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The analysis of bifurcations in the spatially distributed Langford system

Examiners: Professor Heikki Haario
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ABSTRACT

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In this thesis the bifurcational behavior of the solutions of Langford system is analysed. The equilibriums of the Langford system are found, and the stability of equilibriums is discussed. The conditions of loss of stability are found. The periodic solution of the system is approximated. We consider three types of boundary condition for Langford spatially distributed system: Neumann conditions, Dirichlet conditions and Neumann conditions with additional requirement of zero average. We apply the Lyapunov-Schmidt method to Langford spatially distributed system for asymptotic approximation of the periodic mode. We analyse the influence of the diffusion on the behavior of self-oscillations. As well in the present work we perform numerical experiments and compare it with the analytical results.

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Sergey Klimov

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List of Symbols and Abbreviations

ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
c.j.	Complex conjugation
crit	critical value

1 INTRODUCTION

One of the most important problems in mathematical modeling is the prediction of the behavior of an object based on certain information of its initial state. This problem is reduced to finding a law, which allows, by using known information about the object at initial time t_0 in space point x_0, to define its future at any time $t > t_0$. Depending on difficulty of the object this law can be deterministic or probabilistic, can describe evolution of the object only in time, space or spatio-temporal ([15]).

Such systems, whose state changes in time, are called dynamical. In mathematics dynamical systems are related to differential equations. Many fundamental equations such as Newtons' and Hamilton's in mechanics, Maxwell's in electrodynamics and phenomenological equations, for instance, Hodgkin-Huxley model in biophysics of neuron, the Lotka-Volterra equations in biological problem "predator-prey" and Leontiev's model in economics, are written in a form of differential equations and computer representation of the last one — in form of difference equations.

From theoretical and practical points of view one of the most interesting and important problem is the analysis of the local bifurcations of the dynamical system, in particular, the task of restructuring phase portrait of the system near equilibrium, by changing the parameters of the system. Such bifurcations can lead to appearance of new stationary states, periodical oscillations of small amplitude. The theory of local bifurcations is well developed for smooth dynamical systems.

Significantly less studied bifurcations of dynamical systems with non-analytical or discontinuous nonlinearity, but there are a lot of known effective methods for it, such as method of dotted images, methods from the theory of polysemantic images and differential inclusion, methods of mathematical theory of the systems with hysteresis.

Many nonsmooth dynamical systems are characterized by the fact that properties of the smooth (continuous) of functions in the mathematical model can be broken on some manifolds of the system's phase portrait, codimension of which is equal to one. In tasks related to local bifurcations near stationary state of the system such manifolds can contain stationary state or be located near it. In particular, such models, which have rich bifurcation behavior, were suggested by Hopf for the modeling of turbulence in liquid.

In this thesis we consider the Langford system and use Lyapunov-Shmidt method to get the solution of this system. Then we modify the Langford system by adding space distribution and analyse it in the case of Neumann and Dirichlet boundary conditions. After that we study the connection between solutions of the Langford system and its modifications. In the end of the present work we show numerical experiments and solutions to compare it with the analytical one.

2 Theoretical background.

2.1 Turbulence

Turbulence is a complex behavior of a dissipative medium or a field, disordered, stochastic in time and space. Turbulence can be parametrized by several nondimensional quantities. The most often used is Reynolds number. Reynolds number represents the ratio of inertial forces to viscous forces. The viscous forces dominate at low Reynolds numbers and disturbances are damped rapidly. These disturbances begin to amplify as Reynolds number is increased and eventually transition into fully turbulent flows The fluid mixes irregularly during turbulent flow. Constant changes in the flow's behavior (wakes, vortexes, eddies) make flow rates difficult, if not impossible, to accurately measure. More detailed description of this phenomena could be found in ([16]).

The turbulent and laminar flows were described by Reynolds in 1883, when he was studying fluid motion in pipe (for details see [11]).

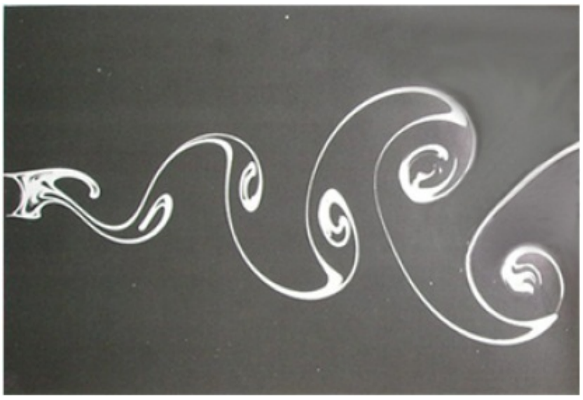

Figure 1: Vortex path

Turbulence is a three-dimensional unsteady viscous phenomenon that occurs at high Reynolds number. Turbulence is not a fluid property, but is a property of the flow itself. Turbulent flow can be highly nonlinear and is random in nature.

Turbulence causes the formation of eddies of many different length scales. Most of the kinetic energy of the turbulent motion is contained in the large-scale structures. The energy "cascades" from these large-scale structures to smaller scale structures by an inertial and essentially inviscid mechanism. This process continues, creating smaller and smaller structures which produces a hierarchy of eddies. Turbulent flow is well described by the system of Navier-Stokes equations.

Turbulence can be created by:

1. Increasing Reynolds number or Rayleigh number or Prandtl number.
2. Irradiating medium by high-intensity sound.
3. the chemical reactions, such as burning. The form of flame can be chaotic like waterfall.

Let us describe a few interesting facts about turbulence and its applications:

1. In a pipe with absolutely smooth sides in any continuum, which has a constant temperature and with velocity higher than critical, under only gravity force always spontaneously formed nonlinear waves and then

turbulence. If additionally create disturbing force or roughness on the pipe's surface, then turbulence also appears.

2. Flies, butterflies and birds use flapping fight. They create eddies during the flight, which help to create very high ascensional power, spending less energy.

3. Aircrafts have winglets. They help to save 4% of fuel, because it cause the decreasing of size and number of eddies behind the wings, which take useful kinetic energy.

4. A jet exhausting from a nozzle into a quiescent fluid. As the flow emerges into this external fluid, shear layers originating at the lips of the nozzle are created. These layers separate the fast moving jet from the external fluid, and at a certain critical Reynolds number they become unstable and break down to turbulence.

2.2 Hopf bifurcation

Let us first consider the classical dynamical Hopf system, which consists of two ordinary differential equations (ODE):

$$\dot{r} = \lambda r - r^3 \tag{1}$$

$$\dot{\varphi} = c \tag{2}$$

System depends on two parameters, one of them λ will be the main for us and the second one $c = const$. Here, we will perform a detailed analysis of the dynamics of the system. We will follow ([10]) in our discussion.

We will use this system for function $r(t)$ and $\varphi(t)$ to define functions $x(t)$ and $y(t)$, which can be written in polar coordinates:

$$x = rcos(\varphi), \qquad y = rsin(\varphi) \tag{3}$$

Let us add initial conditions to this ODE:

$$r\mid_{t=0} = r_0, \qquad \varphi\mid_{t=0} = \varphi_0 \tag{4}$$

and solve the Cauchy problem for some value of parameter λ. Then draw plots for $x(t)$ as function of time t and then draw it on phase plane as plot $x(y)$.

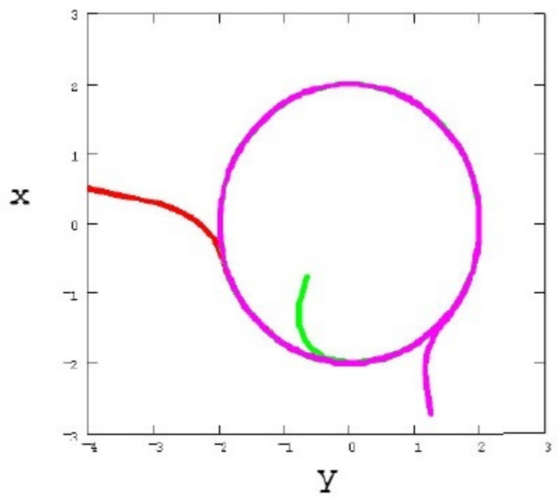

Figure 2: Phase plane as plot x(y) ([10])

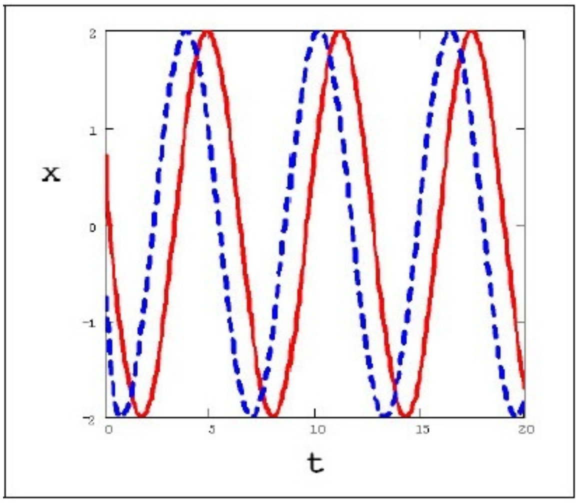

Figure 3: Solution of Hopf equation when $\lambda = 4$ ([10])

Solution is found by using Matlab and $\lambda = 4$.

Let us notice that the solution after some transition process comes to a oscillational mode, which is designated in phase plane as closed curve.

Now change the initial conditions and let us see the solutions. From plots we can see, that the solution, coming from another point, has the same behavior and comes to the same oscillation.

If we continue experiments and solve the Cauchy problem, changing initial conditions, we will get the same results. When $t \to \infty$ any solution will come to the same asymptotic oscillational mode. It is important that these asymptotic oscillations have the same frequency and amplitude as in the first one.

On phase plane all possible solutions of the Cauchy problem "wound" on closed curve. This curve is attractor and is called limit cycle. Oscillation process, describing this limit cycle, is called self-exciting oscillation. Amplitude of such oscillations does not depend on initial conditions and is defined by only equations of the dynamical system.

Solutions in form of self-exciting oscillation are possible only in significantly nonlinear dynamical systems. Dynamical Hopf system has nonlinearity of third order, which is in r^3. Besides, additional nonlinearity is introduced by definition of $x(t)$ and $y(t)$, i. e. if we express them as trigonometric function.

We can show that for this dynamical system amplitude of oscillation in limit cycle is equal $\sqrt{\lambda}$, i.e. depends on parameter of the dynamical system. The increasing of the value of λ causes increasing of the amplitude.

If we solve this system of the differential equations for $\lambda < 0$ we can see, that there is no limit cycle and only one special point is equilibrium point.

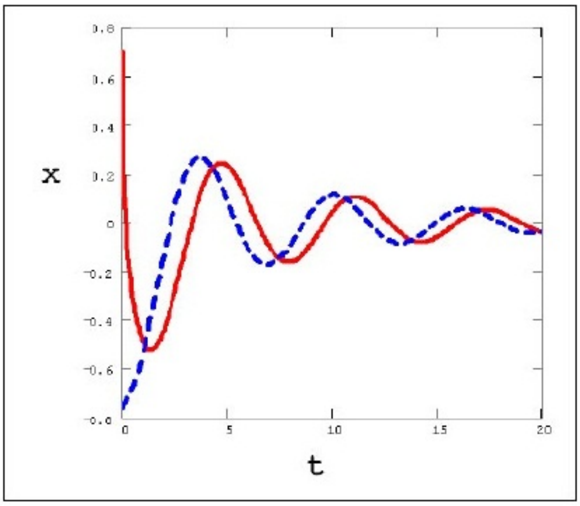

Figure 4: Solution for $\lambda < 0$ ([10])

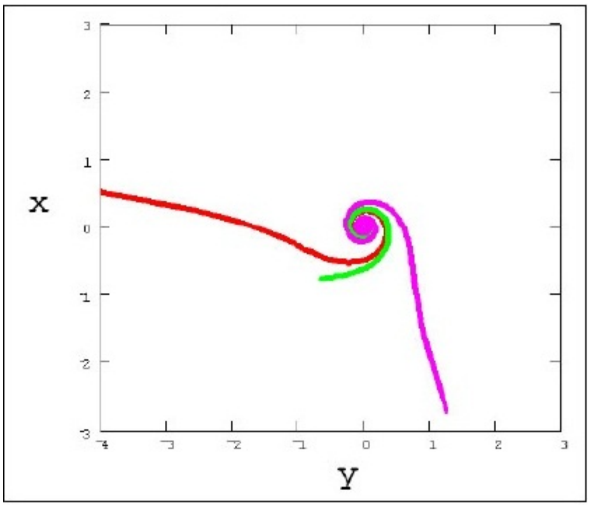

Figure 5: Solution for $\lambda < 0$ on phase plane ([10])

Thus, $\lambda = 0$ is bifurcational value of parameter. In this point "node" loses stability and instead of it the limit cycle appears. This bifurcation of appearance of the limit cycle from fixed point is called Hopf bifurcation. Restructuring of phase plane as the result of Hopf bifurcation is shown on figure.

