIAM Journal of Applied Mathematics, vol. 43, No 4, 1983.

[18] Trenogin V.A. Functional Analysis (in Russian). M:Fizmatlit, 1980.

[19] Tuwankotta J.M. Studies on Rayleigh equation. Integral, vol. 5, No 1, 2000.

[20] Shian O.V. About the dynamics of the travelling waves in Van-der-Pol system with small diffusion. IAMM NASU Papers, vol. 16, 2008.

[21] Yudovich V.I. The onset of auto-oscillations in a fluid. PMM Vol. 35, No 4, 1971.

[22] Yudovich V.I. Investigation of auto-oscillations of a continuous medium, occurring at loss of stability of a stationary mode. PMM Vol. 36, No 3, 1972.

[23] http://www.mathworks.com/help/matlab/

List of Figures

APPENDIX A. Basic terms and definitions

In this appendix we will briefly discuss basic terms and definitions, which are necessary for the understanding of the contents of the present work. For more detailed description of these terms see ([10]).

Let us consider an autonomous system of ODEs:

$$\frac{d\vec{x}}{dt} = f(\vec{x}), \qquad \vec{x} \in \mathbb{R}^n \tag{131}$$

Definition. The point $\vec{x}_0 \in \mathbb{R}^n$ is called an equilibrium point of the system (131) if

$$f(\vec{x}_0) = 0$$

Definition. The equilibrium point $\vec{x}_0$ is called stable if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad ||\vec{x}(0) - \vec{x}_0|| < \delta \rightarrow ||\vec{x}(t) - \vec{x}_0|| < \varepsilon \quad \forall t \geq 0$$

Definition. The equilibrium point $\vec{x}_0$ is called asymptotically stable if it is stable and

$$\exists \delta > 0 : \quad ||\vec{x}(0) - \vec{x}_0|| < \delta \rightarrow \lim_{t \to +\infty} ||\vec{x}(t) - \vec{x}_0|| = 0$$

Definition. The equilibrium point $\vec{x}_0$ is called unstable if

$$\exists \varepsilon > 0 : \quad \forall \delta > 0 : \quad \exists x(t) : ||\vec{x}(0) - \vec{x}_0|| < \delta, \quad ||\vec{x}(t) - \vec{x}_0|| \geq \varepsilon \quad \forall t \geq 0$$

The stability of the equilibrium point $\vec{x}_0$ of the system 131 could be determined by using the following theorem:

Theorem(Lyapunov). Let us consider system (131) and the corresponding linearised system

$$\dot{\vec{x}} = A\vec{x}, \qquad A_{ij} = \frac{\partial f_i}{\partial x_j}|_{\vec{x}=\vec{x}_0} \tag{132}$$

Let us define as λ_j eigenvalues of the linearised system. Then if

$$\forall j \quad Re(\lambda_j) < 0 \rightarrow \vec{x}_0 \quad \text{is asymptotically stable}$$
$$\exists j : Re(\lambda_j) > 0 \rightarrow \vec{x}_0 \quad \text{is unstable}$$
$$\exists j : Re(\lambda_j) = 0 \rightarrow \text{the stability of} \quad \vec{x}_0 \quad \text{cannot be determined}$$

Hopf bifurcation Let us consider an autonomous ODE system, which depends on one real parameter μ

$$\dot{\vec{x}} = f(\vec{x}, \mu), \qquad \vec{x} \in \mathbb{R}^n \tag{133}$$

Let us assume that point $\vec{x}_0$ is the equilibrium point of the system (133). Then, the linearised system

$$\dot{\vec{x}} = A(\mu)\vec{x} \tag{134}$$

will also depend on the parameter μ. We will denote as $\lambda_j(\mu)$ the eigenvalues of the linear system (134).

Definition. The value μ_0 of the parameter μ is called critical if

$$\exists J : \forall j \in J \quad Re(\lambda_j(\mu_0)) = 0, \quad \forall j \notin J \quad Re(\lambda_j(\mu_0)) < 0$$

In the case, when $n = 2$ only two qualitative different situations can occur

$$Re(\lambda_1) = 0, \quad Re(\lambda_2) \leq 0 \quad - \quad \text{monotonic loss of stability}$$
$$Re(\lambda_{1,2}) = 0, \quad \lambda_{1,2} = \pm i\omega_0 \quad - \quad \text{oscillatory loss of stability}$$

In the case of oscillatory instability, Hopf bifurcation could take place in the system.

Definition. The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a bifurcation.

Definition. Hopf bifurcation is a local bifurcation in which an asymptotically stable equilibrium of ODE loses stability as a pair of complex conjugate eigenvalues of the linearised system cross the imaginary axis of the complex plane. Under several assumptions to ODE system, small-amplitude limit cycle will exists near the equilibrium point.

Definition. An isolated periodic solution of the system (133) is called a limit cycle.

Definition. A limit cycle is called stable, if all close solutions of the ODE tends to the periodic solution, when $t \to +\infty$.

Definition. A limit cycle is called unstable, if all close solutions of the ODE tends to the periodic solution, when $t \to -\infty$.

APPENDIX B. Using MATLAB function pdepe() for numerical simulations

In the course of the present work we use function pdepe() for the numerical solution of the PDE system. In this appendix we will describe this process in more detail.