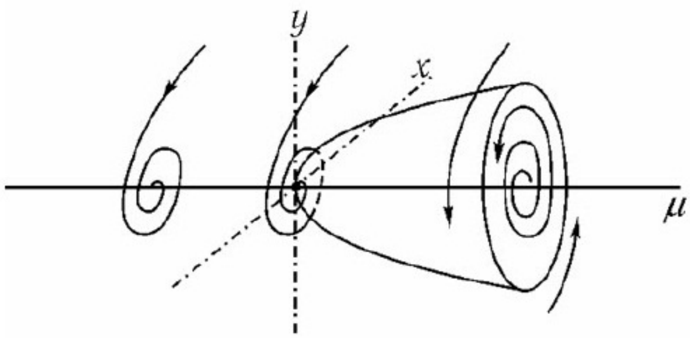

Figure 6: Hopf bifurcation

If the system have dimension of phase plane more than 2 we can get further bifurcation of appearance of the limit cycle.

Loss stability by periodical motion causes appearance of two-dimensional torus. Further, loss stability of this trajectory and increasing of Reynolds number $R > R_{crit}$ leads to the next attractive trajectory, i.e. we come from the two-dimensional torus to the third-dimensional. When we have k bifurcation we will get k-dimensional torus.

Thus, motion acquires difficult and complicated character, it is called turbulent unlike laminar, where fluid flows like layers, having different velocities.

2.3 Hopf model

Hopf model looks as following:

$$\begin{cases} \frac{\partial u}{\partial t} = -z \circ z^* - u \circ 1 + \mu \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial z}{\partial t} = -z \circ u + z \circ F^* + \mu \frac{\partial^2 z}{\partial x^2} \end{cases} \tag{5}$$

where μ is a parameter ,

$u(x + 2\pi, t) = u(x, t)$ is a real periodical function
$z(x + 2\pi, t) = z(x, t)$ is a real periodical function
$F(x) = a(x) + ib(x)$ is a known function

This system depends on two unknown complex functions: $u = u(x, t), z = z(x, t)$. Let us investigate Hopf system in more details (For more information see ([12])). When we write system (5) in complex form we will get system of four equations of four unknown variables. In what follows we confine ourselves to those solutions of (5) for which u, z are even functions of r and for which u is real. If we confine ourselves to the even solutions with u real, (5) splits upon setting $z = v + iw$. And we got:

$z \circ z^* = (v + iw) \circ (v - iw) = v \circ v - v \circ iw + v \circ iw + w \circ w = v \circ v + w \circ w$
$u \circ 1 =< u >$
$z \circ u = (v + iw) \circ u = v \circ u + iw \circ u$
$z \circ F^* = (v + iw) \circ (a - ib) = v \circ a + w \circ b + i(w \circ a - v \circ b)$
$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + i \frac{\partial^2 w}{\partial x^2}$

Then substitute $z, z \circ z^*, z \circ u, z \circ F^*$ in (5) and we will get the following system of real unknown u, v, w:

$$\begin{cases} \frac{\partial u}{\partial t} = -v \circ v - w \circ w - u \circ 1 + \mu \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial v}{\partial t} = v \circ u + v \circ a + w \circ b + \mu \frac{\partial^2 v}{\partial x^2} \\ \frac{\partial w}{\partial t} = w \circ u + w \circ a - v \circ b + \mu \frac{\partial^2 w}{\partial x^2} \end{cases} \tag{6}$$

2.4 Reduction of the Hopf model to a finite system. Langford system

Imagine u, v, w in model (6) as infinite series:
$u = \frac{u_0}{2} + \sum_{n=1}^{\infty} u_n cos(nx)$
$v = \frac{v_0}{2} + \sum_{n=1}^{\infty} v_n cos(nx)$
$w = \frac{w_0}{2} + \sum_{n=1}^{\infty} w_n cos(nx)$

Substitute them in equation for $n \geq 1$ and we get:

$$\begin{cases} \dot{u}_n = -v_n^2 - w_n^2 - n^2\mu u_n \\ \dot{v}_n = v_n u_n + v_n a_n + w_n b_n - n^2\mu v_n \\ \dot{w}_n = w_n u_n + w_n a_n - v_n b_n - n^2\mu w_n \end{cases} \tag{7}$$

Denote $\nu = \mu n^2$ and get:

$$\begin{cases} \dot{v}_n = (a_n - \nu)v_n + w_n b_n + v_n u_n \\ \dot{w}_n = (a_n - \nu)w_n - v_n b_n + w_n u_n \\ \dot{u}_n = -v_n^2 - w_n^2 - \nu u_n \end{cases} \tag{8}$$

Rewrite (5) as:

$$\begin{cases} \dot{v} = (a - \nu)v + wb + vu \\ \dot{w} = (a - \nu)w - vb + wu \\ \dot{u} = -v^2 - w^2 - \nu u \end{cases} \tag{9}$$

System, which was considered by Langford, looks as following:

$$\begin{cases} \dot{x}_1 = (\nu - 1)x_1 - x_2 + x_1 x_3 \\ \dot{x}_2 = x_1 + (\nu - 1)x_2 + x_2 x_3 \\ \dot{x}_3 = \nu x_3 - (x_1^2 + x_2^2 + x_3^2) \end{cases} \tag{10}$$

And can be obtained from Hopf system by adding in third equation quadratic dependence of third parameter. It is used for describing the behavior two-component fluid.

Then we will consider Langford system as:

$$\begin{cases} \dot{x_1} = (2\mu - 1)x_1 + x_1 x_3 - x_2 \\ \dot{x_2} = x_1 + (2\mu - 1)x_2 + x_2 x_3 \\ \dot{x_3} = (1 - 3\mu)x_3 + (2\mu - 1)x_3 - {x_2}^2 - {x_1}^2 - {x_3}^2 \end{cases} \tag{11}$$

Here μ – dimensionless control parameter. It is known that singular zero equilibrium point $1/2 < \mu < 0$ is asymptotically steady, when $\mu > 1/2$ - unsteady. When $\mu = 1/2$ there is bifurcation of the appearance of the limit cycle.

We will use system (10) to analyze equilibrium points. To avoid misunderstanding we will consider system (11) in the Lyapynov-Schmidt method.

If in system (11) take in consideration spatial distribution we come to the system of the partial differential equations (PDE):

$$\begin{cases} u_t = (2\mu - 1)u - v + u_{xx} + uw \\ v_t = (2\mu - 1)v + v_{xx} + u + vw \\ w_t = -\mu w - (u^2 + v^2 + w^2) + w_{xx}, \end{cases} \tag{12}$$

In case of Neumann and Dirichlet boundary conditions in the system the oscillating loss of the equilibrium takes place and auto oscillation mode exists.

2.4.1 The vector representation of the Langford system.

Let us write the Langford system as follows:

$$\begin{cases} \dot{x}_1 = (\nu - 1)x_1 - x_2 + x_1 x_3 \\ \dot{x}_2 = x_1 + (\nu - 1)x_2 + x_2 x_3 \\ \dot{x}_3 = \nu x_3 - (x_1^2 + x_2^2 + x_3^2) \end{cases} \tag{13}$$

Take
$w = (0, 0, 1)$
$x = (x_1, x_2, x_3)$

And calculate

$$[w, x] = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ x_1 & x_2 & x_3 \end{vmatrix} = -ix_2 + jx_1 = (-x_2, x_1, 0) \tag{14}$$

$$[[w, x], x] = \begin{vmatrix} i & j & k \\ -x_2 & x_1 & 0 \\ x_1 & x_2 & x_3 \end{vmatrix} = ix_1x_3 + jx_2x_3 + k(-x_1^2 - x_2^2) = (x_1x_3, x_2x_3, -x_1^2 - x_2^2) \tag{15}$$

Thus system (13) we can rewrite in the form of:

$$\dot{x} = (\nu - 1)x + (w, x)w + [w, x] + [[w, x], x] - (w, x)^2 w \tag{16}$$

System (16) is vector representation of the Langford system.

2.4.2 Equilibrium points of the Langford system.

Now we will find equilibrium points for the Langford system.

It is obviously that $x_0 = (0, 0, 0)$ satisfies the equation (16).

Next Let us find the another solution. Take the scalar product of the system by $[w, x]$:
$|[w, x]|^2 = 0$
$[w, x] = 0 \Leftrightarrow x = cw$

Substitute found x in equation (16) and get:

$(\nu - 1)cw + (w, cw)w + [w, cw] + [[w, cw], cw] - (w, cw)^2 w = 0$
$c(\nu - 1) + c - c^2 = 0$
$c \neq 0 \Rightarrow c = \nu$

Thus we got the second solution $x_1 = \nu w$

2.5 The linear analysis of the equilibriums.

2.5.1 Equilibrium $x = 0$

For studying of the steady zero equilibrium point let us linearise system (16) and get this system in the form of:

$\dot{y} = Ay$

After taking off the nonlinear component from ((16) we come to the system:

$\dot{y} = (\nu - 1)y + (w, y)w + [w, y]$

Let us study steady zero equilibrium point by representing y in complex form:

$\sigma y = Ay$
$\sigma y = (\nu - 1)y + (w, y)w + [w, y]$

Consider 2 cases:

1. When: $[w, y] = 0 \Rightarrow$ vectors w and y are collinear $w \parallel y$

If $y \neq 0 \Rightarrow y = cw \qquad c = 1$

Then we will get:

$\sigma w = (\nu - 1)w + (w, w)w$

Next we get the solution:

$\sigma_1 = \nu$

2. When vectors orthogonal: $y \perp w$

$(y, w) = 0$

Then

$\sigma y = (\nu - 1)y + [w, y]$
$(\sigma - \nu + 1)y = [w, y]$

The last system we can rewrite in the form of:

$\lambda y = \mathrm{B}y$

$\mathrm{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \lambda^2 + 1 = 0$

We get the solution: $\sigma_{2,3} = \pm i - 1 + \nu$

When $\nu < 0$ zero equilibrium is asymptotically steady, if $\nu = 0$ is critical case and if $\nu > 0$ equilibrium is unsteady.

2.5.2 Equilibrium $x = \nu w$

Let us consider:

$\dot{x} = (\nu - 1)x + (w, x)w + [w, x] + [[w, x], x] - (w, x)^2 w$

Then we come to the equation of perturbations, by doing the substitution $x = \nu w + u$

Notice that $x = \nu w$ is equilibrium point, then we get equality:

$(\nu - 1)\nu w + (w, \nu w)w - (w, \nu w)^2 w = 0$
$\dot{u} = (\nu - 1)\nu w + (\nu - 1)u + (w, \nu w)w + (w, u)w + [w, u] + [[w, \nu w + u], \nu w + u] - (w, \nu w + u)^2 w$
$\dot{u} = (\nu - 1)\nu w + (\nu - 1)u + (w, \nu w)w + (w, u)w + [w, u] + \nu u + u(u, w) - \nu w(w, u) - w(u, u) - \nu^2 w - 2\nu w(w, u) - w(w, u)^2$
$\dot{u} = (2\nu - 1)u + (1 - 3\nu)(w, u)w + [w, u] + u(u, w) - w(u, u) - w(w, u)^2$ the last on is called equation of perturbations.

Linearise the equation of perturbations:

$\dot{u} = (2\nu - 1)u + (1 - 3\nu)(w, u)w + [w, u]$

Consider two cases:

1. When: $u \parallel w \qquad u = cw$

$\sigma w = (2\nu - 1)w + (1 - 3\nu)(w, w)w$

We get the solution:

$\sigma_1 = -\nu$

2. When: $u \perp w$

$(\sigma - 2\nu + 1)y = [w, y]$
$\lambda y = [w, y]$

We get the solution:

$\lambda_{2,3} = \pm i \Rightarrow \sigma_{2,3} = 2\nu - 1 \pm i$

When $0 < \nu < \frac{1}{2}$ equilibrium is asymptotically steady, if $\nu = \frac{1}{2}$ is critical case and if $\nu < 0, \nu > \frac{1}{2}$ equilibrium is unsteady.

3 Langford ODE

In this section we will find a periodic solution for equation (11). For this purpose we will be using Lyapunov-Schmidt method. Firstly we will make a brief introduction in Lyapunov-Schmidt method. Lyapunov-Schmidt method was initially developed in the theory of nonlinear equations for the analysis of the bifurcations of the solutions and then was modified V.I.Yudovich for the analysis of periodic solutions in the case of Hopf bifurcation. Let us notice that this method is not only applied to ODEs but to PDEs as well.

Here we present a short scheme of the method:

1. Get the chain of equations
2. Solve first linear equation
3. Go to the next equation in the chain
4. Satisfy the condition of solvability for the equation
5. Solve the inhomogeneous equation

6. Repeat from the step 3

In section 3.3 we will consider Lyapunov-Schmidt method on the example of Langford system.

3.1 Operator form of the Langford system

Let us consider operator A:

$$\mathrm{A}\vec{\phi} = \mathrm{B}\vec{\phi} + \mu\mathrm{C}\vec{\phi} \tag{17}$$

Here $\vec{\phi} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, $\mathrm{B} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathrm{C} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Consider bilinear operator $K(\vec{x}, \vec{x})$, defined by the next rule:

$$K(\vec{\phi}, \vec{\phi}) = \begin{pmatrix} \phi_{11}\phi_{23} \\ \phi_{12}\phi_{23} \\ -\phi_{11}\phi_{21} - \phi_{12}\phi_{22} - \phi_{13}\phi_{23} \end{pmatrix}$$

Then system (11) could be rewritten as:

$$\dot{\vec{\phi}} = \mathrm{A}\vec{\phi} + K(\vec{\phi}, \vec{\phi}) \tag{18}$$

3.2 Eigenvalues and eigenvectors of the linearised system

Let us first find the critical values of the parameter. It can be found from the next two equations:

$$\mathrm{A}\vec{\phi} = \lambda\vec{\phi} \tag{19}$$

and the second one:

$$\vec{\phi} = \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \tag{20}$$

By substituting (20) into (19) we obtain:

$$\begin{pmatrix} 2\mu - 1 & -1 & 0 \\ 1 & 2\mu - 1 & 0 \\ 0 & 0 & -\mu \end{pmatrix} \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} = \lambda \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \qquad k = 0, 1, 2, \ldots \tag{21}$$

From here we can find eigenvalues of the matrix A: $\lambda_{1,2} = 2\mu - 1 \pm i;\ \lambda_3 = -\mu$.

Now it is easily to find the critical value of the parameter μ:

$$\mu_{\text{crit}} = \frac{1}{2}. \tag{22}$$

When $\mu > \frac{1}{2}$ the system is unstable.

Let us consider:

$$\mathrm{A}\vec{\phi} = \lambda_{1,2}\vec{\phi} \tag{23}$$

Where $\mu > \frac{1}{2}$, $\lambda_{1,2} = (2\mu - 1 \pm i)$.

In order to find eigenvectors that corresponding to eigenvalues $\lambda_{1,2}$, solve the next equation:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \pm i \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{24}$$

$$\Rightarrow \begin{cases} \mp ia - b = 0 \\ a \mp ib = 0 \\ \pm ic + \frac{1}{2}c = 0 \end{cases} \tag{25}$$

Therefore, vectors:

$$\vec{\phi}_{1,2} = \begin{pmatrix} \pm i \\ 1 \\ 0 \end{pmatrix} \tag{26}$$

By performing similar actions, we can find the eigenvalues of the matrix A^*. It is clear that matrix A^* will have the same eigenvalues as matrix A. It is easy to find that eigenvectors are:

$$\vec{\psi}_{1,2} = \begin{pmatrix} \mp i \\ 1 \\ 0 \end{pmatrix} \tag{27}$$

3.3 Applying the Lyapunov-Schmidt method to the Langford ODE

Let us consider (18) when $\mu_{\text{crit}} = \frac{1}{2}$.

Let us introduce new definitions:

$\mu = a + \delta$,
$a = \mu_{\text{crit}}$, $\delta \ll 1$.

then, equation (18) can be rewritten as:

$$\dot{\vec{\phi}} = \mathrm{B}\vec{\phi} + a\mathrm{C}\vec{\phi} + \delta\mathrm{C}\vec{\phi} + K(\vec{\phi}, \vec{\phi}) \tag{28}$$

Let us set:

$\tau = \omega t,$
$\delta = \varepsilon^2.$

Then, equation (28) will be rewritten in the next form:

$$\omega \dot{\vec{\phi}} = \mathrm{B}\vec{\phi} + a\mathrm{C}\vec{\phi} + \varepsilon^2 \mathrm{C}\vec{\phi} + K(\vec{\phi}, \vec{\phi}) \tag{29}$$

We will be looking for $\vec{\phi}$ and ω in the form of a series:

$$\vec{\phi} = \sum_{i=1}^{\infty} \varepsilon^i \vec{\phi}_i, \qquad \omega = \sum_{i=0}^{\infty} \varepsilon^i \omega_i \tag{30}$$

$\omega_0 = 1.$

By substituting (30) into (29) and equating the coefficients of like power of ε, we will arrive:

$$\varepsilon^1: \quad \omega_0 \dot{\vec{\phi}}_1 = \mathrm{B}\vec{\phi}_1 + a\mathrm{C}\vec{\phi}_1 \tag{31}$$

$$\varepsilon^2: \quad \omega_0 \dot{\vec{\phi}}_2 = \mathrm{B}\vec{\phi}_2 + a\mathrm{C}\vec{\phi}_2 - \omega_1 \dot{\vec{\phi}}_1 + K(\vec{\phi}_1, \vec{\phi}_1) \tag{32}$$

$$\varepsilon^3: \quad \omega_0 \dot{\vec{\phi}}_3 = \mathrm{B}\vec{\phi}_3 + a\mathrm{C}\vec{\phi}_3 + \mathrm{C}\vec{\phi}_1 - \omega_1 \dot{\vec{\phi}}_2 - \omega_2 \dot{\vec{\phi}}_1 + K(\vec{\phi}_1, \vec{\phi}_2) + K(\vec{\phi}_2, \vec{\phi}_1) \tag{33}$$

$$\begin{aligned} \varepsilon^4: \quad \omega_0 \dot{\vec{\phi}}_4 = {} & \mathrm{B}\vec{\phi}_4 + a\mathrm{C}\vec{\phi}_4 + \mathrm{C}\vec{\phi}_2 - \omega_1 \dot{\vec{\phi}}_3 - \omega_2 \dot{\vec{\phi}}_2 - \omega_3 \dot{\vec{\phi}}_1 + K(\vec{\phi}_2, \vec{\phi}_2) + \\ & + K(\vec{\phi}_1, \vec{\phi}_3) + K(\vec{\phi}_3, \vec{\phi}_1) \end{aligned} \tag{34}$$

(31) is a linear equation. Solution of (31) has the form:

$$\vec{\phi}_1 = \alpha_1 \vec{\varphi} e^{i\tau} + \text{c.j.}, \qquad \alpha_1 > 0 \tag{35}$$

An inhomogeneous equation (32) will have a solution if and only if the condition of solvability is satisfied:

$$\int_0^{2\pi} (-\omega_1 \dot{\vec{\phi}}_1 + K(\vec{\phi}_1, \vec{\phi}_1), \vec{\psi}) e^{-i\tau} d\tau = 0 \tag{36}$$

By performing necessary calculations, we obtain:

$$K(\vec{\phi}_1,\vec{\phi}_1) = (\alpha_1\vec{\varphi}e^{i\tau} + \alpha_1\vec{\varphi^*}e^{-i\tau}, \alpha_1\vec{\varphi}e^{i\tau} + \alpha_1\vec{\varphi^*}e^{-i\tau}) = \begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \tag{37}$$

$$\int\limits_0^{2\pi} (-\omega_1 \dot{\vec{\phi}}_1, \vec{\psi})e^{-i\tau}d\tau = -4i\pi\alpha_1\omega_1 \tag{38}$$

From $\alpha_1 > 0$ follows, that $\omega_1 = 0$. The solution of (32) is:

$$\dot{\vec{\phi}}_2 = \mathrm{B}\vec{\phi}_2 + a\mathrm{C}\vec{\phi}_2 + \begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \tag{39}$$

$$\dot{\vec{\phi}}_2 = \mathrm{A}\vec{\phi}_2 + \begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \tag{40}$$

$$\vec{\phi}_2 = \alpha_2\vec{\varphi}e^{i\tau} + \text{c.j.} - \begin{pmatrix} 0 \\ 0 \\ \frac{4\alpha_1^2}{a} \end{pmatrix} \tag{41}$$

Now, check the solvability condition for (33):

$$\int\limits_0^{2\pi} (-\omega_2 \dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_1 + K(\vec{\phi}_1,\vec{\phi}_2) + K(\vec{\phi}_2,\vec{\phi}_1), \vec{\psi})e^{-i\tau}d\tau = 0 \tag{42}$$

Let us find $K(\vec{\phi}_1,\vec{\phi}_2)$ and $K(\vec{\phi}_2,\vec{\phi}_1)$:

$$K(\vec{\phi}_2,\vec{\phi}_1)=\begin{pmatrix} i\alpha_2\vec{\varphi}e^{i\tau}-i\alpha_1\vec{\varphi^*}e^{-i\tau} & i\alpha_1\vec{\varphi}e^{i\tau}-i\alpha_1\vec{\varphi^*}e^{-i\tau} \\ \alpha_2\vec{\varphi}e^{i\tau}+\alpha_1\vec{\varphi^*}e^{-i\tau} & , \quad \alpha_1\vec{\varphi}e^{i\tau}+\alpha_1\vec{\varphi^*}e^{-i\tau} \\ -\frac{4\alpha_1^2}{a} & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -4\alpha_1\alpha_2 \end{pmatrix} \tag{43}$$

$$K(\vec{\phi}_1,\vec{\phi}_2)=\begin{pmatrix} i\alpha_1\vec{\varphi}e^{i\tau}-i\alpha_1\vec{\varphi^*}e^{-i\tau} & i\alpha_2\vec{\varphi}e^{i\tau}-i\alpha_1\vec{\varphi^*}e^{-i\tau} \\ \alpha_1\vec{\varphi}e^{i\tau}+\alpha_1\vec{\varphi^*}e^{-i\tau} & , \quad \alpha_2\vec{\varphi}e^{i\tau}+\alpha_1\vec{\varphi^*}e^{-i\tau} \\ 0 & -\frac{4\alpha_1^2}{a} \end{pmatrix} = \begin{pmatrix} (e^{i\tau}-e^{-i\tau})(-\frac{1}{a})4i\alpha_1^3 \\ (e^{i\tau}+e^{-i\tau})(-\frac{1}{a})4\alpha_1^3 \\ -4\alpha_1\alpha_2 \end{pmatrix} \tag{44}$$

After calculations, we will get:

$$\int_0^{2\pi}(-\omega_2\dot{\vec{\phi}}_1+\mathrm{C}\vec{\phi}_1+K(\vec{\phi}_1,\vec{\phi}_2)+K(\vec{\phi}_2,\vec{\phi}_1),\vec{\psi})e^{-i\tau}d\tau=2\pi(-\omega_2 i\alpha_1+2\alpha_1-\alpha_1^3\frac{4}{a})$$

By splitting up this expression into real and imaginary part, we will get:

$$\alpha_1^2=\frac{1a}{2},\quad \omega_2=0$$

Finally we get the following solutions:

$$\begin{cases} \vec{\phi}=0.7\varepsilon(e^{i\tau}\vec{\varphi}+\text{c.j.})+O(\varepsilon^2) \\ \omega=1+O(\varepsilon^3) \end{cases} \tag{45}$$

4 Spatially distributed Langford system

4.1 Operator form of the Langford PDE

Let us introduce operator A:

$$\mathrm{A}\vec{\phi} = \vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + \mu\mathrm{C}\vec{\phi} \tag{46}$$

Here $\vec{\phi} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, $\mathrm{B} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathrm{C} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Consider bilinear operator $K(\vec{x}, \vec{x})$, defined by the next rule:

$$K(\vec{\phi}, \vec{\phi}) = \begin{pmatrix} \phi_{11}\phi_{23} \\ \phi_{12}\phi_{23} \\ -\phi_{11}\phi_{21} - \phi_{12}\phi_{22} - \phi_{13}\phi_{23} \end{pmatrix}$$

Then system (12) could be rewritten as:

$$\dot{\vec{\phi}} = \mathrm{A}\vec{\phi} + K(\vec{\phi}, \vec{\phi}) \tag{47}$$

4.2 Neumann boundary conditions

4.2.1 Eigenvectors and eigenvalues of Langford PDE with Neumann boundary conditions

Let us consider (47) in case of Neumann boundary conditions.

System $cos(\pi kx)_{k=0}^{\infty}$ is the basis

Let us consider:

$$\mathrm{A}\vec{\phi} = \lambda\vec{\phi} \tag{48}$$

Writing down the basis decomposition of vector $\vec{\phi}$:

$$\vec{\phi} = \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \cos(\pi k x) \tag{49}$$

By substituting (49) into (48) we will get:

$$\begin{pmatrix} 2\mu - 1 - (\pi k)^2 & -1 & 0 \\ 1 & 2\mu - 1 - (\pi k)^2 & 0 \\ 0 & 0 & -\mu - (\pi k)^2 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} = \lambda \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \quad k = 0, 1, 2, \ldots \tag{50}$$

Now let us find the critical value of the parameter μ:

$$\mu_{\text{crit}} = \frac{1}{2}. \tag{51}$$

When $\mu > \frac{1}{2}$ the system loss the stability.

Let us consider:

$$\mathrm{A}\vec{\phi} = \lambda_{1,2}\vec{\phi} \tag{52}$$

Where $\mu > \frac{1}{2}$, $\lambda_{1,2} = (2\mu - 1 \pm i)$.

In order to find eigenvectors that corresponding to eigenvalues $\lambda_{1,2}$, solve the next equation:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \pm i \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{53}$$

$$\Rightarrow \begin{cases} \mp ia - b = 0 \\ a \mp ib = 0 \\ \pm ic + \frac{1}{2}c = 0 \end{cases} \tag{54}$$

Therefore, vectors:

$$\vec{\phi}_{1,2} = \begin{pmatrix} \pm i \\ 1 \\ 0 \end{pmatrix} \tag{55}$$

By performing similar actions, we can find the eigenvalues of the matrix A^*. It is clear that matrix A^* will have the same eigenvalues as matrix A. It is easy to find that eigenvectors are:

$$\vec{\psi}_{1,2} = \begin{pmatrix} \mp i \\ 1 \\ 0 \end{pmatrix} \tag{56}$$

4.2.2 Applying the Lyapunov-Schmidt method

Let us consider (47) when $\mu_{\text{crit}} = \frac{1}{2}$. New definitions:

$\mu = a + \delta$,
$a = \mu_{\text{crit}}$, $\delta \ll 1$.