According to MATLAB online help ([23]), pdepe() can solve initial-boundary value problems for parabolic-elliptic PDEs in the case of one space variable $x \in [a, b]$ on the time interval $t \in [t_0, t_1]$. The PDE has the following form

$$c(x, t, u, \frac{\partial u}{\partial x})\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}[x^m f(x, t, u, \frac{\partial u}{\partial x})] + s(x, t, u, \frac{\partial u}{\partial x})$$

with the boundary conditions at $x = a, b$

$$p(x, t, u) + q(x, t)f(x, t, u, \frac{\partial u}{\partial x}) = 0$$

The syntax of calling to this function is

```
sol = pdepe(m,pdefun,icfun,bcfun,xmesh,tspan)
```

Here, parameter m corresponds to the symmetry of the problem, $pdefun$ defines the expression for PDE, initial and boundary conditions are defined in $icfun$ and $bcfun$ respectively. The space and time intervals are described by the parameters $xmesh$ and $tspan$.

Function *pdepe*() returns the solution as multidimensional array. Therefore, in order to get the approximation for i-th component, the user should use $sol(:, :, i)$.

APPENDIX C. The idea of Lyapunov-Schmidt method, explained in the case of ODE

Definition. Let us consider nonlinear equation

$$F(x, \lambda) = 0$$

We will assume that this equation has the solution x_0, corresponding to the fixed value of the parameter λ_0. If other solutions, close to x_0 exists when the parameter λ is close to the value λ_0, then it is said that the bifurcation of the solution x_0 takes place.

Lyapunov-Schmidt method was initially developed in the theory of nonlinear equations for the analysis of the bifurcations of the solutions. More detailed description of these applications could be found in ([18]).

However, this method was modified by V.I. Yudovich for the analysis of periodic solutions in the case of Hopf bifurcation. For details see ([21], [22]). In this appendix we will explain the idea of the method in the case of nonlinear ODE.

We will consider the equation

$$\dot{x} = Ax + f(x, \mu), \qquad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}, \qquad \dot{} = \frac{d}{dt}$$

We will rewrite it as

$$\dot{x} = Ax + g(x, \delta), \qquad x \in \mathbb{R}^n, \quad \delta \in \mathbb{R}$$

here $\delta = \mu - \mu_0$, and as μ_0 is denoted the critical value of the parameter μ. A is the linear operator.

We assume that operator A has a pair of purely imaginary eigenvalues $\pm i\omega_0 \neq 0$ when $\mu = \mu_0$.

Then, if we introduce the unknown cyclic frequency ω, i.e. by substituting $\tau = \omega t$, we arrive at

$$\omega\dot{x} = Ax + g(x, \delta), \qquad x \in \mathbb{R}^n, \quad \delta \in \mathbb{R}, \qquad \dot{} = \frac{d}{d\tau}$$

We can look for the periodic solution of this equation in the form of series

$$x = \sum_{i=1}^{+\infty} \varepsilon^i x_i, \qquad \omega = \sum_{i=0}^{+\infty} \varepsilon^i \omega_i$$

After substituting of the series into our equation and equating the coefficients of like powers of ε, we will arrive at a chain of equations. However, all these equations will be linear and therefore will have analytical solutions. The first equation in the chain will be linear homogeneous equation

$$\omega_0 \dot{x}_1 = A x_1$$

The 2π-periodic solution of this equation could be written as

$$x_1 = \alpha_1 \vec{\varphi} e^{i\tau} + \text{c.j}, \qquad A\vec{\varphi} = i\omega_0 \vec{\varphi}$$

We require that $\alpha_1 > 0$. It remains unknown on the first step of the method. However, it will be determined from the condition of solvability in the next steps.

Next equations, in general, will be inhomogeneous linear equations. These equations could be written in the form

$$\omega_0 \dot{x}_n = A x_n + f_n, \qquad n = 2, 3, \ldots$$

These equations will have a periodic solution if and only if the condition of solvability is satisfied

$$\int_0^{2\pi} (f_n, \vec{\psi}) e^{-i\tau} d\tau = 0, \qquad A\vec{\psi} = -i\omega_0 \vec{\psi}$$

We can satisfy this condition of solvability by choosing the unknown values α_i, ω_i.

The meaning of this condition is very simple. It is clear that function f_n can be written as

$$f_n = \sum_{k=0}^{N} C_k e^{ik\tau}$$

Therefore, the solution of the inhomogeneous equation can contain periodic solutions $Me^{ik\tau}$ and "secular" terms $M\tau e^{ik\tau}$, where M is a constant. Roughly speaking, the condition of solvability guarantees that in the right side of the equation there will be no terms, which produce "secular" solutions. This can be seen from the formulation of the method in the terms of functional analysis, however we will not discuss it here. For further information one can see ([21]).

To sum up, the process of finding the periodic solution consists of the following steps:

1. Arrive at the chain of equations.
2. Solve first linear equation.
3. Consider next equation in chain.
4. Satisfy the condition of solvability for the equation, by the selection of α_i, ω_i, available from the previous steps.
5. Solve the inhomogeneous equation.
6. Return to step 3.

We should point out that the method is not limited by ODEs. It can be applied to PDEs as well. The idea remains the same, however the computations will become more difficult.