Then, equation (47) will change the form into:

$$\dot{\vec{\phi}} = \vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + a\mathrm{C}\vec{\phi} + \delta\mathrm{C}\vec{\phi} + K(\vec{\phi}, \vec{\phi}) \tag{57}$$

Let us introduce new definitions:

$\tau = \omega t$,
$\delta = \varepsilon^2$.

(57) can be rewritten in the form:

$$\omega\dot{\vec{\phi}} = \vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + a\mathrm{C}\vec{\phi} + \varepsilon^2\mathrm{C}\vec{\phi} + K(\vec{\phi},\vec{\phi}) \tag{58}$$

We will be looking for $\vec{\phi}$ and ω in the form of a series:

$$\vec{\phi} = \sum_{i=1}^{\infty} \varepsilon^i \vec{\phi}_i, \qquad \omega = \sum_{i=0}^{\infty} \varepsilon^i \omega_i \tag{59}$$

$\omega_0 = 1.$

By substituting (59) into (58) and equating coefficients of like power of ε, we will arrive:

$$\varepsilon^1: \quad \omega_0\dot{\vec{\phi}}_1 = \frac{\partial^2\vec{\phi}_1}{\partial x^2} + \mathrm{B}\vec{\phi}_1 + a\mathrm{C}\vec{\phi}_1 \tag{60}$$

$$\varepsilon^2: \quad \omega_0\dot{\vec{\phi}}_2 = \frac{\partial^2\vec{\phi}_2}{\partial x^2} + \mathrm{B}\vec{\phi}_2 + a\mathrm{C}\vec{\phi}_2 - \omega_1\dot{\vec{\phi}}_1 + K(\vec{\phi}_1,\vec{\phi}_1) \tag{61}$$

$$\varepsilon^3: \quad \omega_0\dot{\vec{\phi}}_3 = \frac{\partial^2\vec{\phi}_3}{\partial x^2} + \mathrm{B}\vec{\phi}_3 + a\mathrm{C}\vec{\phi}_3 + \mathrm{C}\vec{\phi}_1 - \omega_1\dot{\vec{\phi}}_2 - \omega_2\dot{\vec{\phi}}_1 + K(\vec{\phi}_1,\vec{\phi}_2) + K(\vec{\phi}_2,\vec{\phi}_1) \tag{62}$$

$$\varepsilon^4: \quad \omega_0\dot{\vec{\phi}}_4 = \frac{\partial^2\vec{\phi}_4}{\partial x^2} + \mathrm{B}\vec{\phi}_4 + a\mathrm{C}\vec{\phi}_4 + \mathrm{C}\vec{\phi}_2 - \omega_1\dot{\vec{\phi}}_3 - \omega_2\dot{\vec{\phi}}_2 - \omega_3\dot{\vec{\phi}}_1 + K(\vec{\phi}_2,\vec{\phi}_2) + K(\vec{\phi}_1,\vec{\phi}_3) + K(\vec{\phi}_3,\vec{\phi}_1) \tag{63}$$

It should be pointed that there is no dependence on x. So the solution that we obtain for ODE will be solution for spatially distributed system with Neumann boundary conditions.

On the figures below the visualization of asymptotic is presented.

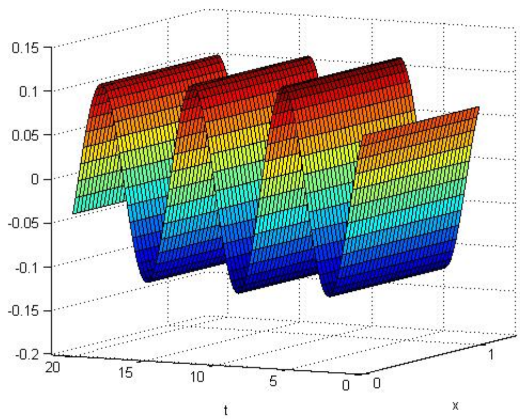

Figure 7: The asymptotic of u(x,t) $\mu = a + 0.01$

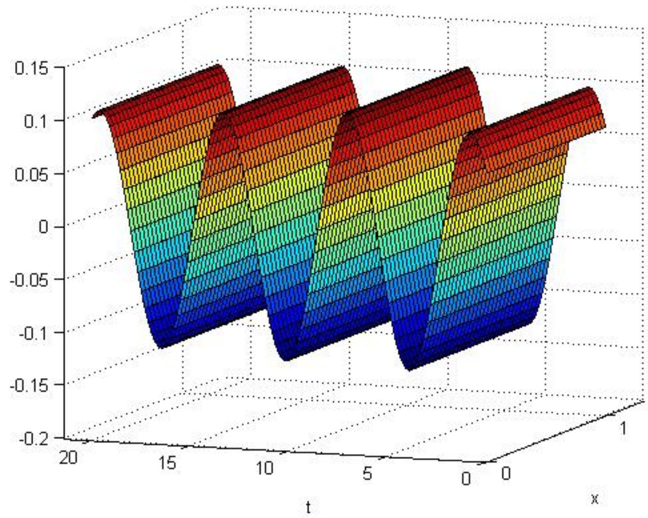

Figure 8: The asymptotic of v(x,t) when $\mu = a + 0.01$

On the next figures we plot numerical solution. Initial conditions were taken from asymptotic.

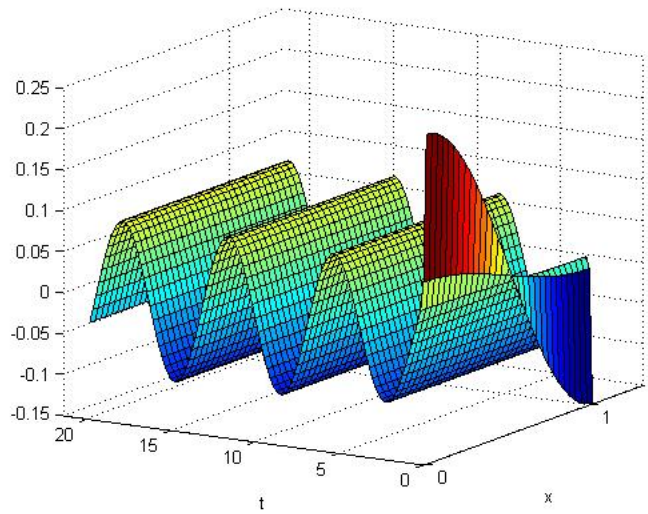

Figure 9: Numerical solution u(x,t) when $\mu = a + 0.01$

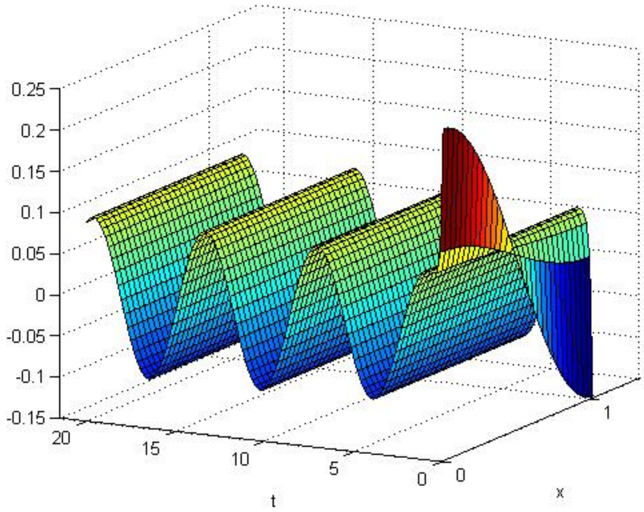

Figure 10: Numerical solution v(x,t) when $\mu = a + 0.01$

4.3 Dirichlet boundary conditions

4.3.1 Eigenvalues the system

Let us find critical value of the parameter:

$$\mathrm{A}\vec{\phi} = \lambda\vec{\phi} \tag{64}$$

Basis decomposition of vector $\vec{\phi}$ will be the next:

$$\vec{\phi} = \sum_{k=1}^{\infty} \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \sin(\pi k x) \tag{65}$$

By substituting (65) into (64) we will get:

$$\begin{pmatrix} 2\mu-1-(\pi k)^2 & -1 & 0 \\ 1 & 2\mu-1-(\pi k)^2 & 0 \\ 0 & 0 & -\mu-(\pi k)^2 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} = \lambda \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \qquad k = 1, 2, \ldots \tag{66}$$

Now we can calculate the critical value of the parameter μ:

$$\mu_{\text{crit}} = \frac{1+\pi^2}{2}. \tag{67}$$

When $\mu > \frac{1+\pi^2}{2}$ the system loss the stability.

4.3.2 Eigenvectors of the system

Let us consider a eigenvector problem:

$$\mathrm{A}\vec{\phi} = \lambda_{1,2}\vec{\phi} \tag{68}$$

Here $\mu > \frac{1+\pi^2}{2}$, $\lambda_{1,2} = (2\mu - 1 - (\pi k)^2 \pm i)$.

Eigenvectors could be find from the system:

$$\begin{pmatrix} 2\mu-1-\pi^2 & -1 & 0 \\ 1 & 2\mu-1-\pi^2 & 0 \\ 0 & 0 & -\mu-\pi^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (2\mu-1-\pi^2 \pm i) \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{69}$$

$$\begin{cases} (2\mu-1-\pi^2)a - b - (2\mu-1-\pi^2 \pm i)a = 0 \\ a + (2\mu-1-\pi^2)b - (2\mu-1-\pi^2 \pm i)a = 0 \\ (-\mu-\pi^2)c - (2\mu-1-\pi^2 \pm i)c = 0 \end{cases} \tag{70}$$

$$\Rightarrow \begin{cases} \mp ia - b = 0 \\ a \mp ib = 0 \\ c = 0 \end{cases} \tag{71}$$

$$\vec{\phi}_{1,2} = \begin{pmatrix} \pm i \\ 1 \\ 0 \end{pmatrix} \sin(\pi x) \tag{72}$$

By performing similar actions, we can find the eigenvalues of the matrix A*. It is clear that matrix A* will have the same eigenvalues as matrix A. It is easy to find that eigenvectors are:

$$\vec{\psi}_{1,2} = \begin{pmatrix} \mp i \\ 1 \\ 0 \end{pmatrix} \sin(\pi x) \tag{73}$$

4.3.3 Applying the Lyapunov-Schmidt method to the Langford system with Dirichlet boundary conditions

Let us consider equation (47) when $\mu_{\text{crit}} = \frac{1+\pi^2}{2}$.

We introduce new notation:

$\mu = a + \delta$,
$a = \mu_{\text{crit}}$, $\delta \ll 1$.

Therefore equation (47) can be rewritten as:

$$\dot{\vec{\phi}} = \vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + a\mathrm{C}\vec{\phi} + \delta\mathrm{C}\vec{\phi} + K(\vec{\phi}, \vec{\phi}) \tag{74}$$

Now introduce new time variable:

$\tau = \omega t$,
$\delta = \varepsilon^2$.

Then equation (74) could be rewritten as:

$$\omega\dot{\vec{\phi}} = \vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + a\mathrm{C}\vec{\phi} + \varepsilon^2\mathrm{C}\vec{\phi} + K(\vec{\phi},\vec{\phi}) \tag{75}$$

We will be looking for $\vec{\phi}$ and ω in the form of a series:

$$\vec{\phi} = \sum_{i=1}^{\infty}\varepsilon^i\vec{\phi}_i, \qquad \omega = \sum_{i=0}^{\infty}\varepsilon^i\omega_i \tag{76}$$

$\omega_0 = 1$.

By substituting (76) into (75) and equating coefficients of like power of ε, we will arrive:

$$\varepsilon^1 : \quad \omega_0\dot{\vec{\phi}}_1 = \frac{\partial^2\vec{\phi}_1}{\partial x^2} + \mathrm{B}\vec{\phi}_1 + a\mathrm{C}\vec{\phi}_1 \tag{77}$$

$$\varepsilon^2 : \quad \omega_0\dot{\vec{\phi}}_2 = \frac{\partial^2\vec{\phi}_2}{\partial x^2} + \mathrm{B}\vec{\phi}_2 + a\mathrm{C}\vec{\phi}_2 - \omega_1\dot{\vec{\phi}}_1 + K(\vec{\phi}_1,\vec{\phi}_1) \tag{78}$$

$$\varepsilon^3 : \quad \omega_0\dot{\vec{\phi}}_3 = \frac{\partial^2\vec{\phi}_3}{\partial x^2} + \mathrm{B}\vec{\phi}_3 + a\mathrm{C}\vec{\phi}_3 + \mathrm{C}\vec{\phi}_1 - \omega_1\dot{\vec{\phi}}_2 - \omega_2\dot{\vec{\phi}}_1 + {} + K(\vec{\phi}_1,\vec{\phi}_2) + K(\vec{\phi}_2,\vec{\phi}_1) \tag{79}$$

$$\varepsilon^4 : \quad \omega_0\dot{\vec{\phi}}_4 = \frac{\partial^2\vec{\phi}_4}{\partial x^2} + \mathrm{B}\vec{\phi}_4 + a\mathrm{C}\vec{\phi}_4 + \mathrm{C}\vec{\phi}_2 - \omega_1\dot{\vec{\phi}}_3 - \omega_2\dot{\vec{\phi}}_2 - \omega_3\dot{\vec{\phi}}_1 + K(\vec{\phi}_2,\vec{\phi}_2) + {} + K(\vec{\phi}_1,\vec{\phi}_3) + K(\vec{\phi}_3,\vec{\phi}_1) \tag{80}$$

Let us define: $\vec{\varphi} = \vec{\phi}_1$, $\vec{\psi} = \vec{\psi}_2$.

And calculate $(\vec{\phi},\vec{\psi})$:

$$(\vec{\phi},\vec{\psi}) = \int_0^1 (1+1)sin^2(\pi x)dx = 2\int_0^1 sin^2(\pi x)dx = 1 \tag{81}$$

Let us start solving equations. Equation (77) is a linear homogeneous equation. It is solution can be written in a form:

$$\vec{\phi}_1 = \alpha_1\vec{\varphi}e^{i\tau} + \text{c.j.}, \qquad \alpha_1 > 0 \tag{82}$$

Inhomogeneous equation (78) has solution if and only if the condition of solvability is satisfied:

$$\int_0^{2\pi} (-\omega_1 \dot{\vec{\phi}}_1 + K(\vec{\phi}_1, \vec{\phi}_1), \vec{\psi}) e^{-i\tau} d\tau = 0 \tag{83}$$

By performing calculations, we get:

$$K(\vec{\phi}_1, \vec{\phi}_1) = (\alpha_1 \vec{\varphi} e^{i\tau} + \alpha_1 \vec{\varphi}^* e^{-i\tau}, \alpha_1 \vec{\varphi} e^{i\tau} + \alpha_1 \vec{\varphi}^* e^{-i\tau}) = \begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \sin^2(\pi x) \tag{84}$$

$$\int_0^{2\pi} (-\omega_1 \dot{\vec{\phi}}_1, \vec{\psi}) e^{-i\tau} d\tau = -4i\pi\alpha_1\omega_1 \tag{85}$$

From the condition $\alpha_1 > 0$, we conclude that $\omega_1 = 0$.

Then the solution (78) can be written as:

$$\dot{\vec{\phi}}_2 = \frac{\partial^2 \vec{\phi}_2}{\partial x^2} + \mathrm{B}\vec{\phi}_2 + a\mathrm{C}\vec{\phi}_2 + \begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \sin^2(\pi x) \tag{86}$$

$$\dot{\vec{\phi}}_2 = \mathrm{A}\vec{\phi}_2 + \begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \sin^2(\pi x) = \mathrm{A}\vec{\phi}_2 + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \cos(2\pi x) \tag{87}$$

$$\vec{\phi}_2 = \vec{\phi}_{pr} + \vec{\phi}_{ob} \tag{88}$$

$$\vec{\phi}_{pr} = \vec{\phi}_0 + \vec{\phi}_1 \cos(2\pi x) \tag{89}$$

$$0 = \mathrm{A}(\vec{\phi}_0 + \vec{\phi}_1 cos(2\pi x)) + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix}\cos(2\pi x) \tag{90}$$

$$\Rightarrow \begin{cases} \mathrm{A}\vec{\phi}_0 = -\frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \\ \mathrm{A}(\vec{\phi}_1 cos(2\pi x)) = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix}\cos(2\pi x) \end{cases} \tag{91}$$

$$\begin{cases} -\vec{\phi}_0^1 - \vec{\phi}_0^2 + 2a\vec{\phi}_0^1 = 0 \\ \vec{\phi}_0^1 - \vec{\phi}_0^2 + 2a\vec{\phi}_0^2 = 0 \\ -a\vec{\phi}_0^3 = 2\alpha_1^2 \end{cases} \Rightarrow \vec{\phi}_0 = \begin{pmatrix} 0 \\ 0 \\ -\frac{2\alpha_1^2}{a} \end{pmatrix} \tag{92}$$

$$\begin{cases} -\vec{\phi}_1^1 - \vec{\phi}_1^2 + 2a\vec{\phi}_1^1 - 4\pi^2\vec{\phi}_1^1 = 0 \\ \vec{\phi}_1^1 - \vec{\phi}_1^2 + 2a\vec{\phi}_1^2 - 4\pi^2\vec{\phi}_1^2 = 0 \\ -a\vec{\phi}_1^3 - 4\pi^2\vec{\phi}_1^3 = -2\alpha_1^2 \end{cases} \Rightarrow \vec{\phi}_1 = \begin{pmatrix} 0 \\ 0 \\ \frac{2\alpha_1^2}{a+4\pi^2} \end{pmatrix} \tag{93}$$

$$\vec{\phi}_2 = \alpha_2\vec{\varphi}e^{i\tau} + \text{c.j.} - \begin{pmatrix} 0 \\ 0 \\ \frac{2\alpha_1^2}{a} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{2\alpha_1^2}{a+4\pi^2} \end{pmatrix} cos(2\pi x) \tag{94}$$

Now, let us satisfy the condition of solvability for the equation (79):

$$\int_0^{2\pi}(-\omega_2\dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_1 + K(\vec{\phi}_1, \vec{\phi}_2) + K(\vec{\phi}_2, \vec{\phi}_1), \vec{\psi})e^{-i\tau}d\tau = 0 \tag{95}$$

Find $K(\vec{\phi}_1, \vec{\phi}_2)$ and $K(\vec{\phi}_2, \vec{\phi}_1)$:

$$K(\vec{\phi}_2,\vec{\phi}_1) = \begin{pmatrix} i\alpha_2\vec{\varphi}e^{i\tau} - i\alpha_1\vec{\varphi}^*e^{-i\tau} & i\alpha_1\vec{\varphi}e^{i\tau} - i\alpha_1\vec{\varphi}^*e^{-i\tau} \\ \alpha_2\vec{\varphi}e^{i\tau} + \alpha_1\vec{\varphi}^*e^{-i\tau} & , \quad \alpha_1\vec{\varphi}e^{i\tau} + \alpha_1\vec{\varphi}^*e^{-i\tau} \\ \frac{2\alpha_1^2}{a+4\pi^2}cos(2\pi x) - \frac{2\alpha_1^2}{a} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \\ -4\alpha_1\alpha_2 \end{pmatrix} \sin^2(\pi x) \qquad (96)$$

$$K(\vec{\phi}_1,\vec{\phi}_2) = \begin{pmatrix} i\alpha_1\vec{\varphi}e^{i\tau} - i\alpha_1\vec{\varphi}^*e^{-i\tau} & i\alpha_2\vec{\varphi}e^{i\tau} - i\alpha_1\vec{\varphi}^*e^{-i\tau} \\ \alpha_1\vec{\varphi}e^{i\tau} + \alpha_1\vec{\varphi}^*e^{-i\tau} & , \quad \alpha_2\vec{\varphi}e^{i\tau} + \alpha_1\vec{\varphi}^*e^{-i\tau} \\ 0 & \frac{2\alpha_1^2}{a+4\pi^2}cos(2\pi x) - \frac{2\alpha_1^2}{a} \end{pmatrix}$$
$$= \begin{pmatrix} (e^{i\tau} - e^{-i\tau})(\frac{1}{a+4\pi^2}cos(2\pi x) - \frac{1}{a})2i\alpha_1^3 sin(\pi x) \\ (e^{i\tau} + e^{-i\tau})(\frac{1}{a+4\pi^2}cos(2\pi x) - \frac{1}{a})2\alpha_1^3 sin(\pi x) \\ -4\alpha_1\alpha_2\sin^2(\pi x) \end{pmatrix} \qquad (97)$$

By performing the calculations, we will get:

$\int\limits_0^{2\pi}(-\omega_2 \dot{\vec{\phi_1}} + \mathrm{C}\vec{\phi}_1 + K(\vec{\phi}_1,\vec{\phi}_2) + K(\vec{\phi}_2,\vec{\phi}_1),\vec{\psi})e^{-i\tau}d\tau = \int\limits_0^{2\pi}(-\omega_2 i\alpha_1\vec{\varphi}e^{i\tau} + \mathrm{C}\alpha_1\vec{\varphi}e^{i\tau} +$ c.j. $+$
$+ (\frac{1}{a+4\pi^2}cos(2\pi x) - \frac{1}{a})2\alpha_1^3 sin(\pi x)e^{i\tau} + ..., \vec{\psi})e^{-i\tau}d\tau = \int\limits_0^{2\pi}(-\omega_2 i\alpha_1 + 2\alpha_1 +$
$+ \int\limits_0^1(\frac{1}{a+4\pi^2}cos(2\pi x) - \frac{1}{a})2\alpha_1^3 sin^2(\pi x)dxd\tau = 2\pi(-\omega_2 i\alpha_1 + 2\alpha_1 - \alpha_1^3\frac{3a+8\pi}{2a(a+4\pi)})$

By splitting up this expression into real and imaginary part, we get:

$$\alpha_1^2 = \frac{4a(a+4\pi)}{3a+8\pi}, \quad \omega_2 = 0$$

Finally, we got the solutions:

$$\begin{cases} \vec{\phi} = 3.07\varepsilon(e^{i\tau}\vec{\varphi} + \text{c.j.}) + O(\varepsilon^2) \\ \omega = 1 + O(\varepsilon^3) \end{cases} \qquad (98)$$

On the figures below the visualization of asymptotic is presented.

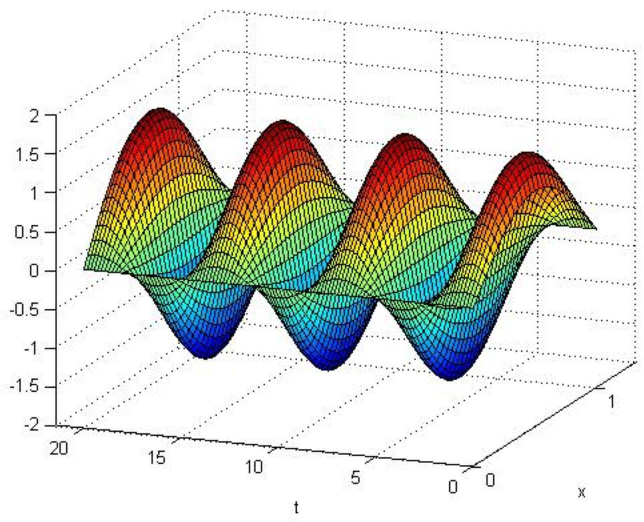

Figure 11: The asymptotic of u(x,t) $\mu = a + 0.01$

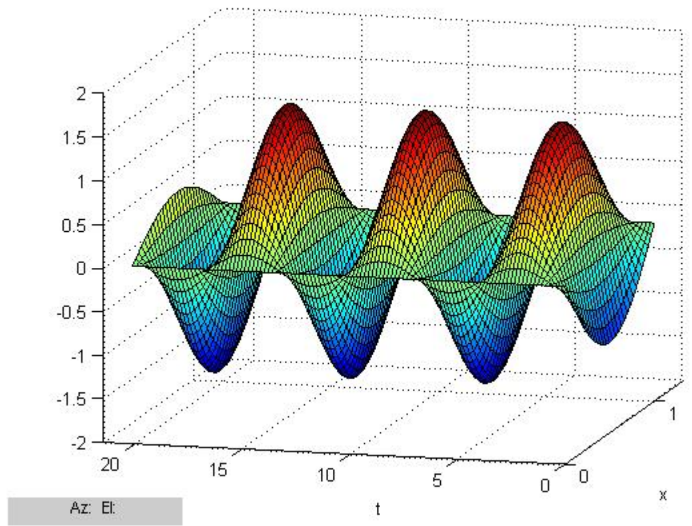

Figure 12: The asymptotic of v(x,t) when $\mu = a + 0.01$

On the next figures we plot numerical solution. Initial conditions were taken from asymptotic.

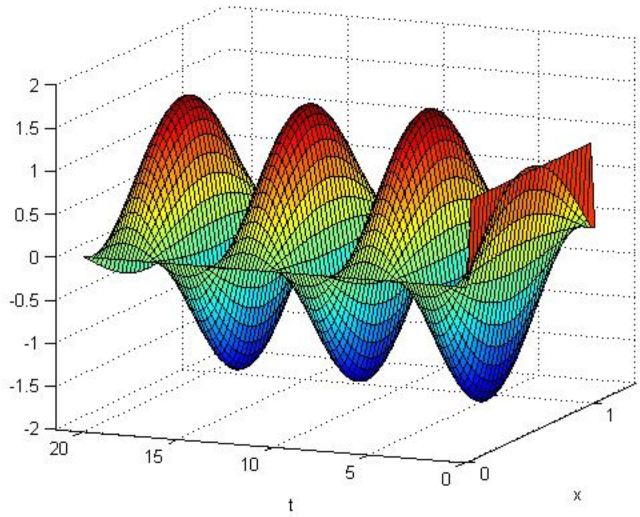

Figure 13: Numerical solution u(x,t) when $\mu = a + 0.01$

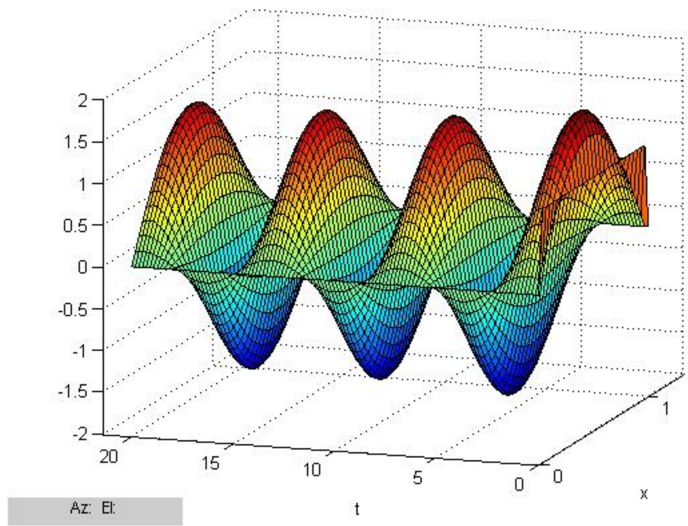

Figure 14: Numerical solution v(x,t) when $\mu = a + 0.01$

4.4 Neumann boundary conditions with additional requirement of zero average

Let us consider equation (47) in case of Neumann boundary conditions with additional requirement of zero average:

$$\int_0^1 u dx = 0 \qquad \int_0^1 v dx = 0 \qquad \int_0^1 w dx = 0 \tag{99}$$

system $cos(\pi kx)_{k=1}^{\infty}$ is base system

4.4.1 Eigenvalues and eigenvectors of Langford PDE

Let us consider:

$$\mathrm{A}\vec{\phi} = \lambda\vec{\phi} \tag{100}$$

We can write base decomposition of vector $\vec{\phi}$ in the form:

$$\vec{\phi} = \sum_{k=1}^{\infty} \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \cos(\pi kx) \tag{101}$$

By substituting (101) into (100) we get:

$$\begin{pmatrix} 2\mu - 1 - (\pi k)^2 & -1 & 0 \\ 1 & 2\mu - 1 - (\pi k)^2 & 0 \\ 0 & 0 & -\mu - (\pi k)^2 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} = \lambda \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \qquad k = 1, 2, \ldots \tag{102}$$

Now let us find the critical value of the parameter μ:

$$\mu_{\text{crit}} = \frac{1 + \pi^2}{2}. \tag{103}$$

When $\mu > \frac{1+\pi^2}{2}$ the system loss the stability.

Let us consider:

$$\mathrm{A}\vec{\phi} = \lambda_{1,2}\vec{\phi} \tag{104}$$

Where $\mu > \frac{1+\pi^2}{2}$, $\lambda_{1,2} = (2\mu - 1 - (\pi k)^2 \pm i)$.

In order to find eigenvectors that corresponding to eigenvalues $\lambda_{1,2}$, solve the next equation:

$$\begin{pmatrix} 2\mu - 1 - \pi^2 & -1 & 0 \\ 1 & 2\mu - 1 - \pi^2 & 0 \\ 0 & 0 & -\mu - \pi^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (2\mu - 1 - \pi^2 \pm i) \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{105}$$

$$\begin{cases} (2\mu - 1 - \pi^2)a - b - (2\mu - 1 - \pi^2 \pm i)a = 0 \\ a + (2\mu - 1 - \pi^2)b - (2\mu - 1 - \pi^2 \pm i)a = 0 \\ (-\mu - \pi^2)c - (2\mu - 1 - \pi^2 \pm i)c = 0 \end{cases} \tag{106}$$

$$\Rightarrow \begin{cases} \mp ia - b = 0 \\ a \mp ib = 0 \\ c = 0 \end{cases} \tag{107}$$

$$\vec{\phi}_{1,2} = \begin{pmatrix} \pm i \\ 1 \\ 0 \end{pmatrix} \cos(\pi x) \tag{108}$$

By performing similar actions, we can find the eigenvalues of the matrix A^*. It is clear that matrix A^* will have the same eigenvalues as matrix A. It is easy to find that eigenvectors are:

$$\vec{\psi}_{1,2} = \begin{pmatrix} \mp i \\ 1 \\ 0 \end{pmatrix} \cos(\pi x) \tag{109}$$

4.4.2 The analysis of nonlinear problem

Let us consider (47) when $\mu_{\text{crit}} = \frac{1+\pi^2}{2}$.

Let us define:

$\mu = a + \delta,$
$a = \mu_{\text{crit}},\ \delta \ll 1.$

Then, equation (47) will change the form into:

$$\dot{\vec{\phi}} = \vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + a\mathrm{C}\vec{\phi} + \delta\mathrm{C}\vec{\phi} + K(\vec{\phi}, \vec{\phi}) \tag{110}$$

Let us introduce new definitions:

$\tau = \omega t,$
$\delta = \varepsilon^2.$

(110) can be rewritten in the form:

$$\omega\dot{\vec{\phi}} = \vec{\phi}_{xx} + \mathrm{B}\vec{\phi} + a\mathrm{C}\vec{\phi} + \varepsilon^2\mathrm{C}\vec{\phi} + K(\vec{\phi}, \vec{\phi}) \tag{111}$$

We will be looking for $\vec{\phi}$ and ω in the form of a series:

$$\vec{\phi} = \sum_{i=1}^{\infty} \varepsilon^i \vec{\phi}_i, \qquad \omega = \sum_{i=0}^{\infty} \varepsilon^i \omega_i \tag{112}$$

$\omega_0 = 1.$

By substituting (112) into (111) and equating coefficients of like power of ε, we will arrive:

$$\varepsilon^1: \quad \omega_0 \dot{\vec{\phi}}_1 = \frac{\partial^2 \vec{\phi}_1}{\partial x^2} + \mathrm{B}\vec{\phi}_1 + a\mathrm{C}\vec{\phi}_1 \tag{113}$$

$$\varepsilon^2: \quad \omega_0 \dot{\vec{\phi}}_2 = \frac{\partial^2 \vec{\phi}_2}{\partial x^2} + \mathrm{B}\vec{\phi}_2 + a\mathrm{C}\vec{\phi}_2 - \omega_1 \dot{\vec{\phi}}_1 + K(\vec{\phi}_1, \vec{\phi}_1) \tag{114}$$

$$\varepsilon^3: \quad \omega_0 \dot{\vec{\phi}}_3 = \frac{\partial^2 \vec{\phi}_3}{\partial x^2} + \mathrm{B}\vec{\phi}_3 + a\mathrm{C}\vec{\phi}_3 + \mathrm{C}\vec{\phi}_1 - \omega_1 \dot{\vec{\phi}}_2 - \omega_2 \dot{\vec{\phi}}_1 + \\ + K(\vec{\phi}_1, \vec{\phi}_2) + K(\vec{\phi}_2, \vec{\phi}_1) \tag{115}$$

$$\varepsilon^4: \quad \omega_0 \dot{\vec{\phi}}_4 = \frac{\partial^2 \vec{\phi}_4}{\partial x^2} + \mathrm{B}\vec{\phi}_4 + a\mathrm{C}\vec{\phi}_4 + \mathrm{C}\vec{\phi}_2 - \omega_1 \dot{\vec{\phi}}_3 - \omega_2 \dot{\vec{\phi}}_2 - \omega_3 \dot{\vec{\phi}}_1 + K(\vec{\phi}_2, \vec{\phi}_2) + \\ + K(\vec{\phi}_1, \vec{\phi}_3) + K(\vec{\phi}_3, \vec{\phi}_1) \tag{116}$$

Let us define: $\vec{\varphi} = \vec{\phi}_1$, $\vec{\psi} = \vec{\psi}_2$.

And calculate $(\vec{\phi}, \vec{\psi})$:

$$(\vec{\phi}, \vec{\psi}) = \int_0^1 (1+1)cos^2(\pi x)dx = 2\int_0^1 cos^2(\pi x)dx = 1 \tag{117}$$

Let us start solving equations. Equation (113) is a linear homogeneous equation. It is solution can be written in a form:

$$\vec{\phi}_1 = \alpha_1 \vec{\varphi} e^{i\tau} + \text{c.j.}, \qquad \alpha_1 > 0 \tag{118}$$

Inhomogeneous equation (114) has solution if and only if the condition of solvability is satisfied:

$$\int_0^{2\pi} (-\omega_1 \dot{\vec{\phi}}_1 + K(\vec{\phi}_1, \vec{\phi}_1), \vec{\psi}) e^{-i\tau} d\tau = 0 \tag{119}$$

By performing calculations, we get:

$$K(\vec{\phi}_1, \vec{\phi}_1) = (\alpha_1 \vec{\varphi} e^{i\tau} + \alpha_1 \vec{\varphi}^* e^{-i\tau}, \alpha_1 \vec{\varphi} e^{i\tau} + \alpha_1 \vec{\varphi}^* e^{-i\tau}) = \begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \cos^2(\pi x) \tag{120}$$

$$\int_0^{2\pi} (-\omega_1 \dot{\vec{\phi}}_1, \vec{\psi}) e^{-i\tau} d\tau = -4i\pi\alpha_1\omega_1 \tag{121}$$

From the condition $\alpha_1 > 0$, we conclude that $\omega_1 = 0$.

Then the solution (114) can be written as:

$$\dot{\vec{\phi}}_2 = \frac{\partial^2 \vec{\phi}_2}{\partial x^2} + \mathrm{B}\vec{\phi}_2 + a\mathrm{C}\vec{\phi}_2 + \begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \cos^2(\pi x) \tag{122}$$

$$\dot{\vec{\phi}}_2 = \mathrm{A}\vec{\phi}_2 + \begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \cos^2(\pi x) = \mathrm{A}\vec{\phi}_2 + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \cos(2\pi x) \tag{123}$$

$$\vec{\phi}_2 = \vec{\phi}_{pr} + \vec{\phi}_{ob} \tag{124}$$

$$\vec{\phi}_{pr} = \vec{\phi}_0 + \vec{\phi}_1 \cos(2\pi x) \tag{125}$$

$$0 = \mathrm{A}(\vec{\phi}_0 + \vec{\phi}_1 cos(2\pi x)) + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \cos(2\pi x) \tag{126}$$

$$\Rightarrow \begin{cases} \mathrm{A}\vec{\phi}_0 = -\frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \\ \mathrm{A}(\vec{\phi}_1 cos(2\pi x)) = -\frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -4\alpha_1 \end{pmatrix} \cos(2\pi x) \end{cases} \tag{127}$$

$$\begin{cases} -\vec{\phi}_0^1 - \vec{\phi}_0^2 + 2a\vec{\phi}_0^1 = 0 \\ \vec{\phi}_0^1 - \vec{\phi}_0^2 + 2a\vec{\phi}_0^2 = 0 \\ -a\vec{\phi}_0^3 = 2\alpha_1^2 \end{cases} \Rightarrow \vec{\phi}_0 = \begin{pmatrix} 0 \\ 0 \\ -\frac{2\alpha_1^2}{a} \end{pmatrix} \tag{128}$$

$$\begin{cases} -\vec{\phi}_1^1 - \vec{\phi}_1^2 + 2a\vec{\phi}_1^1 - 4\pi^2\vec{\phi}_1^1 = 0 \\ \vec{\phi}_1^1 - \vec{\phi}_1^2 + 2a\vec{\phi}_1^2 - 4\pi^2\vec{\phi}_1^2 = 0 \\ -a\vec{\phi}_1^3 - 4\pi^2\vec{\phi}_1^3 = 2\alpha_1^2 \end{cases} \Rightarrow \vec{\phi}_1 = \begin{pmatrix} 0 \\ 0 \\ \frac{2\alpha_1^2}{a+4\pi^2} \end{pmatrix} \tag{129}$$

$$\vec{\phi}_2 = \alpha_2\vec{\varphi}e^{i\tau} + \text{c.j.} - \begin{pmatrix} 0 \\ 0 \\ \frac{2\alpha_1^2}{a} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \frac{2\alpha_1^2}{a+4\pi^2} \end{pmatrix} cos(2\pi x) \tag{130}$$

Now, let us satisfy the condition of solvability for the equation (115):

$$\int_0^{2\pi} (-\omega_2 \dot{\vec{\phi}}_1 + \mathrm{C}\vec{\phi}_1 + K(\vec{\phi}_1, \vec{\phi}_2) + K(\vec{\phi}_2, \vec{\phi}_1), \vec{\psi})e^{-i\tau}d\tau = 0 \tag{131}$$

Find $K(\vec{\phi}_1, \vec{\phi}_2)$ and $K(\vec{\phi}_2, \vec{\phi}_1)$:

$$K(\vec{\phi}_2, \vec{\phi}_1) = \begin{pmatrix} i\alpha_2\vec{\varphi}e^{i\tau} - i\alpha_1\vec{\varphi}^*e^{-i\tau} & i\alpha_1\vec{\varphi}e^{i\tau} - i\alpha_1\vec{\varphi}^*e^{-i\tau} \\ \alpha_2\vec{\varphi}e^{i\tau} + \alpha_1\vec{\varphi}^*e^{-i\tau} & , \quad \alpha_1\vec{\varphi}e^{i\tau} + \alpha_1\vec{\varphi}^*e^{-i\tau} \\ -\frac{2\alpha_1^2}{a+4\pi^2}cos(2\pi x) - \frac{2\alpha_1^2}{a} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \\ -4\alpha_1\alpha_2 \end{pmatrix} \cos^2(\pi x) \tag{132}$$

$$K(\vec{\phi}_1,\vec{\phi}_2)=\begin{pmatrix} i\alpha_1\vec{\varphi}e^{i\tau}-i\alpha_1\vec{\varphi^*}e^{-i\tau} & i\alpha_2\vec{\varphi}e^{i\tau}-i\alpha_1\vec{\varphi^*}e^{-i\tau} \\ \alpha_1\vec{\varphi}e^{i\tau}+\alpha_1\vec{\varphi^*}e^{-i\tau} \quad , & \alpha_2\vec{\varphi}e^{i\tau}+\alpha_1\vec{\varphi^*}e^{-i\tau} \\ 0 & -\frac{2\alpha_1^2}{a+4\pi^2}cos(2\pi x)-\frac{2\alpha_1^2}{a} \end{pmatrix}$$
$$=\begin{pmatrix} (e^{i\tau}-e^{-i\tau})(-\frac{1}{a+4\pi^2}cos(2\pi x)-\frac{1}{a})2i\alpha_1^3cos(\pi x) \\ (e^{i\tau}+e^{-i\tau})(-\frac{1}{a+4\pi^2}cos(2\pi x)-\frac{1}{a})2\alpha_1^3cos(\pi x) \\ -4\alpha_1\alpha_2\cos^2(\pi x) \end{pmatrix} \tag{133}$$

By performing the calculations, we will get:

$\int\limits_0^{2\pi}(-\omega_2\dot{\vec{\phi_1}}+\mathrm{C}\vec{\phi}_1+K(\vec{\phi}_1,\vec{\phi}_2)+K(\vec{\phi}_2,\vec{\phi}_1),\vec{\psi})e^{-i\tau}d\tau=\int\limits_0^{2\pi}(-\omega_2 i\alpha_1\vec{\varphi}e^{i\tau}+\mathrm{C}\alpha_1\vec{\varphi}e^{i\tau}+$ c.j. $-$

$-\,(\frac{1}{a+4\pi^2}cos(2\pi x)-\frac{1}{a})2\alpha_1^3cos(\pi x)e^{i\tau}+...,\vec{\psi})e^{-i\tau}d\tau=\int\limits_0^{2\pi}(-\omega_2 i\alpha_1+2\alpha_1-$

$-\int\limits_0^1(\frac{1}{a+4\pi^2}cos(2\pi x)-\frac{1}{a})2\alpha_1^3cos^2(\pi x)dxd\tau=2\pi(-\omega_2 i\alpha_1+2\alpha_1-\alpha_1^3\frac{3a+8\pi}{2a(a+4\pi)})$

By splitting up this expression into real and imaginary part, we get:

$$\alpha_1^2=\frac{4a(a+4\pi)}{3a+8\pi},\quad \omega_2=0$$

Finally we have the solutions:

$$\begin{cases} \vec{\phi}=3.07\varepsilon(e^{i\tau}\vec{\varphi}+\text{c.j.})+O(\varepsilon^2) \\ \omega=1+O(\varepsilon^3) \end{cases} \tag{134}$$

5 Numerical experiments

In this section we will discuss the results of the numerical experiments, which were performed in order to support our theoretical results. Our main purpose is to visualise the results, obtained in the previous sections and to give graphical explanations of the phenomena, which appear in the system under study. MATLAB software is being used for numerical computations.

5.1 Dirichlet boundary conditions

5.1.1 The behaviour of the system when $\mu < \mu_{cr}$

However, when $\mu < a$, there will be no self-oscillations in the system. The solutions of the system will decay to zero, when $t \to +\infty$.

In the following figures we show the results of numerical simulation. Here we have taken $\mu < a$. Initial conditions for $u(x,t)$, $v(x,t)$ and $w(x,t)$ are taken from asymptotic. We will show only the solution for $u(x,t)$, because $v(x,t)$ and $w(x,t)$ behaves in the similar way.

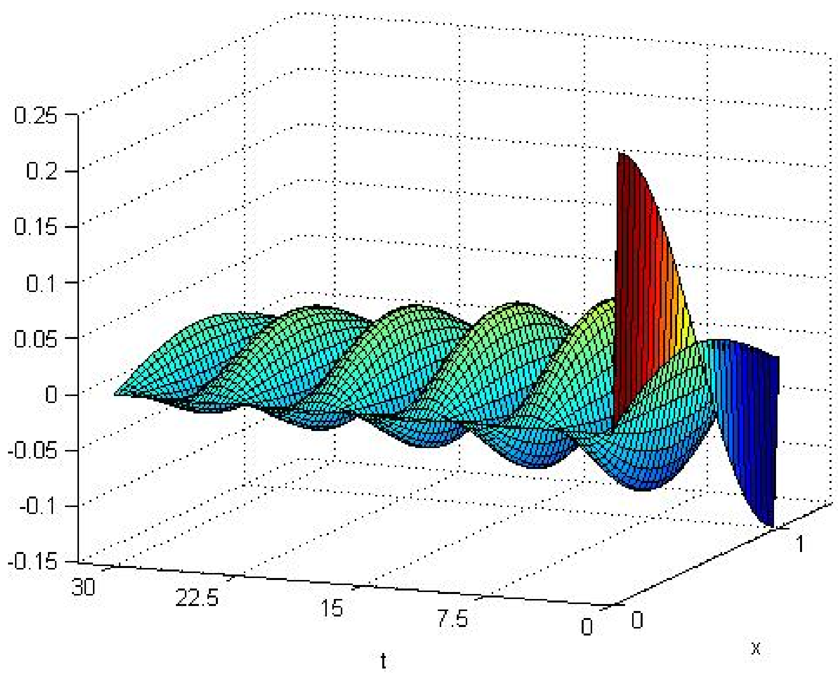

Figure 15: The numerical solution when $\mu = a - 0.01$

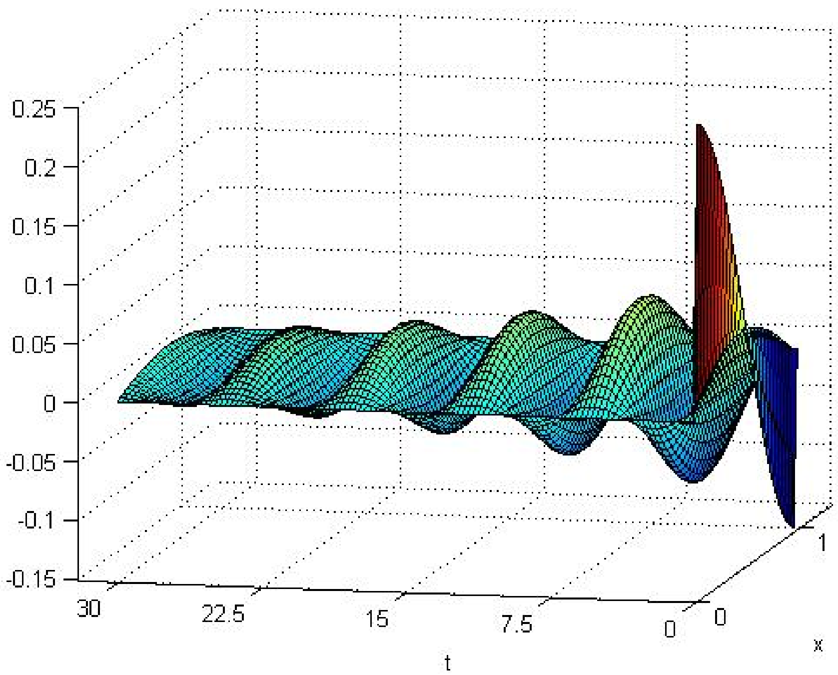

Figure 16: The numerical solution when $\mu = a - 0.05$

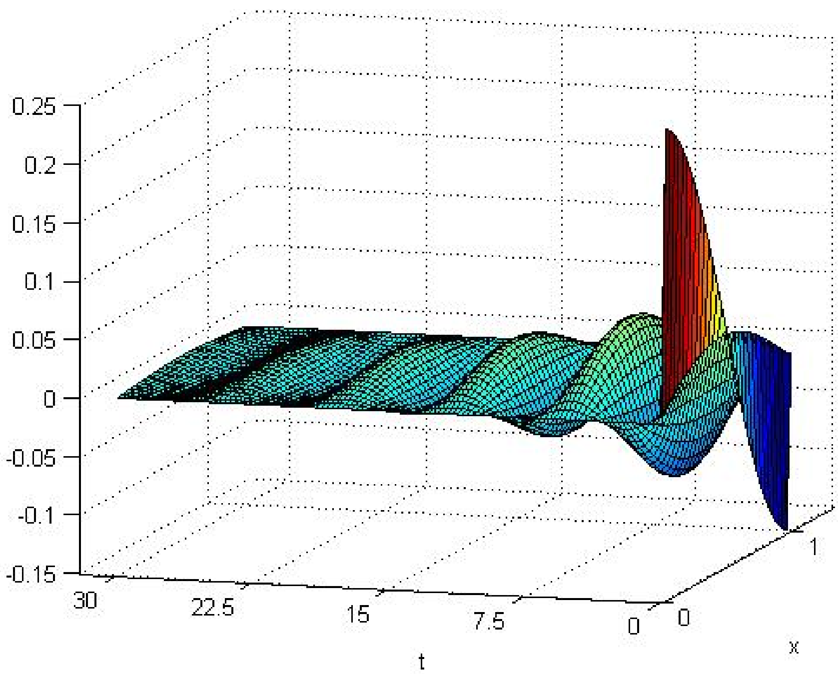

Figure 17: The numerical solution when $\mu = a - 0.1$

We can see from the figures, that the solution decays to zero and that the speed of decay is increasing when $\mu \searrow 0$.

5.1.2 The behavior of the system when $\mu > \mu_{cr}$

Now we will perform several simulations in the case when $\mu > a$ and the self-oscillations are presented in the system. We will illustrate the stability of the periodic mode. We will fix $\mu = a + 0.1$ and perform several numerical simulations with different initial conditions. On the following figures we plot the results.

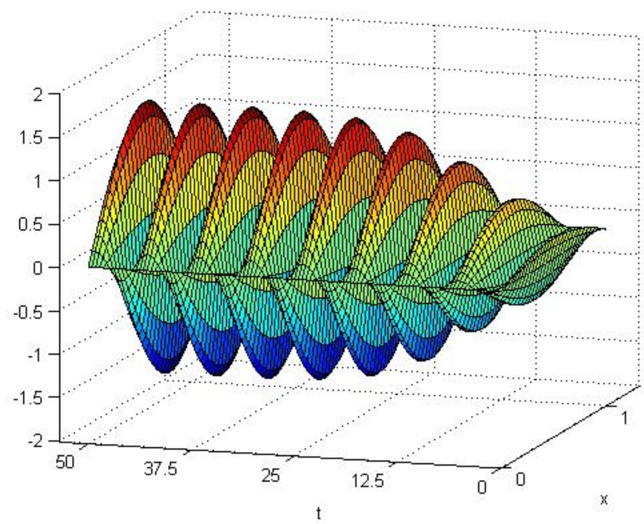

Figure 18: The numerical solution when $u_0(x) = v_0(x) = x(1 - x)$

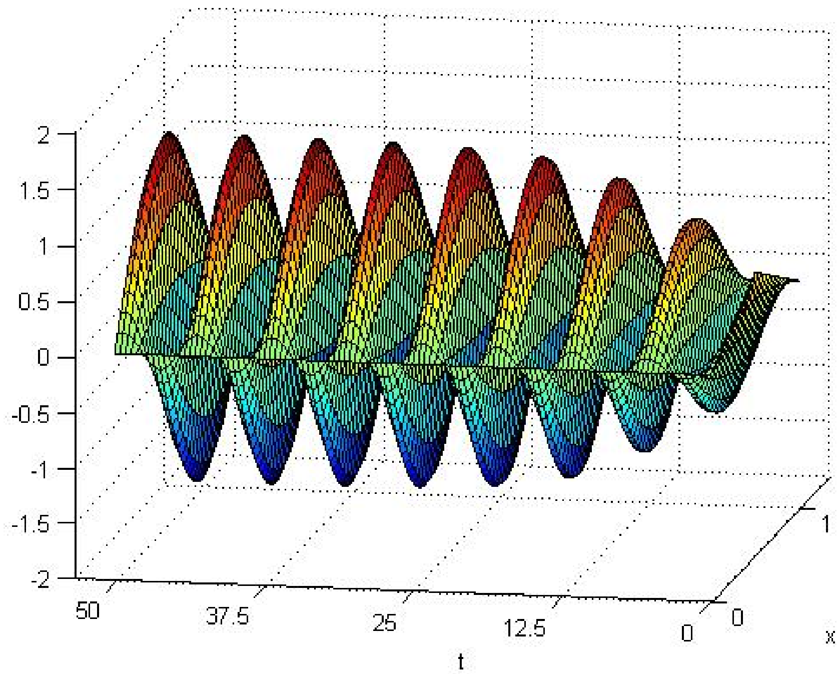

Figure 19: The numerical solution when $u_0(x) = v_0(x) = 0.5 - |x - 0.5|$

We observe from this figures, that the solutions with different initial conditions tends to the stable periodic solution, approximated earlier, when $t \to +\infty$.

5.2 Neumann boundary conditions

5.2.1 The behaviour of the system when $\mu < \mu_{cr}$

Again, we consider our system in the case when $\mu < 0$. In the following figures we show the results of numerical simulation Initial conditions for $u(x,t)$ are taken from asymptotic. We will again show only the solution for $u(x,t)$.

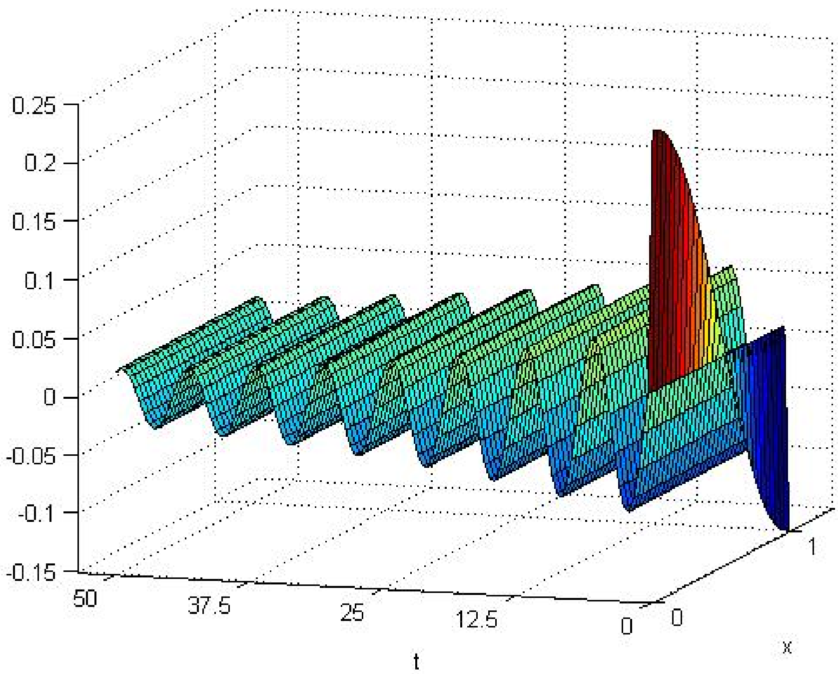

Figure 20: The numerical solution when $\mu = -0.1$

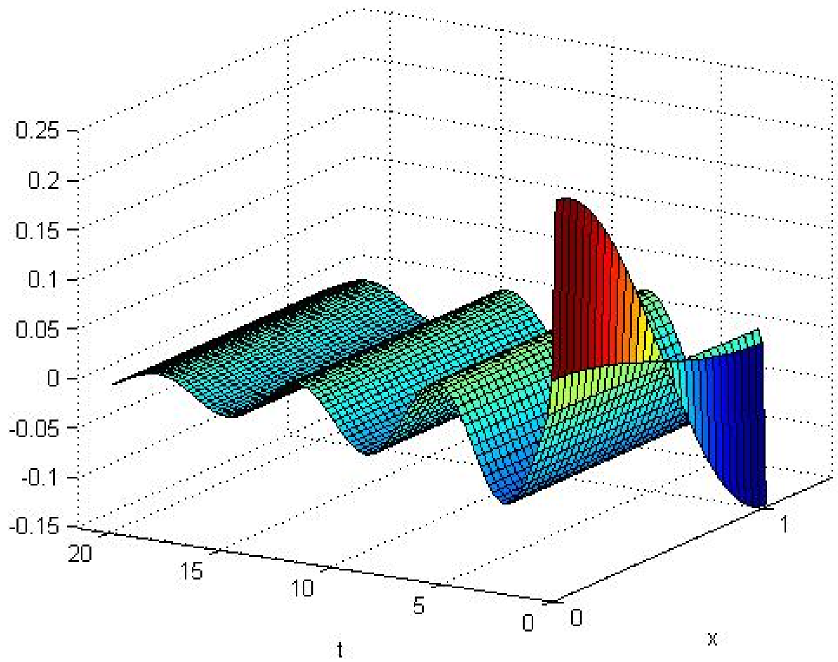

Figure 21: The numerical solution when $\mu = -0.2$

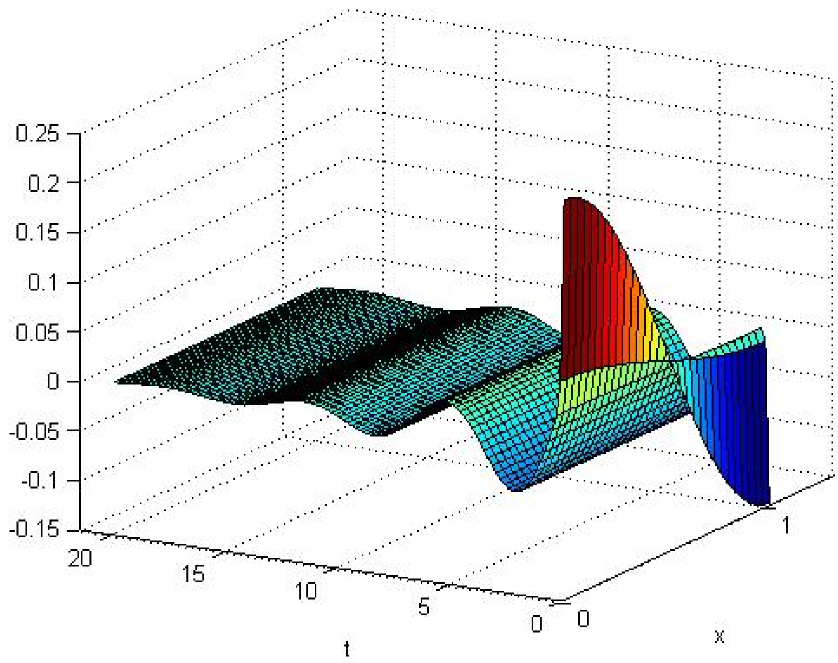

Figure 22: The numerical solution when $\mu = -0.3$

In this case we can see again that the solution tends to zero when $t \to +\infty$.

5.2.2 The behavior of the system when $\mu > \mu_{cr}$

Here we will perform several numerical experiments in the case when $\mu > 0$. We will illustrate the stability of the periodic mode. Perform several numerical simulations with different initial conditions. On the following figures the results are presented.

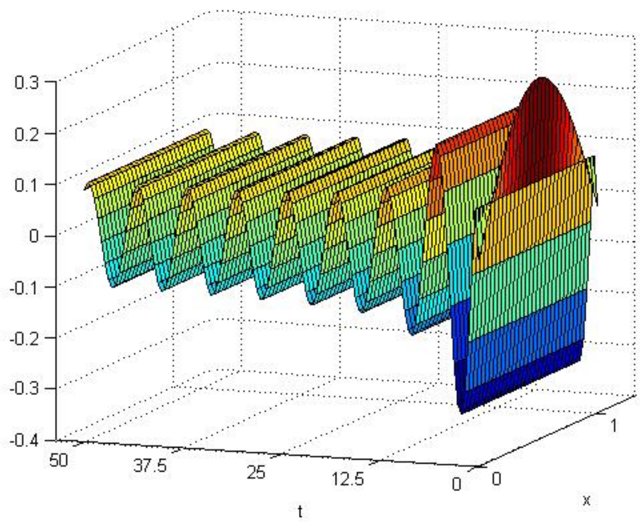

Figure 23: The numerical solution when $u_0(x) = 0.3\sin(x)$

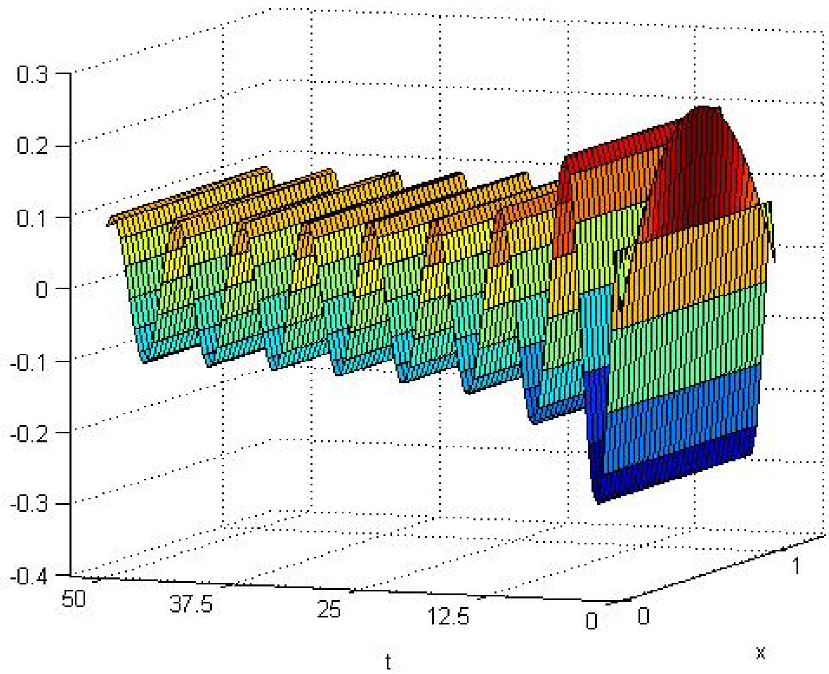

Figure 24: The numerical solution when $u_0(x) = v_0(x) = x(1-x)$

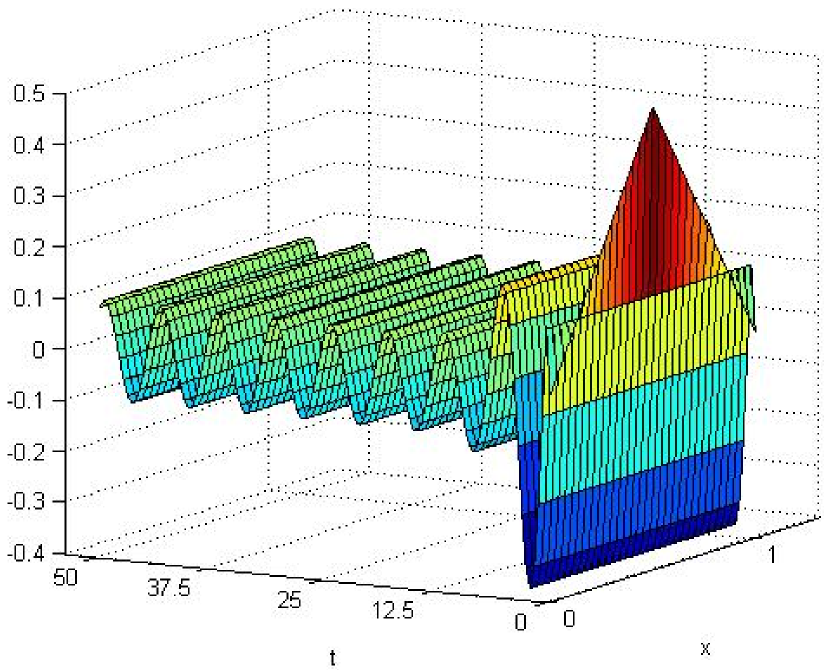

Figure 25: The numerical solution when $u_0(x) = v_0(x) = 0.5 - |x - 0.5|$

We observe that all considered solutions tends to the stable periodic solution when $t \to +\infty$.

6 CONCLUSIONS

In the course of the present work we have performed the analysis of the bifurcations in the spatially distributed Langford system. We have seen that the diffusion in general slows down the frequency of self-oscillations. However, in several special cases of boundary conditions (for example, Neumann boundary conditions), it has no influence on the periodic mode at all. We have found out that the soft loss of stability takes place in the system, which means that a stable limit cycle appears in the system when the equilibrium $x = \omega\nu$ loses the stability.

In the case of Dirichlet boundary conditions and Neumann boundary conditions with additional requirement of zero average,a spatially inhomogeneous periodic mode exists in the system. In that case, the frequency of self-oscillations is lowered by the diffusion. The soft loss of stability takes place in this case in the system as well.

Several numerical experiments were performed in order to support the theoretical results. The case when the parameter value is near the critical value was investigated numerically as well. The simulations were performed in the case of Dirichlet and Neumann boundary conditions. However, when the value of the parameter is significantly greater than critical, the behaviour of the system becomes difficult for the analysis by both analytical and numerical methods. It seems that the system begin to show quasiperiodical oscillations and then chaotic motions start to appear in the system. This case could be investigated further in more detail.

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